Commun. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/CKMS.c170453
pISSN: 1225-1763 / eISSN: 2234-3024

NILPOTENT-DUO PROPERTY ON POWERS

Dong Hwa Kim

Abstract. We study the structure of a generalization of right nilpotent-duo rings in relation with powers of elements. Such a ring property is said to be *weakly right nilpotent-duo*. We find connections between weakly right nilpotent-duo and weakly right duo rings, in several algebraic situations which have roles in ring theory. We also observe properties of weakly right nilpotent-duo rings in relation with their subrings and extensions.

Throughout this paper all rings are associative with identity unless otherwise specified. Let R be a ring. We use $N(R)$ and $U(R)$ to denote the set of all nilpotent elements, and the group of all units in R, respectively. A nilpotent element is also called a nilpotent for simplicity. Denote the n by n full (resp., upper triangular) matrix ring over R by $\text{Mat}_n(R)$ (resp., $T_n(R)$). Write $D_n(R) = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn}\}$ and use E_{ij} for the matrix with (i, j)-entry 1 and elsewhere 0. $R[x]$ denotes the polynomial ring with an indeterminate x over R. $\mathbb{Z}$ ($\mathbb{Z}_n$) denotes the ring of integers (modulo n).

1. Weakly right nilpotent-duo rings

In this section we study the basic properties of weakly right nilpotent-duo ring. Due to Feller [6], a ring is called *right* (resp. *left*) duo if every right (resp. left) ideal is an ideal; and a ring is called *duo* if it is both right and left duo. One may find many useful results for right duo rings in [3,6,15,17]. Due to Bell [2], a ring R is called *IFP* if $ab = 0$ for $a, b \in R$ implies $aRb = 0$. It is easily shown that right (or left) duo rings are IFP. A ring is usually called *Abelian* if every idempotent is central. IFP rings are easily shown to be Abelian.

Following Hong et al. [8], a ring R (possibly without identity) is called *right* (resp., *left*) *nilpotent-duo* if $N(R)a \subseteq aN(R)$ (resp., $aN(R) \subseteq N(R)a$) for every $a \in R$; and a ring is called *nilpotent-duo* if it is both left and right nilpotent-duo. One-sided nilpotent-duo rings are Abelian by [8, Lemma 1.6(1)]. The concepts of right nilpotent-duo and IFP are independent of each other by the arguments in [8].

Received November 13, 2017; Accepted February 1, 2018.
2010 *Mathematics Subject Classification.* 16N40, 16U80.
Key words and phrases. weakly right nilpotent-duo ring, nilpotent, weakly right duo ring, right nilpotent-duo ring, Abelian ring.

©0 Korean Mathematical Society

Ahead of Print

A reduced ring R is right duo if and only if $D_2(R)$ is right nilpotent-duo by [8, Proposition 1.2]. However, by [8, Example 1.7], the nilpotent-duo property of $D_n(R)$ is not valid for the case of $n \geq 3$, where R is a reduced ring. But we obtain an affirmative situation when this argument is related to powers.

Example 1.1. Let A be a division ring and $R = D_n(A)$ for $n \geq 3$. Then R is neither right nor left nilpotent-duo by [8, Example 1.7]. Note $U(R) = \{(a_{ij}) \in R \mid a_{ii} \neq 0\}$ and $N(R) = \{(a_{ij}) \in R \mid a_{ii} = 0\} = N_*(R) = N^*(R)$. Then $R = U(R) \cup N(R)$. Let $0 \neq M \in R$ and $N \in N(R)$.

Suppose $M \in U(R)$. Then $M^k \in U(R)$ for all $k \geq 1$ and so $NM^k = M^k M^{-k} N M^k$, noting that $M^{-k} N M^k \in N(R)$. Suppose $M \in N(R)$. Then $M^n = 0$, and $NM^n = 0 = M^n N$ follows. Therefore $N(R)M^n \subseteq M^n N(R)$.

Based on this example, we introduce the following.

Definition 1.2. A ring R is said to be *weakly right nilpotent-duo* if for every $a \in R$ there exists $n = n(a) \geq 1$ such that $N(R)a^n \subseteq a^n N(R)$. A *weakly left nilpotent-duo* ring is defined similarly. A ring is called *weakly nilpotent-duo* if it is both weakly left and weakly right nilpotent-duo.

