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DISTRIBUTIONAL FRACTIONAL POWERS OF SIMILAR OPERATORS WITH APPLICATIONS TO THE BESSEL OPERATORS

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Abstract. This paper provides a method to study the nonnegativity of certain linear operators, from other operators with similar spectral properties. If these new operators are formally self-adjoint and nonnegative, we can study the complex powers using an appropriate locally convex space. In this case, the initial operator also will be nonnegative and we will be able to study its powers. In particular, we have applied this method to Bessel-type operators.

1. Introduction

Operators of Bessel type appear in the literature related with different versions of Hankel transform (see [1, 3, 4, 11]). We are going to consider Bessel operators on $\mathbb{R}_+ = (0, \infty)$ given by

$$\Delta_\mu = \frac{d^2}{dx^2} + (2\mu + 1)(x^{-1}\frac{d}{dx}) = x^{-2\mu-1}\frac{d}{dx}x^{2\mu+1}\frac{d}{dx} \tag{1}$$

and

$$S_\mu = \frac{d^2}{dx^2} - \frac{4\mu^2 - 1}{4x^2} = x^{-\mu-\frac{1}{2}}\frac{d}{dx}x^{2\mu+1}\frac{d}{dx}x^{-\mu-\frac{1}{2}}, \tag{2}$$

which are related through

$$S_\mu = x^{\mu+\frac{1}{2}}\Delta_\mu x^{-\mu-\frac{1}{2}}.$$

This feature has inspired us to develop a method to study its fractional powers based in a concept of similar operator. Similar operators have the same spectral properties and also that of being nonnegative if one of them has this property. This method will be applied in the contexts of Banach spaces and locally convex spaces as follows: Let X and Y be Banach spaces. Suppose that we have an isometric isomorphism $T : X \to Y$ and let $T^{-1} : Y \to X$ be its inverse. Let A

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be a linear operator $A : D(A) \subset X \to X$. Then we can consider the operator $B = TAT^{-1}$, $B : D(B) \subset Y \to Y$ with domain $D(B) = \{x \in Y : T^{-1}x \in D(A)\}$ given by

$$B = TAT^{-1}. \tag{3}$$

Under these conditions we will say that A and B are *similar*. If A and B are similar operators, then

$$(zId - B)^{-1} = T(zId - A)^{-1}T^{-1}$$

for a complex number z, and we deduce immediately that A is a nonnegative operator if and only if so is B.

If A is a nonnegative operator, then for $\alpha \in \mathbb{C}$ such that $\operatorname{Re}\alpha > 0$, $n > \operatorname{Re}\alpha$, $n \in \mathbb{N}$, and $\phi \in D(A^n)$, the Balakrisnahn operator associated with A, can be represented by

$$J_A^\alpha \phi = \frac{\Gamma(m)}{\Gamma(\alpha)\Gamma(m-\alpha)} \int_0^\infty \lambda^{\alpha-1}[A(\lambda + A)^{-1}]^m \phi \, d\lambda,$$

(see [5, Proposition 3.1.3, p. 59]).

If A is bounded, J_A^α can be considered as the fractional power of A, and in another case we can consider the following representation for the fractional power (see [5, Theorem 5.2.1, p. 114]),

$$A^\alpha = (A + \lambda)^n J_A^\alpha (A + \lambda)^{-n},$$

with α, n as above and $\lambda \in \rho(-A)$.

When two operators are similar, the fractional powers also meet this property. Thus, we have the following result:

Proposition 1.1. *Let A and B be similar nonnegative operators. If $\alpha \in \mathbb{C}$ such that $Re\,\alpha > 0$, then*

$$J_B^\alpha = TJ_A^\alpha T^{-1}, \tag{4}$$

and

$$B^\alpha = TA^\alpha T^{-1}, \tag{5}$$

where T is the isometric isomorphism that verifies $B = TAT^{-1}$.

The Bessel operator (1) appears when one considers the Laplacian operator in polar coordinates for radial functions. In [2], the regularity of solutions to fractional nonlocal Bessel equation given by

$$(-\Delta_\mu)^\alpha u = f,$$

is studied in $\mathbb{R}_+$. In this paper, the fractional Bessel operator $(-\Delta_\mu)^\alpha$ considered in the above formula is defined by

$$(-\Delta_\mu)^\alpha u = H_\mu(x^{2\alpha} H_\mu u), \tag{6}$$

where H_μ is the Hankel transform given by (39) (see Appendix). In [7], we obtained the following representation in $L^2(\mathbb{R}^n_+)$ for fractional powers of Bessel operator (2)

$$(-S_\mu)^\alpha u = h_\mu(|x|^{2\alpha} h_\mu u), \tag{7}$$

where h_μ is the Hankel transform given by (18) and is related to H_μ through $h_\mu(\phi)(y) = y^{\mu+\frac{1}{2}} H_\mu(x^{-\mu-\frac{1}{2}}\phi)(y)$. Note that the equality (6) in $L^2(\mathbb{R}^n_+)$, can be obtained from (7) using the last equality, similarity of operators S_μ and Δ_μ and Proposition 1.1.

