

STRUCTURE OF 3-PRIME NEAR-RINGS SATISFYING SOME IDENTITIES

ABDELKARIM BOUA

ABSTRACT. In this paper, we investigate commutativity of 3-prime near-rings \mathcal{N} in which $(1, \alpha)$ -derivations satisfy certain algebraic identities. Some well-known results characterizing commutativity of 3-prime near-rings have been generalized. Furthermore, we give some examples show that the restriction imposed on the hypothesis is not superfluous.

1. Introduction

In the present paper, \mathcal{N} will denote a left near-ring with center $Z(\mathcal{N})$. A near-ring \mathcal{N} is called zero-symmetric if $0x = 0$ for all $x \in \mathcal{N}$ (recall that left distributivity yields $x0 = 0$). \mathcal{N} is 3-prime, that is, for $a, b \in \mathcal{N}$, $a\mathcal{N}b = \{0\}$ implies $a = 0$ or $b = 0$. A non empty subset U of \mathcal{N} is said to be a semigroup left (resp. right) ideal of \mathcal{N} if $\mathcal{N}U \subseteq U$ (resp. $U\mathcal{N} \subseteq U$) and if U is both a semigroup left ideal and a semigroup right ideal, it is called a semigroup ideal of \mathcal{N} . As usual for all x, y in \mathcal{N} , the symbol $[x, y]$ stands for Lie product (commutator) $xy - yx$ and $x \circ y$ stands for Jordan product (anticommutator) $xy + yx$. We note that for a left near-ring, $-(x+y) = -y-x$ and $-xy = x(-y)$. For terminologies concerning near-rings we refer to G. Pilz [9].

An additive mapping $d : \mathcal{N} \rightarrow \mathcal{N}$ is said to be a derivation if $d(xy) = xd(y) + d(x)y$ for all $x, y \in \mathcal{N}$, or equivalently, as noted in [11], that $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{N}$. An additive mapping $d : \mathcal{N} \rightarrow \mathcal{N}$ is called a semiderivation if there exists a map $g : \mathcal{N} \rightarrow \mathcal{N}$ such that $d(xy) = d(x)g(y) + xd(y) = d(x)y + g(x)d(y)$ and $d(g(x)) = g(d(x))$ hold for all $x, y \in \mathcal{N}$. An additive mapping $d : \mathcal{N} \rightarrow \mathcal{N}$ is called a two sided α -derivation if there exists a map $\alpha : \mathcal{N} \rightarrow \mathcal{N}$ such that $d(xy) = d(x)y + \alpha(x)d(y)$ and $d(xy) = d(x)\alpha(y) + xd(y)$ hold for all $x, y \in \mathcal{N}$. An additive mapping $d : \mathcal{N} \rightarrow \mathcal{N}$ is called $(1, \alpha)$ -derivation if there exists a map $\alpha : \mathcal{N} \rightarrow \mathcal{N}$ such that $d(xy) = d(x)y + \alpha(x)d(y)$ holds for all $x, y \in \mathcal{N}$. An additive mapping $d : \mathcal{N} \rightarrow \mathcal{N}$ is called $(\alpha, 1)$ -derivation if there exists a map $\alpha : \mathcal{N} \rightarrow \mathcal{N}$ such that $d(xy) = d(x)\alpha(y) + xd(y)$ holds for all

Received November 7, 2017; Revised March 25, 2018; Accepted April 11, 2018.

2010 *Mathematics Subject Classification*. Primary 16N60, 16W25, 16Y30.

Key words and phrases. 3-prime near-rings, commutativity, $(1, \alpha)$ -derivations, two sided α -derivations.

$x, y \in \mathcal{N}$. Obviously, a two sided α -derivation is both a $(1, \alpha)$ -derivation as well as an $(\alpha, 1)$ -derivation. Also, any derivation on \mathcal{N} is a $(1, \alpha)$ -derivation, but the converse is not true in general (see [6]). The study of commutativity of 3-prime near-rings by using derivations was initiated by H. E. Bell and G. Mason in 1987 (see [5]). In [8] A. A. M. Kamal generalizes some results of Bell and Mason by studying the commutativity of 3-prime near-rings using σ -derivations instead of the usual derivations, where σ is an automorphism on the near-ring. M. Ashraf, A. Ali and Shakir Ali in [1] and N. Aydin and Ö. Gölbası in [7] generalize Kamal's work by using a (s, t) -derivation instead of a s -derivation, where s and t are automorphisms. Recently many authors (see [2], [5], [4] for reference where further references can be found) have studied commutativity of 3-prime near-rings satisfying certain identities involving derivations, semiderivations, two sided α -derivations. Now our aim in this paper is to study the commutativity behavior of 3-prime near-ring which admits $(1, \alpha)$ -derivations satisfying certain properties. In fact, our results generalize, extend and complement several results obtained earlier in [2], [6], [10] on derivations, semiderivations and two sided α -derivations for 3-prime near-rings.