Right (resp., left) nilpotent-duo rings are clearly weakly right (resp., left) nilpotent duo. But the converse need not hold by the ring R in Example 1.1.

The following contains basic facts about weakly right nilpotent-duo rings.

Lemma 1.3. (1) *Weakly right* (*left*) *nilpotent-duo rings are Abelian.*

(2) *Let R be a weakly right* (*left*) *nilpotent-duo ring. Then R/I is Abelian for every nil ideal I of R.*

Proof. (1) The proof is almost similar to one of [8, Lemma 6.1(1)]. But we write it here for completeness. Let R be a weakly right nilpotent-duo ring and assume on the contrary that there exist $r, e^2 = e \in R$ with $er(1-e) \neq 0$. Then $er(1-e) \in N(R)$. Set $a = 1-e$ and $b = er(1-e)$. Since R is weakly right nilpotent-duo, $ba^n = a^n c$ for some $n \geq 1$ and $c \in N(R)$. But

$$er(1-e) = (er(1-e))(1-e) = (er(1-e))(1-e)^n = ba^n = a^n c = (1-e)c$$

and

$$0 \neq er(1-e) = e[er(1-e)] = e[a^n c] = e[(1-e)c] = 0,$$

a contradiction. Thus R is Abelian. The proof of the left case is similar.

(2) is proved by [14, Proposition 3.6.1] and (1). □

By Lemma 1.3(2), every factor R/N is Ableian for any nilradical N of a weakly right nilpotent-duo ring R.

Following Yao [18], a ring R is called *weakly right duo* if for each $a \in R$ there exists $n = n(a) \geq 1$ such that $a^n R$ is a two-sided ideal of R, i.e., $Ra^n \subseteq a^n R$. Weakly left duo rings are defined similarly. A ring is called weakly duo if it is both weakly left and weakly right duo. Weakly right duo rings are Abelian by [18, Lemma 4]. Right duo rings are clearly weakly right duo, but the converse

Ahead of Print

need not hold by the next argument. Let R be the ring in Example 1.1. Then each element of R is either a unit or a nilpotent, so R is easily shown to be weakly duo. But R is neither right nor left duo as can be seen by

$$RE_{23} = AE_{13} + AE_{23} \nsubseteq E_{23}R = AE_{23} + AE_{24} + \cdots + AE_{2n}$$

and

$$E_{12}R = AE_{12} + AE_{13} + \cdots + AE_{1n} \nsubseteq RE_{12} = AE_{12}.$$

In the following we see a connection, between weakly right duo rings and weakly right nilpotent-duo rings, that is similar to [8, Proposition 1.2].

Proposition 1.4. *Let R be a reduced ring. Then R is weakly right (resp., left) duo if and only if $D_2(R)$ is weakly right (resp., left) nilpotent-duo.*

Proof. Let $E = D_2(R)$. Since R is reduced, we have $N(E) = \left(\begin{smallmatrix} 0 & R \\ 0 & 0 \end{smallmatrix}\right)$. We apply the proof of [8, Proposition 1.2]. Suppose that R is weakly right duo, and consider $0 \neq A = \left(\begin{smallmatrix} a & c \\ 0 & a \end{smallmatrix}\right) \in E$ and $0 \neq B = \left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right) \in N(E)$.

Since R is weakly right duo, $Ra^k \subseteq a^kR$ for some $k \geq 1$. So $ba^k = a^kb_1$ for some $b_1 \in R$. Let $B_1 = \left(\begin{smallmatrix} 0 & b_1 \\ 0 & 0 \end{smallmatrix}\right)$. Then $B_1 \in N(E)$ and

$$\begin{aligned} BA^k &= \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a^k & c_1 \\ 0 & a^k \end{pmatrix} \\ &= \begin{pmatrix} 0 & ba^k \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a^kb_1 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} a^k & c_1 \\ 0 & a^k \end{pmatrix} \begin{pmatrix} 0 & b_1 \\ 0 & 0 \end{pmatrix} \\ &= A^kB_1, \end{aligned}$$

where $A^k = \left(\begin{smallmatrix} a^k & c_1 \\ 0 & a^k \end{smallmatrix}\right)$. So E is weakly right nilpotent-duo.