In order to apply the method described above for similar operators to Bessel operators (1) and (2), we study the nonnegativity of Bessel operator (2) in weighted L^p-spaces, for $n = 1$. Let $L^p(\mathbb{R}_+, sr^p)$ for $1 \leq p < \infty$ and $L^\infty(\mathbb{R}_+, r)$ be the Lebesgue spaces with the norms

$$\|f\|_{L^p(\mathbb{R}_+, sr^p)} = \left[\int_0^\infty |f(x)|^p \, s(x) r^p(x) dx\right]^{1/p},$$

and

$$\|f\|_{L^\infty(\mathbb{R}_+, r)} = \|rf\|_\infty \, .$$

respectively, and s and r given by

$$s = x^{2\mu+1}/c_\mu, \tag{8}$$

$$r = x^{-\mu-\frac{1}{2}} \tag{9}$$

with $c_\mu = 2^\mu \Gamma(\mu+1)$. For $p = 2$, the $L^p(\mathbb{R}_+, sr^p)$ coincides with $L^2(\mathbb{R}_+)$.

We will denote by $S_{\mu,p}$ the part of S_μ in $L^p(\mathbb{R}_+, sr^p)$; namely, the operator S_μ with domain

$$D(S_{\mu,p}) = \{f \in L^p(\mathbb{R}_+, sr^p) : S_\mu f \in L^p(\mathbb{R}_+, sr^p)\}$$

and given by $S_{\mu,p} f = S_\mu f$.

Analogously, by $S_{\mu,\infty}$ we will denote the part of S_μ in $L^\infty(\mathbb{R}_+, r)$; namely, the operator S_μ with domain

$$D(S_{\mu,\infty}) = \{f \in L^\infty(\mathbb{R}_+, r) : S_\mu f \in L^\infty(\mathbb{R}_+, r)\}$$

and $S_{\mu,\infty} f = S_\mu f$. Under these conditions we obtained the following result:

Theorem 1.2. *Given $\mu > -\frac{1}{2}$. Then*

1. *The operators $S_{\mu,p}$ and $S_{\mu,\infty}$ are closed.*
2. *The operators $-S_{\mu,p}$ and $-S_{\mu,\infty}$ are nonnegative.*

Moreover, another feature of operators S_μ and Δ_μ is that one of them is formally self-adjoint (considering the inner product in the usual $L^2(\mathbb{R}_+)$), and the other is not. In order to define the complex powers of a differential operator in distributional spaces is important that this operator be formally self-adjoint.

It would therefore be interesting to obtain an operator similar and formally self-adjoint from a given initial operator. In this case S_μ is formally self-adjoint, and we obtain the following duality formula

$$((-(S_\mu))^\alpha T, \phi) = (T, (-(S_\mu))^\alpha \phi), \;\; (\phi \in \mathcal{B}, T \in \mathcal{B}'),$$

where $\mathcal{B}$ is a suitable locally convex space and $\mathcal{B}'$ is its corresponding strong dual defined in Sections 5 and 6. In Theorem 5.5 we established the nonnegativity of $-(S_\mu)$ in $\mathcal{B}$ from where we infer immediately the nonnegativity of $-(S_\mu)$ in $\mathcal{B}'$.

In Section 2 we will review some of the standard facts about Hankel transforms, convolution and Bessel operators in distributional and Lebesgue spaces, which are fundamental to establish the nonnegativity of Bessel operator. In Section 3 we will provide the proof of Proposition 1.1. Moreover, we will extend this idea to locally convex spaces and apply these ideas to the Bessel operators.

In Section 4 we will establish a series of lemmas which will be used in the proof of Theorem 1.2. In Sections 5 and 6 we will establish the nonnegativity of S_μ in a suitable locally convex space and in its dual space.

For the convenience of the reader, we have added an Appendix with the proofs of some results about the theory related with Hankel transform, thus making our exposition self-contained.

2. Some preliminaries on Hankel transform and convolution

In this section we introduce the distributional spaces necessary for our purposes.

By $\mathcal{D}(\mathbb{R}_+)$ we denote the space of functions in $C^\infty(\mathbb{R}_+)$ with compact support in $\mathbb{R}_+$ and with the usual topology, and by $\mathcal{D}'(\mathbb{R}_+)$ the space of classical distributions in $\mathbb{R}_+$.

Throughout this paper we assume $\mu > -\frac{1}{2}$. We will consider the Hankel transform defined in a suitable functional space denoted by $\mathcal{H}_\mu$ and given by

$$\mathcal{H}_\mu = \left\{ \phi \in C^\infty(\mathbb{R}_+) : \sup_{x\in\mathbb{R}_+} \left| x^m (x^{-1}D)^k x^{-\mu-\frac{1}{2}} \phi(x) \right| < \infty : m, k = 0, 1, 2, \ldots \right\}$$

endowed with the family of seminorms $\left\{ \gamma^\mu_{m,k} \right\}$, given by

$$\gamma^\mu_{m,k}(\phi) = \sup_{x\in\mathbb{R}_+} \left| x^m (x^{-1}D)^k x^{-\mu-\frac{1}{2}} \phi(x) \right|, \tag{10}$$

$\mathcal{H}_\mu$ is a Fréchet space (see [11, Lemma 5.2-2, p. 131]).

Now we consider the weighted Lebesgue spaces $L^p(\mathbb{R}_+, sr^p)$ with $1 \le p < \infty$ and $L^\infty(\mathbb{R}_+, r)$ given in the Introduction. We have the following lemma:

Lemma 2.1. *It holds that*

$$\mathcal{H}_\mu \subset L^1(\mathbb{R}_+, sr) \cap L^\infty(\mathbb{R}_+, r) \subset L^p(\mathbb{R}_+, sr^p), \quad 1 \le p < \infty, \tag{11}$$

with s and r given by (8) *and* (9).