2. Some preliminaries

In this section, we give some well-known results and we add some new lemmas which will be used throughout the next sections of the paper.

Lemma 2.1 ([4, Theorem 2.9]). *Let \mathcal{N} be a 3-prime near-ring. If \mathcal{U} is a nonzero semigroup ideal of \mathcal{N} , then the following assertions are equivalent:*

- (i) $[x, y] \in Z(\mathcal{N})$ for all $x, y \in \mathcal{U}$;
- (ii) \mathcal{N} is a commutative ring.

Lemma 2.2 ([4, Theorem 2.10]). *Let \mathcal{N} be a 2-torsion free 3-prime near-ring and \mathcal{U} be a nonzero semigroup ideal. If $u \circ v \in Z(\mathcal{N})$ for all $u, v \in \mathcal{U}$, then \mathcal{N} is a commutative ring.*

Lemma 2.3. *Let \mathcal{N} be a 3-prime near-ring and \mathcal{U} be a nonzero semigroup ideal of \mathcal{N} .*

- (i) [3, Lemma 1.5] *If $\mathcal{U} \subseteq Z(\mathcal{N})$, then \mathcal{N} is a commutative ring.*
- (ii) [3, Lemma 1.4 (i)] *If $x, y \in \mathcal{N}$ and $x\mathcal{U}y = \{0\}$, then $x = 0$ or $y = 0$.*
- (iii) [3, Lemma 1.3 (i)] *If x is an element of \mathcal{N} such that $\mathcal{U}x = \{0\}$ (resp. $x\mathcal{U} = \{0\}$), then $x = 0$.*
- (iv) *If z centralizes a nonzero semigroup ideal \mathcal{U} , then $z \in Z(\mathcal{N})$.*

Lemma 2.4. *Let \mathcal{N} be a near-ring and d an additive map.*

(i) *If d is a $(1, \alpha)$ -derivation associated with a map α . Then \mathcal{N} satisfies the following property:*

$$\left(d(x)y + \alpha(x)d(y) \right) z = d(x)yz + \alpha(x)d(y)z + \alpha(x)\alpha(y)d(z) - \alpha(xy)d(z)$$

for all $x, y, z \in \mathcal{N}$.

(ii) If d is an $(\alpha, 1)$ -derivation, then \mathcal{N} satisfies the following relation:

$$\left(d(x)\alpha(y) + xd(y) \right) \alpha(t) = d(x)\alpha(yt) + xd(y)\alpha(t) \text{ for all } t, x, y \in \mathcal{N}$$

Proof. (i) We have

$$\begin{aligned} d((xy)z) &= d(xy)z + \alpha(xy)d(z) \\ &= \left(d(x)y + \alpha(x)d(y) \right) z + \alpha(xy)d(z) \text{ for all } x, y, z \in \mathcal{N}. \end{aligned}$$

Also

$$\begin{aligned} d(x(yz)) &= d(x)yz + \alpha(x)d(yz) \\ &= d(x)yz + \alpha(x)d(y)z + \alpha(x)\alpha(y)d(z) \text{ for all } x, y, z \in \mathcal{N}. \end{aligned}$$

Combining the above two equalities, we find that

$$\left(d(x)y + \alpha(x)d(y) \right) z + \alpha(xy)d(z) = d(x)yz + \alpha(x)d(y)z + \alpha(x)\alpha(y)d(z)$$

for all $x, y, z \in \mathcal{N}$, which gives the required result.

(ii) Using the same proof of [10, Lemma 2.1], we find the required result. \square

Lemma 2.5. *Let \mathcal{N} be a 3-prime near-ring and \mathcal{U} be a nonzero semigroup ideal of \mathcal{N} . If \mathcal{N} admits a nonzero $(1, \alpha)$ -derivation d associated with a map α , the following properties are satisfied:*

- (i) If $d(\mathcal{U}) = \{0\}$, then $d = 0$.
- (ii) If $ad(\mathcal{U}) = \{0\}$, $a \in \mathcal{N}$ and $\alpha(\mathcal{U}) = \mathcal{U}$, then $a = 0$.
- (iii) If $d(\mathcal{U})a = \{0\}$, $a \in \mathcal{N}$ and $\alpha(xy) = \alpha(x)\alpha(y)$ for all $x, y \in \mathcal{U}$, then $a = 0$.