Conversely let E be weakly right nilpotent-duo, and consider $c, d \in R$. Let $C = \left(\begin{smallmatrix} c & 0 \\ 0 & c \end{smallmatrix}\right)$ and $D = \left(\begin{smallmatrix} 0 & d \\ 0 & 0 \end{smallmatrix}\right)$. Then $D \in N(E)$. Since E is weakly right nilpotent-duo, there exists $h \geq 1$ such that $N(E)C^h \subseteq C^hN(E)$. So $DC^h = C^hD_1$ for some $D_1 \in N(E)$. Since $N(E) = \left(\begin{smallmatrix} 0 & R \\ 0 & 0 \end{smallmatrix}\right)$, $D_1 = \left(\begin{smallmatrix} 0 & d_1 \\ 0 & 0 \end{smallmatrix}\right)$ for some $d_1 \in R$. Then, from

$$\begin{aligned} \begin{pmatrix} 0 & dc^h \\ 0 & 0 \end{pmatrix} &= \begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c^h & 0 \\ 0 & c^h \end{pmatrix} = DC^h = C^hD_1 \\ &= \begin{pmatrix} c^h & 0 \\ 0 & c^h \end{pmatrix} \begin{pmatrix} 0 & d_1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & c^hd_1 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

we obtain $Rc^h \subseteq c^hR$. Thus R is weakly right duo. The proof of the left case is similar. □

The concepts of weakly right nilpotent-duo and IFP are independent of each other as we see in the following.

Ahead of Print

Example 1.5. Consider $D_n(R)$ over a division ring R for $n \geq 4$. Then $D_n(R)$ is weakly nilpotent-duo by the argument in Example 1.1. However $D_n(R)$ in not IFP by [13, Example 1.3].

Consider next the converse argument. Let $R = K\langle x, y\rangle$ be the free algebra with noncommuting indeterminates x, y over a field K. Note that $xy^n \notin y^n R$ for any $n \geq 1$, so R is not weakly right duo. It then follows that $D_2(R)$ is not weakly right nilpotent-duo by Proposition 1.4. But $D_2(R)$ is IFP by [13, Proposition 1.2].

The following shows that the weak nilpotent-duo property is not left-right symmetric.

Example 1.6. We follow the construction in [12, Example 1] which states that there exists a left duo ring but not weakly right duo. Let $S = F(t)$ be the quotient field of the polynomial ring $F[t]$ with an indeterminate t over a field F, and consider the field monomorphism $\sigma : S \to S$ defined by $\sigma\left(\frac{f(t)}{g(t)}\right) = \frac{f(t^2)}{g(t^2)}$. Next set $R = S[[x; \sigma]]$ be the skew power series ring in which every element is of the form $\sum_{i=0}^{\infty} a_i x^i$, only subject to $xa = \sigma(a)x$ for all $a \in S$. Then R is left duo but not weakly right duo by the argument in [12, Example 1]. This implies that $E = D_2(R)$ is (weakly) left nilpotent-duo by [8, Proposition 1.2] but not weakly right nilpotent-duo by Proposition 1.4.

In the following we examine the property that $D_2(R)$ is not weakly left nilpotent-duo, via a direct computation. Note first that

$$U(E) = \left\{ \left(\begin{smallmatrix} f(x) & r(x) \\ 0 & f(x) \end{smallmatrix}\right) \in E \mid \text{the constant term of } f(x) \text{ is nonzero and } r(x) \in R \right\}$$

and

$$N(E) = \left\{ \begin{pmatrix} 0 & s(x) \\ 0 & 0 \end{pmatrix} \in E \mid s(x) \in R \right\}.$$