Proof. The inclusion $\mathcal{H}_\mu \subset L^\infty(\mathbb{R}_+, r)$ is immediate and also

$$\|\phi\|_{L^\infty(\mathbb{R}_+,r)} = \gamma^\mu_{0,0}(\phi), \quad \phi \in \mathcal{H}_\mu. \tag{12}$$

It also verifies that $\mathcal{H}_\mu \subset L^1(\mathbb{R}_+, sr)$ as

$$\int_0^\infty |\phi|\, sr\, dx = \int_0^1 |x^{-\mu-\frac{1}{2}}\phi| x^{2\mu+1} c_\mu^{-1}\, dx + \int_1^\infty x^m |x^{-\mu-\frac{1}{2}}\phi| x^{-m+2\mu+1} c_\mu^{-1}\, dx < \infty$$

if $m > 2\mu + 2$, and

$$\|\phi\|_{L^1(\mathbb{R}_+,sr)} \le C\{\gamma^\mu_{0,0}(\phi) + \gamma^\mu_{m,0}(\phi)\}, \quad \phi \in \mathcal{H}_\mu. \tag{13}$$

It also verifies that

$$\begin{aligned} \|\phi\|_{L^p(\mathbb{R}_+,sr^p)} &= \left\{\int_0^\infty |\phi|^{p-1} r^{p-1} |\phi| rs\right\}^{\frac{1}{p}} \\ &\le \left\{\|\phi\|_{L^\infty(\mathbb{R}_+,r)}\right\}^{\frac{p-1}{p}} \left\{\|\phi\|_{L^1(\mathbb{R}_+,sr)}\right\}^{\frac{1}{p}} \end{aligned} \tag{14}$$

and by (12) and (13) we can consider a constant C' such that

$$\|\phi\|_{L^\infty(\mathbb{R}_+,r)} \le C' \left[\gamma^\mu_{0,0}(\phi) + \gamma^\mu_{m,0}(\phi)\right], \quad \phi \in \mathcal{H}_\mu, \tag{15}$$

$$\|\phi\|_{L^1(\mathbb{R}_+,sr)} \le C' \left[\gamma^\mu_{0,0}(\phi) + \gamma^\mu_{m,0}(\phi)\right], \quad \phi \in \mathcal{H}_\mu \tag{16}$$

and by (14), (15) and (16) we finally conclude that

$$\|\phi\|_{L^p(\mathbb{R}_+,sr^p)} \le C' \left[\gamma^\mu_{0,0}(\phi) + \gamma^\mu_{m,0}(\phi)\right], \quad \phi \in \mathcal{H}_\mu. \tag{17}$$

□

If J_μ denote the Bessel function of first kind and order μ, we consider the Hankel transform h_μ given by

$$h_\mu\phi(x) = \int_0^\infty \sqrt{xy} J_\mu(xy)\phi(y) dy \tag{18}$$

for $\phi \in \mathcal{H}_\mu$.

Remark 2.2. If $\phi \in L^1(\mathbb{R}_+, sr)$, then Hankel transform $h_\mu\phi$ is well defined because the kernel $(xy)^{-\mu} J_\mu(xy)$ is bounded if $\mu > -\frac{1}{2}$ (see [10, (1), p. 49]). By Lemma 2.1, $h_\mu\phi$ is well defined for all $\phi \in \mathcal{H}_\mu$ and is an automorphism of $\mathcal{H}_\mu$(see [11, Theorem 5.4-1, p. 141]).

The space of the continuous linear functions $T : \mathcal{H}_\mu \to \mathbb{C}$ is denoted by $\mathcal{H}'_\mu$. We call a function $f \in L^1_{loc}(\mathbb{R}_+)$ a *regular element* of $\mathcal{H}'_\mu$ if the application $T_f \in \mathcal{H}'_\mu$, where $T_f(\phi) = \int_0^\infty f\phi$, $\phi \in \mathcal{H}_\mu$.

Remark 2.3. Given $T \in \mathcal{H}'_\mu$, we can consider the restriction of T to $\mathcal{D}(\mathbb{R}_+)$ as a member of $\mathcal{D}'(\mathbb{R}_+)$, because convergence in $\mathcal{D}(\mathbb{R}_+)$ implies convergence in $\mathcal{H}_\mu$. But $\mathcal{D}(\mathbb{R}_+)$ is not dense in $\mathcal{H}_\mu$ (see [11]), consequently the behavior of an element $u \in \mathcal{H}'_\mu$ over $\mathcal{D}(\mathbb{R}_+)$ not determines univocally the behavior of u as element of $\mathcal{H}'_\mu$. If a locally integrable function f defines a regular element of

$\mathcal{H}'_\mu$ and $T_f\big|_{\mathcal{D}(\mathbb{R}_+)} = 0$, then $f = 0$, a.e. in $\mathbb{R}_+$. So, regular distributions in $\mathcal{H}'_\mu$ are included in injective way in $\mathcal{D}'(\mathbb{R}_+)$.

Lemma 2.4. *Suppose that* $1 \le p < \infty$. *A function in* $L^p(\mathbb{R}_+, sr^p)$ *or* $L^\infty(\mathbb{R}_+, r)$ *is a regular element of* $\mathcal{H}'_\mu$. *In particular, the functions in* $\mathcal{H}_\mu$ *can be considered as regular elements of* $\mathcal{H}'_\mu$.