Proof. (i) Suppose that $d(\mathcal{U}) = \{0\}$. Then

$$\begin{aligned} 0 &= d(mnu) \\ &= d(m)nu + \alpha(m)d(nu) \\ &= d(m)nu \text{ for all } u \in \mathcal{U}, m \in \mathcal{N}, \end{aligned}$$

which implies that $d(m)\mathcal{N}u = \{0\}$ for all $u \in \mathcal{U}$, $m \in \mathcal{N}$. But \mathcal{N} is 3-prime and $\mathcal{U} \neq \{0\}$, then $d = 0$.

(ii) If $ad(\mathcal{U}) = \{0\}$ and $a \in \mathcal{N}$, then $ad(xy) = 0$ for all $x, y \in \mathcal{U}$. This implies that $ad(x)y + a\alpha(x)d(y) = 0$ for all $x, y \in \mathcal{U}$, and hence $a\alpha(x)d(y) = 0$ for all $x, y \in \mathcal{U}$. But $\alpha(\mathcal{U}) = \mathcal{U}$, then $a\mathcal{U}d(y) = \{0\}$ for all $y \in \mathcal{U}$. Using (i) and Lemma 2.3 (ii), we obtain $a = 0$.

(iii) If $d(\mathcal{U})a = \{0\}$, then $d(xy)a = 0$ for all $x, y \in \mathcal{U}$. By Lemma 2.4, we get $d(x)ya + \alpha(x)d(y)a + \alpha(x)\alpha(y)d(a) - \alpha(xy)d(a) = 0$ for all $x, y \in \mathcal{U}$. Using the given hypothesis, we find that $d(x)ya = 0$ for all $x, y \in \mathcal{U}$, i.e., $d(x)\mathcal{U}a = \{0\}$ for all $x \in \mathcal{U}$. Since $d \neq 0$, we arrive at $a = 0$. \square

3. Main results

In [5], H. E. Bell and G. Mason proved that a 3-prime near-ring \mathcal{N} must be commutative if it admits a derivation d such that $d(\mathcal{N}) \subseteq Z(\mathcal{N})$. This result was generalized by the authors in [6], [10]. They replaced the derivation by a semiderivation or a two sided α -derivation where α is an homomorphism. Our objective in the following theorem is to generalize this result by treating the cases of $(1, \alpha)$ -derivations, $(\alpha, 1)$ -derivations and two sided α -derivations where α is an additive map.

Theorem 3.1. *Let \mathcal{N} be a 3-prime near-ring. If \mathcal{N} admits a nonzero map d such that $d(\mathcal{N}) \subseteq Z(\mathcal{N})$, then \mathcal{N} is a commutative ring if d has one of the following properties:*

- (i) d is a $(1, \alpha)$ -derivation associated with an additive map α .
- (ii) d is a $(\alpha, 1)$ -derivation associated with an additive map α .
- (iii) d is a two sided α -derivation associated with an additive map α .

Proof. (i) Using our assumptions, we have $zd(xy) = d(xy)z$ and $d(z)d(xy) = d(xy)d(z)$ for all $x, y, z \in \mathcal{N}$. By Lemma 2.4, we obtain

$$(3.1) \quad zd(x)y + \alpha(x)d(y) = d(x)yz + \alpha(x)d(y)z + \alpha(x)\alpha(y)d(z) - \alpha(xy)d(z) \text{ and}$$

$$(3.2) \quad \begin{aligned} & d(z)d(x)y + d(z)\alpha(x)d(y) \\ &= d(x)y d(z) + \alpha(x)d(y)d(z) + \alpha(x)\alpha(y)d^2(z) - \alpha(xy)d^2(z). \end{aligned}$$

Since $d^2(z) = d(d(z)) \in Z(\mathcal{N})$, (3.2) becomes

$$(3.3) \quad d^2(z)\mathcal{N}(\alpha(xy) - \alpha(x)\alpha(y)) = \{0\} \text{ for all } x, y, z \in \mathcal{N}.$$

Since \mathcal{N} is 3-prime, we have

$$(3.4) \quad d^2(z) = 0 \text{ or } \alpha(xy) = \alpha(x)\alpha(y) \text{ for all } x, y, z \in \mathcal{N}.$$