Let $A = \left(\begin{smallmatrix} f(x) & f_1(x) \\ 0 & f(x) \end{smallmatrix}\right)$ with $f_1(x) \in R$ and $B = \left(\begin{smallmatrix} 0 & g(x) \\ 0 & 0 \end{smallmatrix}\right)$. If $f(x) = 0$, then $A^2 = 0$ and $BA^2 = 0 = A^2 B$ follows. So we assume $f(x) \neq 0$. We can write $f(x) = a_0 x^k + a_1 x^{k+1} + \cdots \in R$ with $a_0 \neq 0$ and $k \geq 0$. Write $h(x) = \sum_{i=0}^{\infty} a_i x^i$. Then $f(x) = h(x)x^k$ and $h(x) \in U(R)$. Let $s \geq 1$ be any. For $f(x)$ we have

$$\begin{aligned} f(x)^s &= [a_0 \sigma^k(a_0) \sigma^{2k}(a_0) \cdots \sigma^{(s-1)k}(a_0)] x^{sk} + h_1(x) \\ &= [a_0 \sigma^k(a_0) \sigma^{2k}(a_0) \cdots \sigma^{(s-1)k}(a_0) + h_2(x)] x^{sk} \end{aligned}$$

for some $h_1(x), h_2(x) \in R$. Let $v(x) = a_0 \sigma^k(a_0) \sigma^{2k}(a_0) \cdots \sigma^{(s-1)k}(a_0) + h_2(x)$, i.e., $f(x) = v(x)x^{sk}$. Then $v(x) \in U(R)$ because $a_0 \sigma^k(a_0) \sigma^{2k}(a_0) \cdots \sigma^{(s-1)k}(a_0)$ is nonzero. So the argument in [4, Lemma 1.3(3)] is applicable to this case.

Let $g(x) = t^{2l+1} + \sum_{j=1}^{\infty} b_j x^j \in R$ with $l \geq 0$. Then there cannot exist $k(x) \in R$ such that $g(x)f(x)^s = f(x)^s k(x)$ for any $s \geq 1$. This implies that there cannot exist $C = \left(\begin{smallmatrix} 0 & k(x) \\ 0 & 0 \end{smallmatrix}\right) \in E$ such that

$$\begin{pmatrix} 0 & g(x)f(x)^s \\ 0 & 0 \end{pmatrix} = BA^s = A^s C = \begin{pmatrix} 0 & f(x)^s k(x) \\ 0 & 0 \end{pmatrix}.$$

Ahead of Print

(2) In the construction of (1), let $R = S[[x;\sigma]]$ be the skew power series ring, with the same S and σ, in which every element is of the form $\sum_{i=0}^{\infty} x^i a_i$, only subject to $ax = x\sigma(a)$ for $a \in S$. Then $D_2(R)$ can be shown to be (weakly) right nilpotent-duo but not weakly left nilpotent-duo, through a similar computation to the one of (1).

We observe a condition under which the weakly nilpotent-duo property can be left-right symmetric. Recall that an involution on a ring R is a function $* : R \to R$ which satisfies the properties that $(x+y)^* = x^* + y^*$, $(xy)^* = y^*x^*$, $1^* = 1$, and $(x^*)^* = x$ for all $x, y \in R$. Note that $(x^k)^* = (x^*)^k$ for $k \geq 1$ and $0^* = 0$, and hence $x \in N(R)$ implies $x^* \in N(R)$.

Proposition 1.7. (1) *Let R be a ring with an involution $*$. Then R is weakly left nilpotent-duo if and only if R is weakly right nilpotent-duo.*

(2) *Let K be a commutative ring and G be a group. Then the group ring KG is weakly one-sided nilpotent-duo if and only if KG is nilpotent-duo.*

Proof. We extend the proof of [8, Proposition 2.1] to this case.

(1) Suppose that R is weakly left nilpotent-duo. Let $a \in N(R)$ and $b \in R$. Then $(b^*)^k N(R) \subseteq N(R)(b^*)^k$ for some $k \geq 1$. Note $a^* \in N(R)$. So $(b^*)^k a^* = c(b^*)^k$ for some $c \in N(R)$, entailing $(b^k)^* a^* = c(b^k)^*$. This yields

$$ab^k = ((ab^k)^*)^* = ((b^k)^* a^*)^* = (c(b^k)^*)^* = b^k c^*,$$

noting $c^* \in N(R)$. Therefore R is weakly right nilpotent-duo. The proof of the converse is analogous.