Proof. Let $f \in L^\infty(\mathbb{R}_+, r)$ and $\phi \in \mathcal{H}_\mu$. Since

$$\mathcal{H}_\mu \subset L^1(\mathbb{R}_+, sr) = L^1(\mathbb{R}_+, r^{-1}/c_\mu),$$

then $\phi \in L^1(\mathbb{R}_+, r^{-1})$ and $(T_f, \phi) = \int_0^\infty f\phi$ is well defined. So, by (13)

$$\begin{aligned} |(T_f, \phi)| &\le \|f\|_{L^\infty(\mathbb{R}_+,r)} \|\phi\|_{L^1(\mathbb{R}_+,r^{-1})} = c_\mu \|f\|_{L^\infty(\mathbb{R}_+,r)} \|\phi\|_{L^1(\mathbb{R}_+,sr)} \\ &\le Cc_\mu \|f\|_{L^\infty(\mathbb{R}_+,r)} \left[\gamma^\mu_{0,0}(\phi) + \gamma^\mu_{m,0}(\phi)\right]. \end{aligned}$$

Consequently, f is a regular element of $\mathcal{H}'_\mu$.

Now, let $f \in L^p(\mathbb{R}_+, sr^p)$ with $1 \le p < \infty$ and $\phi \in \mathcal{H}_\mu$, then

$$|(T_f, \phi)| \le \int_0^\infty |f\phi| = \int_0^\infty (r\,|f|)\left(s^{-1}r^{-1}\,|\phi|\right) s = \int_0^\infty (r\,|f|)\,(c_\mu r\,|\phi|)\, s. \tag{19}$$

Since $r\,|f| \in L^p(\mathbb{R}_+, s)$ and $r\,|\phi| \in L^q(\mathbb{R}_+, s)$, being q the conjugate of p, by Hölder inequality and (17) we obtain that

$$\begin{aligned} |(T_f, \phi)| &\le c_\mu \|f\|_{L^p(\mathbb{R}_+,sr^p)} \|\phi\|_{L^q(\mathbb{R}_+,sr^q)} \\ &\le Cc_\mu \|f\|_{L^p(\mathbb{R}_+,sr^p)} \left[\gamma^\mu_{0,0}(\phi) + \gamma^\mu_{m,0}(\phi)\right] \end{aligned}$$

with m a positive integer $m > 2\mu + 2$. So, f is a regular element of $\mathcal{H}'_\mu$. ☐

Given f, g defined in $\mathbb{R}_+$, the Hankel convolution $f\sharp g$ is defined formally by

$$(f\sharp g)(x) = \int_0^\infty \int_0^\infty D_\mu(x,y,z) f(y) g(z)\, dydz, \tag{20}$$

where, for every $x, y, z \in \mathbb{R}_+$, $D_\mu(x,y,z)$ is given by

$$D_\mu(x,y,z) = \begin{cases} \dfrac{2^{\mu-1}(xyz)^{-\mu+\frac12}}{\Gamma(\mu+\frac12)\sqrt{\pi}}(A(x,y,z))^{2\mu-1} & \text{if } \; |x-y| < z < x+y \\ 0 & \text{if } \; 0 < z < |x-y| \; \text{ or } \; z > x+y. \end{cases} \tag{21}$$

$A(x,y,z)$ is the measure of area of the triangle with sides $x, y, z \in \mathbb{R}_+$. Note that $|x-y| < z < x+y$ is the condition for such a triangle to exist, and in this case $A(x,y,z) = \frac14\sqrt{[(x+y)^2 - z^2][z^2 - (x-y)^2]}$.

The two following lemmas arise from adapting the classical result of [3] to this context.

Lemma 2.5. *Let* $f \in L^1(\mathbb{R}_+, sr)$.

(1) *If* $g \in L^\infty(\mathbb{R}_+, r)$, *then the convolution* $f\sharp g(x)$ *exists for every* $x \in \mathbb{R}_+$, *and* $f\sharp g \in L^\infty(\mathbb{R}_+, r)$ *with*

$$\|f\sharp g\|_{L^\infty(\mathbb{R}_+,r)} \le \|f\|_{L^1(\mathbb{R}_+,sr)} \|g\|_{L^\infty(\mathbb{R}_+,r)}. \tag{22}$$

(2) *If* $g \in L^p(\mathbb{R}_+, sr^p)$ $(1 \le p < \infty)$, *then the convolution* $f\sharp g(x)$ *exists for a.e.* $x \in \mathbb{R}_+$, *and* $f\sharp g \in L^p(\mathbb{R}_+, sr^p)$ *with*

$$\|f\sharp g\|_{L^p(\mathbb{R}_+, sr^p)} \le \|f\|_{L^1(\mathbb{R}_+, sr)} \|g\|_{L^p(\mathbb{R}_+, sr^p)}. \tag{23}$$

Lemma 2.6. *Let* $f, g \in L^1(\mathbb{R}_+, sr)$. *Then*

$$h_\mu (f\sharp g) = r h_\mu(f) h_\mu(g). \tag{24}$$

The following lemma may be found dispersed in different classical papers of Hankel transforms, for which we consider it convenient to add a prove in the Appendix.

Lemma 2.7. *Let* $\{\phi_n\} \subset L^1(\mathbb{R}_+, rs)$ *such that*

(1) $\phi_n \ge 0$ *in* $\mathbb{R}_+$,
(2) $\int_0^\infty \phi_n(x) r(x) s(x)\, dx = 1$ *for all* n,
(3) *For* $\delta > 0$, $\lim_{n\to\infty} \int_\delta^\infty \phi_n(x) r(x) s(x)\, dx = 0$.