Assume that $\alpha(xy) = \alpha(x)\alpha(y)$ for all $x, y \in \mathcal{N}$. For $z = y$, (3.1) implies $d(y)\mathcal{N}[\alpha(x), y] = \{0\}$ for all $x, y \in \mathcal{N}$. Using \mathcal{N} is 3-prime, we obtain $d(y) = 0$ or $y\alpha(x) = \alpha(x)y$ for all $x, y \in \mathcal{N}$. The last two cases give the following equation

$$d(x)\mathcal{N}[y, z] = \{0\} \text{ for all } x, y, z \in \mathcal{N}.$$

Since \mathcal{N} is 3-prime, for each $y \in \mathcal{N}$, either y centralizes \mathcal{N} or $d(\mathcal{N}) = \{0\}$; and by Lemma 2.3(iv) together with Lemma 2.5(i), we conclude that \mathcal{N} is a commutative ring.

Suppose that $d^2(z) = 0$ for all $z \in \mathcal{N}$. We have

$$\begin{aligned} 0 &= d^2(xy) \\ &= d(d(x)y + \alpha(x)d(y)) \\ &= \alpha(d(x)) + d(\alpha(x))d(y) \text{ for all } x, y \in \mathcal{N}, \end{aligned}$$

this implies that $(\alpha(d(x)) + d(\alpha(x)))d(y) = 0$ for all $x, y \in \mathcal{N}$. Replacing y by yt in the preceding equation and using it again, we arrive at

$$d(y)\mathcal{N}(\alpha(d(x)) + d(\alpha(x))) = \{0\} \quad \text{for all } x, y \in \mathcal{N}.$$

By 3-primeness of \mathcal{N} and $d \neq 0$, we obtain

$$(3.5) \quad \alpha(d(x)) + d(\alpha(x)) = 0 \quad \text{for all } x \in \mathcal{N}.$$

Using our hypothesis, we have $d(xd(y)) = d(d(y)x)$ for all $x, y \in \mathcal{N}$. By the definition of d , we get

$$d(x)d(y) + \alpha(x)d^2(y) = d^2(y)x + \alpha(d(y))d(x) \quad \text{for all } x, y \in \mathcal{N},$$

which can be rewritten as

$$(\alpha(d(y)) - d(y))d(x) = 0 \quad \text{for all } x, y \in \mathcal{N}.$$

From the above, one can easily see that

$$(3.6) \quad \alpha(d(y)) = d(y) \quad \text{for all } y \in \mathcal{N}.$$

Using (3.5) and (3.6), we obtain

$$(3.7) \quad d(x) + d(\alpha(x)) = 0 \quad \text{for all } x \in \mathcal{N}.$$

Taking $d(u)$ instead of y in (3.1), we find that

$$\left(\alpha(xd(u)) - \alpha(x)d(u) \right) d(z) = 0 \quad \text{for all } x, u, z \in \mathcal{N}.$$

Which implies that

$$(3.8) \quad \alpha(xd(u)) = \alpha(x)d(u) \quad \text{for all } x, u \in \mathcal{N}.$$

Using (3.6) with the definition of d , we get

$$(3.9) \quad \alpha(d(x)y + \alpha(x)d(y)) = d(x)y + \alpha(x)d(y) \quad \text{for all } x, y \in \mathcal{N}.$$

Since α is an additive map, using the fact that $d(\mathcal{N}) \subseteq Z(\mathcal{N})$ and (3.8), we arrive at

$$(3.10) \quad d(x)\alpha(y) + \alpha^2(x)d(y) = d(x)y + \alpha(x)d(y) \quad \text{for all } x, y \in \mathcal{N}.$$

Setting $x = y$ in (3.10) and using our hypothesis, we obviously get

$$(3.11) \quad (\alpha^2(x) - x)\mathcal{N}d(x) = \{0\} \quad \text{for all } x \in \mathcal{N}.$$

Since \mathcal{N} is 3-prime, by (3.11) we can easily find that

$$(3.12) \quad \alpha^2 = id_{\mathcal{N}} \text{ or } d(x) = 0 \quad \text{for all } x \in \mathcal{N}.$$