(2) Consider the standard involution $*$ on KG, defined by $(\sum a_i g_i)^* = \sum a_i g_i^{-1}$ for all $a_i \in R$ and $g_i \in G$. Then KG is weakly left nilpotent-duo if and only if KG is weakly right nilpotent-duo if and only if KG is weakly nilpotent-duo by (1). □

Let K be a commutative ring and G be a group. If G is Abelian then the group ring KG is commutative, and hence is both duo and nilpotent-duo by Proposition 1.7. We argue about a case of non-Abelian of G in the following.

Proposition 1.8. *Let K be a field of characteristic zero and R be the group ring KQ_8, where Q_8 is the quaternion group. Then the following conditions are equivalent*:

(1) *R is reduced*;
(2) *R is nilpotent-duo*;
(3) *R is right* (*left*) *nilpotent-duo*;
(4) *R is weakly right* (*left*) *nilpotent-duo*;
(5) *R is Abelian*;
(6) *R is right* (*left*) *duo.*

Proof. The proof is done by help of the proof of [8, Proposition 2.2] and the fact that weakly one-sided nilpotent-duo rings are Abelian by Lemma 1.3(1). □

Ahead of Print

2. More properties of weakly right nilpotent-duo rings

In this section, we observe properties of weakly right nilpotent-duo rings in relation with subrings and extensions. We first show that the class of weakly right nilpotent-duo rings is not closed under subrings and homomorphic images.

Example 2.1. (1) The class of weakly right nilpotent-duo rings is not closed under subrings. We take the ring $S = F(t)$ with the monomorphism σ that is constructed in Example 1.6(1). We apply the argument in [8, Example 1.11]. Consider the skew polynomial ring $R_0 = S[x;\sigma]$ in which every element is of the form $\sum_{i=0}^{n} a_i x^i$, only subject to $xa = \sigma(a)x$ for all $a \in S$. Then R_0 is a principal left ideal domain by the left-handed version of [16, Theorem 1.2.9(i, ii)]. So R_0 is a left Noetherian domain and hence R_0 has a left quotient division ring by [16, Theorem 2.1.14], $Q(R_0)$ say.

$D_2(Q(R_0))$ is weakly nilpotent-duo by Proposition 1.4. For our purpose, consider $R = D_2(R_0)$ that is a subring of $D_2(Q(R_0))$. Clearly $N(R) = \left(\begin{smallmatrix} 0 & R_0 \\ 0 & 0 \end{smallmatrix}\right)$. Let $A = \left(\begin{smallmatrix} x & 0 \\ 0 & x \end{smallmatrix}\right) \in R$ and $B = \left(\begin{smallmatrix} 0 & t+x \\ 0 & 0 \end{smallmatrix}\right) \in N(R)$. Let $n \geq 1$ be arbitrary and consider next

$$A^n = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}^n = \begin{pmatrix} x^n & 0 \\ 0 & x^n \end{pmatrix}.$$

Then $BA^n = \left(\begin{smallmatrix} 0 & tx^n + x^{n+1} \\ 0 & 0 \end{smallmatrix}\right)$. Assume on the contrary that there exists $C \in N(R)$ such that $BA^n = A^n C$. $C = \left(\begin{smallmatrix} 0 & c(x) \\ 0 & 0 \end{smallmatrix}\right)$ for some $c(x) \in R_0$, and so $A^n C = \left(\begin{smallmatrix} 0 & x^n c(x) \\ 0 & 0 \end{smallmatrix}\right)$, entailing $tx^n + x^{n+1} = x^n c(x)$. But since R_0 is a domain, $c(x) = a_0 + a_1 x$ for some nonzero $a_0, a_1 \in S$. This yields

$$tx^n + x^{n+1} = x^n(a_0 + a_1 x) = \sigma^n(a_0)x^n + \sigma^n(a_1)x^{n+1},$$

entailing $t = \sigma^n(a_0)$. But this equality is impossible because $\sigma^n(a_0)$ is of the form $\frac{f(t^{2n})}{g(t^{2n})}$ (hence $t \neq \sigma^n(a_0)$). Therefore R is not weakly right nilpotent-duo.

(2) The class of weakly right nilpotent-duo rings is not closed under homomorphic images. Let R be the ring of quaternions with integer coefficients. Then R is a domain and so (weakly) nilpotent-duo. However for any odd prime integer q, the factor ring R/qR is isomorphic to $\mathrm{Mat}_2(\mathbb{Z}_q)$ by the argument in [7, Exercise 2A]. But $\mathrm{Mat}_2(\mathbb{Z}_q)$ is not Abelian, and thus R/qR is not weakly right nilpotent-duo by Lemma 1.3(1).