Let $f \in L^\infty(\mathbb{R}_+, r)$ *and continuous in* $x_0 \in \mathbb{R}_+$. *Then* $\lim_{n\to\infty} f\sharp\phi_n(x_0) = f(x_0)$. *Further, if* rf *is uniformly continuous in* $\mathbb{R}_+$, *then* $\lim_{n\to\infty} \|f\sharp\phi_n(x) - f(x)\|_{L^\infty(\mathbb{R}_+, r)} = 0$

2.1. The Bessel operator S_μ

In this section we summarize some elementary properties of S_μ on the spaces $\mathcal{H}_\mu$ and $\mathcal{H}'_\mu$. For most of the proofs we refer the reader to [11].

Lemma 2.8. (1) *The operator* $S_\mu : \mathcal{H}_\mu \longrightarrow \mathcal{H}_\mu$ *is continuous.*
(2) *If* $\lambda \ge 0$, *the operator*

$$\begin{array}{ccc} \mathcal{H}_\mu & \longrightarrow & \mathcal{H}_\mu \\ \phi & \longrightarrow & (\lambda + x^2)\phi \end{array}$$

is continuous.
(3) *If* $\lambda > 0$, *the operator*

$$\begin{array}{ccc} \mathcal{H}_\mu & \longrightarrow & \mathcal{H}_\mu \\ \phi & \longrightarrow & (\lambda + x^2)^{-1}\phi \end{array}$$

is continuous.

Lemma 2.9. *Let* $\phi \in \mathcal{H}_\mu$. *Then*

(1) $h_\mu S_\mu \phi = -y^2 (h_\mu \phi)$.
(2) $S_\mu h_\mu \phi = h_\mu(-x^2\phi)$.

Lemma 2.10. *The following continuous operators in* $\mathcal{H}_\mu$ *can be extended to* $\mathcal{H}'_\mu$ *in the following way*:

(1) *The Hankel transform* h_μ

$$(h_\mu u, \phi) = (u, h_\mu \phi), \quad u \in \mathcal{H}'_\mu, \phi \in \mathcal{H}_\mu,$$

and $h_\mu : \mathcal{H}'_\mu \to \mathcal{H}'_\mu$ *is a bijective mapping.*

(2) *The differential operator* S_μ

$$(S_\mu u, \phi) = (u, S_\mu \phi), \quad u \in \mathcal{H}'_\mu, \phi \in \mathcal{H}_\mu.$$

(3) *The product by* $(\lambda + x^2)$ *for* $\lambda \geq 0$

$$((\lambda + x^2)u, \phi) = (u, (\lambda + x^2)\phi), \quad u \in \mathcal{H}'_\mu, \phi \in \mathcal{H}_\mu.$$

(4) *The product by* $(\lambda + x^2)^{-1}$ *for* $\lambda > 0$

$$((\lambda + x^2)^{-1}u, \phi) = (u, (\lambda + x^2)^{-1}\phi), \quad u \in \mathcal{H}'_\mu, \phi \in \mathcal{H}_\mu.$$

Lemma 2.11. *If* $u \in \mathcal{H}'_\mu$, *then*

(1) $h_\mu S_\mu u = -x^2 h_\mu u$.
(2) $S_\mu h_\mu u = h_\mu(-y^2 u)$.

Lemma 2.12. *The following equalities are valid in* $\mathcal{H}_\mu$ *and* $\mathcal{H}'_\mu$ *for* $n = 1, 2, \dots,$ $\lambda \in \mathbb{C}$

(1) $(-S_\mu + \lambda)^n h_\mu = h_\mu(y^2 + \lambda)^n$.
Moreover if $\lambda > 0$,
(2) $h_\mu(-S_\mu + \lambda)^{-n} = (y^2 + \lambda)^{-n} h_\mu$.
(3) $h_\mu(-S_\mu(-S_\mu + \lambda)^{-1})^n = y^{2n}(y^2 + \lambda)^{-n} h_\mu$.

Proof. (1) is immediate consequence of item (2) of Lemma 2.11.

Since $(y^2 + \lambda)$ and $(y^2 + \lambda)^{-1}$ are multipliers in $\mathcal{H}_\mu$ and $\mathcal{H}'_\mu$, (Lemma 2.8), then $h_\mu(y^2 + \lambda)^{-1} h_\mu$ is inverse operator of $-S_\mu + \lambda$. Thus, (2) is obtained by a simple application of Proposition 7.2 (see Appendix) and induction over n.

Equality (3) follows immediately by item (2) of Lemma 2.11 and induction over n. □

3. Similar operators and nonnegativity

In this section we include a brief review of nonnegative operators in Banach spaces and in locally convex spaces.

Let X be a Banach space (real or complex). Let A be a closed linear operator $A : D(A) \subset X \to X$ and $\rho(A)$ the resolvent set of A. We say that A is nonnegative if $(-\infty, 0) \subset \rho(A)$ and

$$\sup_{\lambda \in \mathbb{R}_+} \{\|\lambda(\lambda + A)^{-1}\|\} < \infty.$$

Now, we will give the definition of nonnegative operator in the context of locally convex spaces. Let X be a locally convex space with a Hausdorff topology generated by a directed family of seminorms $\{\| \, \|_\alpha\}_{\alpha \in \Lambda}$. A family of linear operators $\{A_\lambda\}_{\lambda \in \Gamma}$, $A_\lambda : D(A_\lambda) \subset X \to X$, is equicontinuous if for each $\alpha \in \Lambda$ there are $\beta = \beta(\alpha) \in \Lambda$ and a constant $C = C_\alpha \geq 0$ such that for all $\lambda \in \Gamma$

$$\|A_\lambda \phi\|_\alpha \leq C \|\phi\|_\beta, \quad \phi \in X.$$

Under the above conditions, we say that a closed linear operator $A : D(A) \subset X \to X$ is nonnegative if $(-\infty, 0) \subset \rho(A)$ and the family of operators

$$\{\lambda(\lambda + A)^{-1}\}_{\lambda \in \mathbb{R}_+}$$

is equicontinuous.