Suppose there exists an element $u \in \mathcal{N}$ such that $d(u) = 0$ and writing u instead of y in (3.10), we find that $d(x)\alpha(u) = d(x)u$ for all $x \in \mathcal{N}$ which forces that

$$(3.13) \quad d(\mathcal{N})\mathcal{N}(\alpha(u) - u) = \{0\}.$$

Since $d \neq 0$ and \mathcal{N} is 3-prime, we conclude that $\alpha(u) = u$ so that $\alpha^2(u) = \alpha(u) = u$. In this case, (3.12) forces that $\alpha^2 = id_{\mathcal{N}}$. Using Lemma 2.4, we get

$$(3.14) \quad \alpha(z)d(x)\alpha(y) + \alpha(z)xd(y) = d(x)\alpha(yz) + xd(y)\alpha(z) \quad \text{for all } x, y, z \in \mathcal{N}.$$

For $z = \alpha(x)$, (3.14) becomes

$$(3.15) \quad d(x)\mathcal{N}(\alpha(y\alpha(x)) - x\alpha(y)) = \{0\} \quad \text{for all } x, y \in \mathcal{N}.$$

Since \mathcal{N} is 3-prime, either $d(x) = 0$ or $\alpha(y\alpha(x)) = x\alpha(y)$ for all $x, y \in \mathcal{N}$.

Suppose there exists $x_0 \in \mathcal{N}$ such that $d(x_0) = 0$. By (3.14), we arrive at $d(y)\mathcal{N}[x_0, \alpha(z)] = \{0\}$ for all $y, z \in \mathcal{N}$. Replacing z by $\alpha(u)$ and using the 3-primeness of \mathcal{N} and $d \neq 0$, we get $x_0u = ux_0$ for all $u \in \mathcal{N}$.

If there exists $x_0 \in \mathcal{N}$ such that $\alpha(y\alpha(x_0)) = x_0\alpha(y)$ for all $y \in \mathcal{N}$. Replace z by $\alpha(x_0)$ in (3.14), we get $d(y)\mathcal{N}(x_0x - xx_0) = \{0\}$ for all $x, y \in \mathcal{N}$. Since \mathcal{N} is 3-prime and $d \neq 0$, we obtain $x_0x = xx_0$ for all $x \in \mathcal{N}$. In both cases, we conclude that x centralizes \mathcal{N} which forces that \mathcal{N} is a commutative ring by Lemma 2.3(iv).

(ii) Assume that $d(xy) = d(x)\alpha(y) + xd(y)$ for all $x, y \in \mathcal{N}$. By hypothesis, we have $d(xy) = \alpha(y)d(x) + d(y)x$ for all $x, y \in \mathcal{N}$. Calculating $d(x(y+y))$ in two different ways, we obtain

$$\begin{aligned} d(x)\alpha(y) + xd(y) &= xd(y) + d(x)\alpha(y) \\ &= d(y)x + \alpha(y)d(x) \quad \text{for all } x, y \in \mathcal{N}. \end{aligned}$$

From the last expression, we remark that d plays a role of a $(1, \alpha)$ -derivation, in this case, using the same proof of (i), we find that \mathcal{N} is a commutative ring.

(iii) It is clear that if d is a two sided α -derivation, then d is both a $(1, \alpha)$ -derivation and an $(\alpha, 1)$ -derivation, which proves that \mathcal{N} is a commutative ring by (i) and (ii). \square

In the following, we study the commutativity of a near-ring \mathcal{N} admitting nonzero two sided α -derivations ($(1, \alpha)$ -derivations) d satisfying the condition $d(xy) = d(yx)$ ($d(xy) = -d(yx)$) for all $x, y \in \mathcal{N}$. These results have been demonstrated by several authors in cases the derivations, semiderivations and two sided α -derivations on 3-prime near-rings for more details see the following references [2], [3], [5], [6] and [10]. Our goal in the next part is to generalize these results in the case of $(1, \alpha)$ -derivations and two sided α -derivations where α is an additive map instead of a homomorphism.

Theorem 3.2. *Let \mathcal{N} be a 3-prime near-ring and \mathcal{U} be a nonzero semigroup of \mathcal{N} . If \mathcal{N} admits a nonzero map d such that $d([x, y]) = 0$ for all $x, y \in \mathcal{U}$, then \mathcal{N} is a commutative ring if d has one of the following properties:*

- (i) d is a $(1, \alpha)$ -derivation associated with a map α .
- (ii) d is a two sided α -derivation associated with a map α .