To see another example, we follow the construction of Antoine [1, Example 4.8]. Let K be a field and $A = K\langle x, y\rangle$ be the free algebra with noncommuting indeterminates x, y over F; and consider the factor ring $R = A/I$ with I the ideal of A generated by x^2. Then A is a domain and so nilpotent-duo. But R is neither left nor right nilpotent-duo as can be seen by $xy^n \notin y^n N(R)$ and $y^n x \notin N(R)y^n$ for all $n \geq 1$, noting $x \in N(R)$. Assume $xy^n = y^n r$ for some $r \in N(R)$. Then $0 = xxy^n = xy^n r \neq 0$ because $xy^n = y^n r \neq 0$, a contradiction.

We consider next a kind of weakly right nilpotent-duo ring whose subrings inherit the weakly right nilpotent-duo property. Let A be an algebra over a commutative ring S. Due to Dorroh [5], the *Dorroh extension* of A by S is the Abelian group $A \oplus S$ with multiplication given by

$$(r_1, s_1)(r_2, s_2) = (r_1 r_2 + s_1 r_2 + s_2 r_1, s_1 s_2)$$

for $r_i \in A$ and $s_i \in S$. We use $A \times_D S$ for the Dorroh extension of A by S. Here A is clearly a subring of $A \times_D S$.

Proposition 2.2. *Let R be a unitary algebra over a commutative reduced ring S. Then R is weakly right nilpotent-duo if and only if the Dorroh extension $R \times_D S$ is weakly right nilpotent-duo.*

Proof. Note that every $s \in S$ is identified with $s1 \in R$ and so we have $R = \{r + s \mid (r, s) \in D\}$. $N(D) = (N(R), 0)$ because S is a commutative reduced ring. Let $D = R \times_D S$.

We apply the proof of [8, Proposition 1.13]. Suppose that R is weakly right nilpotent-duo. Let $(r, s) \in D$ and $(n, 0) \in N(D)$. Then there exists $k \geq 1$ such that $N(R)(r + s)^k \subseteq (r + s)^k N(R)$. So $n(r + s)^k = (r + s)^k n'$ for some $n' \in N(R)$. From this, we obtain

$$\begin{aligned}
(n, 0)(r, s)^k &= (n, 0)((r + s)^k - s^k, s^k) \\
&= (n(r + s)^k - ns^k + ns^k, 0) = (n(r + s)^k, 0) \\
&= ((r + s)^k n', 0) = ((r + s)^k n' - s^k n' + s^k n', 0) \\
&= ((r + s)^k - s^k, s^k)(n', 0) = (r, s)^k (n', 0).
\end{aligned}$$

Since $(n', 0) \in N(D)$, D is weakly right nilpotent-duo.

Conversely, suppose that D is weakly right nilpotent-duo. Let $a \in R$ and $n \in N(R)$. Then $N(D)(a, 0)^h \subseteq (a, 0)^h N(D)$ for some $h \geq 1$. Since $(n, 0) \in N(D)$, there exists $(n'', 0) \in N(D)$ such that $(n, 0)(a, 0)^h = (a, 0)^h (n'', 0)$. This yields

$$\begin{aligned}
(na^h, 0) &= (n, 0)(a^h, 0) = (n, 0)(a, 0)^h = (a, 0)^h (n'', 0) \\
&= (a^h, 0)(n'', 0) = (a^h n'', 0),
\end{aligned}$$

entailing $na^h = a^h n''$. But $(n'', 0) \in N(D)$ implies $n'' \in N(R)$. Therefore R is weakly right nilpotent-duo. □

Following [11], a ring is called *locally finite* if every finite subset generates a finite multiplicative semigroup. A ring is locally finite if and only if every subring generated by a finite subset is finite by [10, Theorem 2.2(1)]. Finite rings are clearly locally finite, and there exist locally finite rings but not finite by the existence of algebraic closures of finite fields.