With the same notation as the Introduction, let A and B be similar operators, i.e.,

$$B = TAT^{-1}$$

with T an isometric isomorphism $T : X \to Y$ and $A : D(A) \subset X \to X$ $B = TAT^{-1}$, $B : D(B) \subset Y \to Y$ and X, Y Banach spaces. In this section we prove Proposition 1.1.

Proof of Proposition 1.1. Let $\alpha \in \mathbb{C}$ such that $\operatorname{Re}\alpha > 0$ and $n \in \mathbb{N}$, $n > \operatorname{Re}\alpha$. We observe that if $B = TAT^{-1}$, then $B^n = TA^nT^{-1}$ and

$$\begin{aligned} D(TJ_A^\alpha T^{-1}) &= \{x \in Y : T^{-1}x \in D(J_A^\alpha)\} = \{x \in Y : T^{-1}x \in D(A^n)\} \\ &= D(B^n) = D(J_B^\alpha), \end{aligned}$$

and (4) is immediate from properties of Bochner integral. In (5) the equality of domains is evident and

$$\begin{aligned} B^\alpha &= (B + \lambda)^n J_B^\alpha (B + \lambda)^{-n} = (TAT^{-1} + \lambda)^n TJ_A^\alpha T^{-1} (TAT^{-1} + \lambda)^{-n} \\ &= T(A + \lambda)^n T^{-1} TJ_A^\alpha T^{-1} T(A + \lambda)^{-n} T^{-1} = TA^\alpha T^{-1}. \end{aligned}$$

□

In the same way as for Banach spaces, one can consider similar operators in locally convex spaces, i.e., given $A : D(A) \subset X \to X$ and $B : D(B) \subset Y \to Y$ linear operators in the locally convex spaces X and Y, and

$$B = TAT^{-1},$$

with $T : X \to Y$ an isomorphism of locally convex spaces, then A and B are similar operators. In this case we obtain again the nonnegativity of B from that of A and Theorem 1.1 can be easily extended to the case of nonnegative operators in locally convex spaces.

3.1. Applications to Bessel operator

Given $\mu > -\frac{1}{2}$, we consider the differential operator Δ_μ given by (1) in $\mathbb{R}_+$.

We are now going to apply the observations considered in the previous section to the operator Δ_μ. First, we calculate the Sturm-Liouville form of Δ_μ, thereby obtaining the operator

$$T_\mu = x^{2\mu+1}\Delta_\mu$$

which is formally self-adjoint (in the same sense as in Section 1). Operators of type $fT_\mu f$, with $f \in C^\infty(\mathbb{R}_+)$, are still formally self-adjoint. If we want the new operator to be similar to Δ_μ, namely type $r^{-1}\Delta_\mu r$, we have to consider $r = x^{-\mu-\frac{1}{2}}$. Thus the operator

$$S_\mu = x^{\mu+\frac{1}{2}}\,\Delta_\mu\, x^{-\mu-\frac{1}{2}} \tag{25}$$

is formally self-adjoint and similar to Δ_μ and hence with the same spectral properties.

Since mappings $L_r : L^p(\mathbb{R}_+, r^p s) \to L^p(\mathbb{R}_+, s)$ with $1 \le p < \infty$ (or $L_r : L^\infty(\mathbb{R}_+, r) \to L^\infty(\mathbb{R}_+)$) given by $L_r(f) = rf$ are isometric isomorphisms, if we consider the part of the distributional operator Δ_μ in the spaces $L^p(\mathbb{R}_+, s)$ (or $L^\infty(\mathbb{R}_+)$), i.e., the operator with domain

$$D((\Delta_\mu)_{L^p(\mathbb{R}_+,s)}) = \{f \in L^p(\mathbb{R}_+, s) : \Delta_\mu f \in L^p(\mathbb{R}_+, s)\}\,,$$

and given by $(\Delta_\mu)_{L^p(\mathbb{R}_+,s)} f = \Delta_\mu f$. Then, applying the ideas developed in the previous section, it is enough to study the operator S_μ in the spaces $L^p(\mathbb{R}_+, sr^p)$ (or $L^\infty(\mathbb{R}_+, r)$).

4. Fractional powers of S_μ in Lebesgue spaces

In this section we will prove Theorem 1.2 enunciated in the Introduction. This theorem establish the nonnegativity of the parts in $L^p(\mathbb{R}_+, sr^p)$ and in $L^\infty(\mathbb{R}_+, r)$ of distributional differential operator S_μ given by (2).

Let $1 \le p < \infty$. We will denote by $S_{\mu,p}$ the part of S_μ in $L^p(\mathbb{R}_+, sr^p)$; i.e., the operator S_μ with domain

$$D\left(S_{\mu,p}\right) = \{f \in L^p(\mathbb{R}_+, sr^p) : S_\mu f \in L^p(\mathbb{R}_+, sr^p)\}$$

and given by $S_{\mu,p} f = S_\mu f$.

Analogously, by $S_{\mu,\infty}$ we will denote the part of S_μ in $L^\infty(\mathbb{R}_+, r)$; namely, the operator S_μ with domain

$$D\left(S_{\mu,\infty}\right) = \{f \in L^\infty(\mathbb{R}_+, r) : S_\mu f \in L^\infty(\mathbb{R}_+, r)\}$$

and $S_{\mu,\infty} f = S_\mu f$.