Proof. (i) By our assumptions, we have $d([x, y]) = 0$ for all $x, y \in \mathcal{U}$. Replacing y by xy , then

$$\begin{aligned} 0 &= d([x, xy]) \\ &= d(x)[x, y] + \alpha(x)d([x, y]) \\ &= d(x)[x, y] \quad \text{for all } x, y \in \mathcal{U} \end{aligned}$$

which implies that $d(x)xy = d(x)yx$ for all $x, y \in \mathcal{U}$. Taking yz instead of y where $z \in \mathcal{N}$, we obtain $d(x)\mathcal{U}[x, z] = \{0\}$ for all $x \in \mathcal{U}, z \in \mathcal{N}$. Invoking Lemma 2.3(ii), we get

$$(3.16) \quad d(x) = 0 \quad \text{or} \quad x \in Z(\mathcal{N}) \quad \text{for all } x \in \mathcal{U}.$$

Suppose there is an element $x_0 \in \mathcal{U}$ such that $d(x_0) = 0$. Using the fact that $d(x_0y) = d(yx_0)$ for all $y \in \mathcal{U}$, we obtain $\alpha(x_0)d(y) = d(y)x_0$ for all $y \in \mathcal{U}$. Putting yt instead of y and using Lemma 2.4, we get

$$\begin{aligned} \alpha(x_0)d(y)t + \alpha(x_0)\alpha(y)d(t) &= d(yt)x_0 \\ &= d(y)tx_0 + \alpha(y)d(t)x_0 \quad \text{for all } y \in \mathcal{U}, t \in \mathcal{N} \end{aligned}$$

which can be rewritten as

$$d(y)x_0t + \alpha(x_0)\alpha(y)d(t) = d(y)tx_0 + \alpha(y)d(t)x_0 \quad \text{for all } y \in \mathcal{U}, t \in \mathcal{N}.$$

Taking $t = [u, v]$ in last equation, we obviously get $d(y)(x_0[u, v] - [u, v]x_0) = 0$ for all $y, u, v \in \mathcal{U}$. Calculating the expression $d(y(x_0[u, v] - [u, v]x_0))$, one can easily find that $d(y(x_0[u, v] - [u, v]x_0)) = d(y)(x_0[u, v] - [u, v]x_0) = 0$ for all $y, u, v \in \mathcal{U}$. Substituting yt for y , where $t \in \mathcal{U}$ in the above equation, we arrive at $d(y)\mathcal{U}(x_0[u, v] - [u, v]x_0) = \{0\}$ for all $y, u, v \in \mathcal{U}$. Lemma 2.5(ii) and Lemma 2.3(ii) force that $x_0[u, v] = [u, v]x_0$ for all $u, v \in \mathcal{U}$, in this case, (3.16) forces that $x[u, v] = [u, v]x$ for all $x, u, v \in \mathcal{U}$. Replacing x by xt where $t \in \mathcal{N}$ in the preceding equation and using it again, we arrive at $\mathcal{U}([u, v], t) = \{0\}$ for all $u, v \in \mathcal{U}, t \in \mathcal{N}$. By virtue of Lemma 2.3(iii), we obtain $[u, v] \in Z(\mathcal{N})$ for all $u, v \in \mathcal{U}$ which together with Lemma 2.1, yields that \mathcal{N} is a commutative ring.

(ii) It is clear that if d is a two sided α -derivation, then d is a $(1, \alpha)$ -derivation, which proves that \mathcal{N} is a commutative ring by (i). \square

As an application of Theorem 3.1, we obtain the following corollaries.

Corollary 3.1. *Let \mathcal{N} be a 2-torsion free 3-prime near-ring and d a nonzero derivation.*

- (i) [5, Theorem 2] *If $d(\mathcal{N}) \subseteq Z(\mathcal{N})$, then \mathcal{N} is a commutative ring.*
- (ii) [2, Theorem 4.1] *If $d[x, y] = 0$ for all $x, y \in \mathcal{N}$, then \mathcal{N} is a commutative ring.*

Corollary 3.2. *Let \mathcal{N} be a 2-torsion free 3-prime near-ring and d a nonzero semi-derivation.*

- (i) [6, Theorem 1] *If $d(\mathcal{N}) \subseteq Z(\mathcal{N})$, then \mathcal{N} is a commutative ring.*
- (ii) [6, Theorem 2] *If $d([x, y]) = 0$ for all $x, y \in \mathcal{N}$, then \mathcal{N} is a commutative ring.*

Corollary 3.3. *Let \mathcal{N} be a 2-torsion free 3-prime near-ring and d a nonzero two sided α -derivation.*

- (i) [10, Theorem 1] *If $d(\mathcal{N}) \subseteq Z(\mathcal{N})$, then \mathcal{N} is a commutative ring.*
- (ii) [10, Theorem 2] *If $d([x, y]) = 0$ for all $x, y \in \mathcal{N}$, then \mathcal{N} is a commutative ring.*

The following example shows the necessity of the 3-primeness of \mathcal{N} in the previous theorems.