Proposition 2.3. (1) *Every locally finite Abelian ring is weakly right nilpotent-duo.*

(2) *Let R be a locally finite Abelian ring and I be a proper ideal of R. If I is nil, then R/I is weakly right nilpotent-duo.*

Ahead of Print

(3) *Let R be a reduced locally finite ring. Then $D_2(R)$ is weakly right nilpotent-duo.*

Proof. (1) Let R be a locally finite Abelian ring. Let $a \in R$ and $b \in N(R)$. By the proof of [11, Proposition 16], there exists $n \geq 1$ such that $a^n \in I(R)$. Since R is Abelian, $ba^n = a^n b$. So R is weakly right nilpotent-duo.

(2) R is weakly right nilpotent-duo by (1). Suppose that I is nil. Then R/I is Abelian by Lemma 1.3. R/I is also locally finite, and so R/I is weakly right nilpotent-duo by (1).

(3) Applying the proof of (1), R is weakly right duo. So $D_2(R)$ is weakly right nilpotent-duo by Proposition 1.4. □

In the following we see that the weakly right nilpotent-duo property does not pass to polynomial rings.

Proposition 2.4. (1) *Let R be a ring with $N(R) \neq 0$. Suppose that $R[x]$ is weakly right nilpotent-duo. Then for every $a \in R$, there exists $k \geq 1$ such that $ba^k = a^k b$ for all $b \in N(R)$. Especially, R is weakly right nilpotent-duo.*

(2) *There exists a weakly right nilpotent-duo ring over which the polynomial ring is not weakly right nilpotent-duo.*

(3) *There exists a noncommutative division ring over which the polynomial ring is not weakly right duo.*

Proof. We extend the method in the proof of [8, Theorem 2.9] to the case of power. (1) Suppose that $R[x]$ is weakly right nilpotent-duo. Let $a \in R$ and $b \in N(R)$. Then there exists $k \geq 1$ such that $N(R[x])(a+x)^k \subseteq (a+x)^k N(R[x])$. So $b(a+x)^k = (a+x)^k g(x)$ for some $g(x) = \sum_{i=0}^{m} b_i x^i \in N(R[x])$. Comparing the degrees of both sides of the equality

$$b(a+x)^k = ba^k + \cdots + bx^k = (a+x)^k \Big(\sum_{i=0}^{m} b_i x^i\Big) = a^k b_0 + \cdots + b_m x^{k+m},$$

we obtain $m = 0$, i.e., $g(x) = b_0$. This yields $b = b_0$. Thus $ba^k = a^k b_0 = a^k b$. This implies that R is weakly right nilpotent-duo.

(2) Let S be the ring $Q(R_0)$ in Example 2.1. Then S is a noncommutative division ring in which $tx^k \neq t^{2k} x^k = x^k t$ for all $k \geq 1$. Consider next $R = D_2(S)$. Then R is weakly right nilpotent-duo by Proposition 1.4. Consider two matrices

$$A = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix} \in R \text{ and } B = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \in N(R).$$

Then we obtain

$$A^k B = \begin{pmatrix} 0 & x^k t \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & tx^k \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x^k & 0 \\ 0 & x^k \end{pmatrix} = BA^k$$

for all $k \geq 1$. So $R[x]$ is not weakly right nilpotent-duo by (1).

Ahead of Print

(3) Let S be the noncommutative division ring in the proof of (2). Then $D_2(S)[x]$ is not weakly right nilpotent-duo by the proof of (2). It is well-known that $D_2(S[x])$ is isomorphic to $D_2(S)[x]$ via

$$\begin{pmatrix} \sum_{i=0}^m a_i x^i & \sum_{i=0}^m b_i x^i \\ 0 & \sum_{i=0}^m a_i x^i \end{pmatrix} \mapsto \sum_{i=0}^m \begin{pmatrix} a_i & b_i \\ 0 & a_i \end{pmatrix} x^i,$$

noting that any given two polynomials can be assumed to have the same number of terms by using zero coefficients if necessary. $S[x]$ is clearly a reduced ring, so $S[x]$ is not weakly right duo by Proposition 1.4 because $D_2(S[x])$ ($\cong D_2(S)[x]$) is not weakly right nilpotent-duo. □

Next we observe a kind of factor ring of polynomial rings to which the weakly right nilpotent-duo property is able to pass. Recall the subring $V_n(R) = \{(m_{ij}) \in D_n(R) \mid m_{st} = m_{(s+1)(t+1)}$ for $s = 1, \ldots, n-2$ and $t = 2, \ldots, n-1\}$ of $D_n(R)$ for given a ring R and $n \geq 2$.