In order to study the nonnegativity of operators $-S_{\mu,\infty}$ and $-S_{\mu,p}$ we consider the following function:

$$\mathcal{K}_\nu(x) = \frac{1}{2}\Big(\frac{1}{2}x\Big)^\nu \int_0^\infty e^{-t-\frac{x^2}{4t}} \frac{dt}{t^{\nu+1}} \tag{26}$$

for $x \in \mathbb{R}_+$. Since for $\nu < 0$

$$\int_0^\infty e^{-t-\frac{x^2}{4t}} t^{-\nu-1}\, dt \le \int_0^\infty e^{-t} t^{-\nu-1}\, dt < \infty$$

and for $\nu \ge 0$ the function $e^{-t-\frac{x^2}{4t}} t^{-\nu-1}$ is bounded in a neighborhood of zero, $\mathcal{K}_\nu$ is well defined for $\nu \in \mathbb{R}$ and $\mathcal{K}_\nu > 0$.

Remark 4.1. For noninteger values of ν, $\mathcal{K}_\nu$ (see [10, (15), p. 183]), coincides with the Macdonald's function K_ν (see [10, (6) and (7), p. 78]) given by

$$\mathrm{K}_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi} \quad x \in \mathbb{R}_+,$$

with I_υ is the modified Bessel function over $\mathbb{R}_+$ (see [10, (2), p. 77]). For integers values of ν, K_ν is defined by

$$\mathrm{K}_n(x) = \lim_{\nu\to n} \mathrm{K}_\nu(x), \quad x \in \mathbb{R}_+.$$

Now, given $\lambda > 0$, we consider the function

$$N_\lambda(x) = \lambda^{\frac{\mu}{2}} x^{\frac{1}{2}} \mathcal{K}_\mu(\sqrt{\lambda}\, x), \quad x \in \mathbb{R}_+.$$

The following lemmas describe properties of the kernel N_λ which are crucial for the study of the nonnegativity of Bessel operator (for proofs see Appendix).

Lemma 4.2. *Given $\mu > -\frac{1}{2}$ and $\lambda > 0$. Then*

a) *$N_\lambda \in L^1(\mathbb{R}_+, sr) = L^1(\mathbb{R}_+, \frac{r^{-1}}{c_\mu})$ and*

$$\|N_\lambda\|_{L^1(\mathbb{R}_+,sr)} = \frac{1}{\lambda}.$$

b)

$$h_\mu N_\lambda(y) = \frac{y^{\mu+\frac{1}{2}}}{\lambda + y^2}.$$

Lemma 4.3. *Let $1 \le p < \infty$. If $f \in L^p(\mathbb{R}_+, sr^p)$ or $L^\infty(\mathbb{R}_+, r)$. Then the following equality holds on $\mathcal{H}'_\mu$*

$$h_\mu(N_\lambda \sharp f) = \frac{1}{\lambda + y^2}\, h_\mu(f). \tag{27}$$

Now, we can prove Theorem 1.2:

Proof of Theorem 1.2. (1) Let $\{f_n\}_{n=1}^\infty \subset D\left(S_{\mu,\infty}\right)$ such that

$$\lim_{n\to\infty} f_n = f \qquad \text{and} \qquad \lim_{n\to\infty} S_{\mu,\infty} f_n = g$$

in $L^\infty(\mathbb{R}_+, r)$. Since convergence in $L^\infty(\mathbb{R}_+, r)$ implies convergence in $\mathcal{D}'(\mathbb{R}_+)$, then given $\phi \in \mathcal{D}(\mathbb{R}_+)$

$$(S_\mu f, \phi) = (f, S_\mu \phi) = \lim_{n\to\infty} (f_n, S_\mu\phi) = \lim_{n\to\infty} (S_\mu f_n, \phi) = (g, \phi),$$

so, $S_\mu f = g$ and $S_{\mu,\infty}$ is closed. The case of $S_{\mu,p}$ is similar.

(2) Let $\lambda > 0$ and $f \in D\left(S_{\mu,\infty}\right)$ such that $\left(\lambda - S_{\mu,\infty}\right) f = 0$. Then $\left(\lambda - S_{\mu,\infty}\right) f \in L^\infty(\mathbb{R}_+, r)$ and is null as regular element of $\mathcal{H}'_\mu$, so

$$h_\mu\big(\lambda - S_{\mu,\infty} f\big) = 0$$

in $\mathcal{H}'_\mu$. By Lemma 2.11, we obtain that

$$(\lambda + y^2) h_\mu f = 0$$

in $\mathcal{H}'_\mu$, and hence by Lemma 2.10

$$h_\mu f = (\lambda + y^2)^{-1} (\lambda + y^2) h_\mu f = 0.$$

Then, $f = 0$ as element of $\mathcal{H}'_\mu$ and by Remark 2.3 we conclude that $f = 0$ a.e. in $(0, \infty)$ and, $\lambda - S_{\mu,\infty}$ is injective. Now, let $f \in L^\infty(\mathbb{R}_+, r)$ and $g = N_\lambda \sharp f$. Then, by Lemma 2.5, $g \in L^\infty(\mathbb{R}_+, r)$ and

$$h_\mu\big((\lambda - S_{\mu,\infty})\, g\big) = (\lambda + y^2)\, h_\mu g = (\lambda + y^2)\, h_\mu(N_\lambda \sharp\, f) = h_\mu f.$$