Example 3.1. Let S be a 2-torsion free near-ring. Let us define \mathcal{N} and $d, \alpha : \mathcal{N} \rightarrow \mathcal{N}$ by:

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & y \\ 0 & 0 & 0 \end{pmatrix} \mid x, y \in S \right\},$$

$$d \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \alpha \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{pmatrix}.$$

It is clear that \mathcal{N} is not a 3-prime near-ring and d is a nonzero two sided α -derivation associated with an additive map α satisfying the following properties:

- (i) $d(\mathcal{N}) \subseteq Z(\mathcal{N})$,
- (ii) $d([A, B]) = 0$ for all $A, B \in \mathcal{N}$,

but, since the addition in \mathcal{N} is not commutative, \mathcal{N} cannot be a commutative ring.

The conclusion of Theorem 3.2 no remains valid if we replace the product $[x, y]$ by $x \circ y$, provided that \mathcal{N} is 2-torsion free. In fact, we obtain the following result.

Theorem 3.3. *Let \mathcal{N} be a 2-torsion free 3-prime near-ring and \mathcal{U} be a nonzero semigroup ideal of \mathcal{N} . Then there exists no nonzero map d on \mathcal{N} such that $d(x \circ y) = 0$ for all $x, y \in \mathcal{U}$ in the following cases:*

- (i) *d is a $(1, \alpha)$ -derivation associated with a map α .*
- (ii) *d is a two sided α -derivation associated with a map α .*

Proof. (i) Suppose that d is a $(1, \alpha)$ -derivation associated with an additive map such that $d(x \circ y) = 0$ for all $x, y \in \mathcal{U}$. Replacing y by xy , then

$$\begin{aligned} 0 &= d(x \circ xy) \\ &= d(x)(x \circ y) + \alpha(x)d(x \circ y) \\ &= d(x)(x \circ y) \quad \text{for all } x, y \in \mathcal{U} \end{aligned}$$

which implies that $d(x)xy = -d(x)yx$ for all $x, y \in \mathcal{U}$. Taking yz instead of y where $z \in \mathcal{N}$, we obtain $d(x)\mathcal{U}(-z(-x) + (-x)z) = \{0\}$ for all $x \in \mathcal{U}$, $z \in \mathcal{N}$. Using Lemma 2.3(ii), we get

$$(3.17) \quad d(x) = 0 \quad \text{or} \quad -x \in Z(\mathcal{N}) \quad \text{for all } x \in \mathcal{U}.$$

Suppose there exists an element $x_0 \in \mathcal{U}$ such that $-x_0 \in Z(\mathcal{N})$. We have

$$\begin{aligned} 0 &= -d(x_0 \circ x_0) \\ &= d(-x_0 \circ x_0) \\ &= d(2(-x_0)x_0). \end{aligned}$$

By the 2-torsion freeness of \mathcal{N} , we get $d((-x_0)x_0) = d(x_0(-x_0)) = 0$. On the other hand, we have

$$\begin{aligned} 0 &= d(x_0 \circ x_0(-x_0)) \\ &= d((-x_0)(x_0^2 + x_0^2)) \\ &= 2d((-x_0)x_0^2) \\ &= d((-x_0)x_0^2) \\ &= d(x_0x_0(-x_0)) \\ &= d(x_0)x_0(-x_0) + \alpha(x_0)d(x_0(-x_0)) \\ &= d(x_0)x_0(-x_0) \end{aligned}$$

which implies that $d(x_0)x_0\mathcal{N}(-x_0) = \{0\}$. In light of 3-primeness of \mathcal{N} , we conclude that $d(x_0)x_0 = 0 = d(x_0)(-x_0)$ and $d(x_0)\mathcal{N}(-x_0) = \{0\}$. By the 3-primeness of \mathcal{N} , we obtain $d(x_0) = 0$. In all cases (3.17) becomes $d(x) = 0$ for all $x \in \mathcal{U}$ which is a contradiction with Lemma 2.5(i).

(ii) It is clear that if d is a two sided α -derivation, then d is a $(1, \alpha)$ -derivation, which proves that \mathcal{N} is a commutative ring by (i). \square

The following example shows the necessity of the 3-primeness of \mathcal{N} in the previous theorems.