Proposition 2.5. *Let R be a locally finite Abelian ring. Then $R[x]/x^nR[x]$ is weakly right nilpotent-duo for every $n \geq 2$.*

Proof. Since R is Abelian, $D_n(R)$ is also Abelian by [9, Lemma 2]. It is well-known that $R[x]/x^nR[x]$ is isomorphic to $V_n(R)$. But $V_n(R)$ is a subring of $D_n(R)$, so $V_n(R)$ is Abelian. Thus $R[x]/x^nR[x]$ is Abelian. Moreover since R is locally finite, $R[x]/x^nR[x]$ is clearly locally finite, entailing that $R[x]/x^nR[x]$ is a locally finite Abelian ring. Therefore $R[x]/x^nR[x]$ is weakly right nilpotent-duo by Proposition 2.3(1). □

References

[1] R. Antoine, *Nilpotent elements and Armendariz rings*, J. Algebra **319** (2008), no. 8, 3128–3140.

[2] H. E. Bell, *Near-rings in which each element is a power of itself*, Bull. Austral. Math. Soc. **2** (1970), 363–368.

[3] H. H. Brungs, *Three questions on duo rings*, Pacific J. Math. **58** (1975), no. 2, 345–349.

[4] Y. W. Chung and Y. Lee, *Structures concerning group of units*, J. Korean Math. Soc. **54** (2017), no. 1, 177–191.

[5] J. L. Dorroh, *Concerning adjunctions to algebras*, Bull. Amer. Math. Soc. **38** (1932), no. 2, 85–88.

[6] E. H. Feller, *Properties of primary noncommutative rings*, Trans. Amer. Math. Soc. **89** (1958), 79–91.

[7] K. R. Goodearl and R. B. Warfield, Jr., *An Introduction to Noncommutative Noetherian Rings*, London Mathematical Society Student Texts, **16**, Cambridge University Press, Cambridge, 1989.

[8] C. Y. Hong, H. K. Kim, N. K. Kim, T. K. Kwak, and Y. Lee, *One-sided duo property on nilpotents*, (submitted).

[9] C. Huh, H. K. Kim, and Y. Lee, *p.p. rings and generalized p.p. rings*, J. Pure Appl. Algebra **167** (2002), no. 1, 37–52.

[10] C. Huh, N. K. Kim, and Y. Lee, *Examples of strongly π-regular rings*, J. Pure Appl. Algebra **189** (2004), no. 1-3, 195–210.

[11] C. Huh, Y. Lee, and A. Smoktunowicz, *Armendariz rings and semicommutative rings*, Comm. Algebra **30** (2002), no. 2, 751–761.

Ahead of Print

[12] H. K. Kim, N. K. Kim, and Y. Lee, *Weakly duo rings with nil Jacobson radical*, J. Korean Math. Soc. **42** (2005), no. 3, 457–470.
[13] N. K. Kim and Y. Lee, *Extensions of reversible rings*, J. Pure Appl. Algebra **185** (2003), no. 1-3, 207–223.
[14] J. Lambek, *Lectures on Rings and Modules*, With an appendix by Ian G. Connell, Blaisdell Publishing Co. Ginn and Co., Waltham, MA, 1966.
[15] G. Marks, *Reversible and symmetric rings*, J. Pure Appl. Algebra **174** (2002), no. 3, 311–318.
[16] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian Rings*, Pure and Applied Mathematics (New York), John Wiley & Sons, Ltd., Chichester, 1987.
[17] G. Thierrin, *On duo rings*, Canad. Math. Bull. **3** (1960), 167–172.
[18] X. Yao, *Weakly right duo rings*, Pure Appl. Math. Sci. **21** (1985), no. 1-2, 19–24.

Dong Hwa Kim
Department of Mathematics Education
Pusan National University
Busan 46241, Korea
Email address: dhgim@pusan.ac.kr

Ahead of Print