Example 3.2. Let S be a 2-torsion free near-ring. Let us define \mathcal{N} , d and $\alpha : \mathcal{N} \rightarrow \mathcal{N}$ by:

$$\mathcal{N} = \left\{ \left(\begin{array}{ccc} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{array} \right) \mid x, y \in S \right\}$$

$$d \left(\begin{array}{ccc} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{array} \right) = \left(\begin{array}{ccc} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \text{ and } \alpha \left(\begin{array}{ccc} 0 & 0 & x \\ 0 & 0 & y \\ 0 & 0 & 0 \end{array} \right) = \left(\begin{array}{ccc} 0 & 0 & y \\ 0 & 0 & y \\ 0 & 0 & 0 \end{array} \right).$$

It is clear that \mathcal{N} is a non 3-prime near-ring and d is a nonzero two sided α -derivation such that $d(A \circ B) = 0$ for all $A, B \in \mathcal{N}$, but \mathcal{N} is not a commutative ring because the addition is not commutative.

Example 3.3. Let $\mathcal{N} = M_2(\mathbb{Z}_3)$ the noncommutative prime ring and d the nonzero map on \mathcal{N} such that $d \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) = \left(\begin{array}{cc} c & d-a \\ 0 & -c \end{array} \right)$. Taking $x = \left(\begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right)$ and $y = \left(\begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array} \right)$. Then $d(x \circ y) = \left(\begin{array}{cc} 1 & 2 \\ 0 & 2 \end{array} \right) \neq 0$, which shows that the condition “ $d(x \circ y) = 0$ for all $x, y \in \mathcal{N}$ ” is necessary.

Example 3.4. Let $\mathcal{N} = \mathbb{Z}_2[x]$. Then \mathcal{N} is an integral domain which means that \mathcal{N} is a commutative prime ring. Also, we have $2\mathcal{N} = \{0\}$. If we take d to be the identical application on N and $\alpha = 0$. Then d is a nonzero $(1, \alpha)$ -derivation and also is a nonzero two sided α -derivation on \mathcal{N} and $d(p \circ q) = 2d(pq) = 0$ for all $p, q \in \mathcal{N}$. But \mathcal{N} is not 2-torsion free.

References

- [1] M. Ashraf, A. Ali, and S. Ali, (σ, τ) -derivations on prime near rings, Arch. Math. (Brno) **40** (2004), no. 3, 281–286.
- [2] M. Ashraf and S. Ali, On (σ, τ) -derivations of prime near-rings. II, Sarajevo J. Math. **4(16)** (2008), no. 1, 23–30.
- [3] H. E. Bell, On derivations in near-rings. II, in Nearrings, nearfields and K -loops (Hamburg, 1995), 191–197, Math. Appl., 426, Kluwer Acad. Publ., Dordrecht, 1997.
- [4] H. E. Bell, A. Boua, and L. Oukhtite, Semigroup ideals and commutativity in 3-prime near rings, Comm. Algebra **43** (2015), no. 5, 1757–1770.
- [5] H. E. Bell and G. Mason, On derivations in near-rings, in Near-rings and near-fields (Tübingen, 1985), 31–35, North-Holland Math. Stud., 137, North-Holland, Amsterdam, 1987.
- [6] A. Boua and L. Oukhtite, Semiderivations satisfying certain algebraic identities on prime near-rings, Asian-Eur. J. Math. **6** (2013), no. 3, 1350043, 8 pp.
- [7] Ö. Gölbaşı and N. Aydin, Results on prime near-ring with (σ, τ) -derivation, Math. J. Okayama Univ. **46** (2004), 1–7.
- [8] A. A. M. Kamal, σ -derivations on prime near-rings, Tamkang J. Math. **32** (2001), no. 2, 89–93.
- [9] G. Pilz, Near-Rings, 2nd ed., North Holland /American Elsevier, Amsterdam, 1983.
- [10] M. S. Samman, L. Oukhtite, A. Raji, and A. Boua, Two sided α -derivations in 3-prime near-rings, Rocky Mountain J. Math. **46** (2016), no. 4, 1379–1393.
- [11] X. K. Wang, Derivations in prime near-rings, Proc. Amer. Math. Soc. **121** (1994), no. 2, 361–366.

ABDELKARIM BOUA
 SIDI MOHAMMED BEN ABDELLAH UNIVERSITY
 POLYDISCIPLINARY FACULTY, LSI, TAZA, MOROCCO
 Email address: abdelkarimboua@yahoo.fr