By injectivity of Hankel transform in $\mathcal{H}'_\mu$ we obtain that

$$(\lambda - S_{\mu,\infty})\, g = f,$$

so, $\lambda - S_{\mu,\infty}$ is onto. Also,

$$\begin{aligned}\left\|(\lambda - S_{\mu,\infty})^{-1} f\right\|_{L^\infty(\mathbb{R}_+,r)} &= \|g\|_{L^\infty(\mathbb{R}_+,r)} = \|N_\lambda \sharp f\|_{L^\infty(\mathbb{R}_+,r)} \\ &\le \|N_\lambda\|_{L^1(\mathbb{R}_+,rs)}\, \|f\|_{L^\infty(\mathbb{R}_+,r)} = \frac{1}{\lambda}\, \|f\|_{L^\infty(\mathbb{R}_+,r)}\end{aligned}$$

hence $-S_{\mu,\infty}$ is nonnegative. The proof of nonnegativity of $-S_{\mu,p}$ is similar. □

Remark 4.4. In [7], the result of theorem above has been obtained for the particular case $p = 2$ and in $\mathbb{R}^n_+$.

Now, in view of nonnegativity of $-S_{\mu,\infty}$ and $-S_{\mu,p}$ we can consider the complex fractional powers. If $\alpha \in \mathbb{C}$, $\operatorname{Re}\alpha > 0$ and $n > \operatorname{Re}\alpha$, then the fractional power of $-S_{\mu,\infty}$ can be represented by:

$$(-S_{\mu,\infty})^\alpha = (-S_{\mu,\infty} + 1)^n \mathcal{J}^\alpha_\infty (-S_{\mu,\infty} + 1)^{-n}$$

(see [5, (5.20), p. 114]), where with $\mathcal{J}^\alpha_\infty$ we denote the Balakrishnan operator associated to $-S_{\mu,\infty}$ given by:

$$\mathcal{J}^\alpha_\infty \phi = \frac{\Gamma(n)}{\Gamma(\alpha)\Gamma(n-\alpha)} \int_0^\infty \lambda^{\alpha-1} \Big[-S_{\mu,\infty}(\lambda - S_{\mu,\infty})^{-1}\Big]^n \phi\, d\lambda$$

for α and n in the previous conditions and $\phi \in D((-S_{\mu,\infty})^n)$ (see [5, (3.4), p. 59]). The case of $(-S_{\mu,p})^\alpha$ is analogous.

5. Nonnegativity of Bessel operator S_μ in the space $\mathcal{B}$

In order to study non-negativity of Bessel operator in a locally convex space, we begin with the following observation:

Remark 5.1. The continuous operator $-S_\mu : \mathcal{H}_\mu \to \mathcal{H}_\mu$ is not nonnegative.

Indeed, if we suppose that $-S_\mu$ is nonnegative in $\mathcal{H}_\mu$, by the continuity of $-S_\mu$ in $\mathcal{H}_\mu$, given $\alpha \in \mathbb{C}$, $0 < \alpha < 1$ and according to [5, Chapter 5, p. 105 and 134]), we have that fractional power $(-S_\mu)^\alpha$ would be given by

$$(-S_\mu)^\alpha \phi = \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{\alpha-1}(-S_\mu)(\lambda - S_\mu)^{-1}\phi\, d\lambda \tag{28}$$

and $D((-S_\mu)^\alpha) = D(-S_\mu) = \mathcal{H}_\mu$. Applying the Hankel transform in (28) we obtain

$$h_\mu\Big((-S_\mu)^\alpha \phi\Big)(y) = \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{\alpha-1} h_\mu\Big((-S_\mu)(\lambda - S_\mu)^{-1}\phi\Big)\, d\lambda$$

$$= \frac{\sin \alpha\pi}{\pi} \int_0^\infty \lambda^{\alpha-1} y^2 (\lambda + y^2)^{-1} h_\mu \phi(y)\, d\lambda$$
$$= (y^2)^\alpha h_\mu \phi(y),$$

where we have consider the interchange between the Bochner integral and continuous operators in the first equality. We have used (3) of Lemma 2.12 in the second equality and [5, Remark 3.1.1]) in the last equality. In this case it would mean that $(y^2)^\alpha h_\mu \phi(y) \in \mathcal{H}_\mu$ which is false in general (just consider $\phi(y) = y^{\mu+\frac{1}{2}} e^{-y^2}$ and $\alpha = \frac{1}{4}$).

Now, we consider the Banach space $Y = L^1(\mathbb{R}_+, sr) \cap L^\infty(\mathbb{R}_+, r)$ with the norm

$$\|f\|_Y = \max\left(\|f\|_{L^1(\mathbb{R}_+,sr)}, \|f\|_{L^\infty(\mathbb{R}_+,r)}\right),$$

and the part of the Bessel operator in Y, $(S_\mu)_Y$, with domain

$$D\big[(S_\mu)_Y\big] = \{f \in Y : S_\mu f \in Y\}.$$

From Theorem 1.2 it is evident that $-(S_\mu)_Y$ is closed and nonnegative. We have the following proposition:

Proposition 5.2. $D\big[(S_\mu)_Y\big] \subset C_0(\mathbb{R}_+)$.

Proof. By (11) and (41), $L^1(\mathbb{R}_+, sr) \cap L^\infty(\mathbb{R}_+, r) \subset L^1(\mathbb{R}_+) \bigcap L^2(\mathbb{R}_+)$, then for $f \in D\big[(S_\mu)_Y\big]$, f and $S_\mu f$ are in $L^1(\mathbb{R}_+)$. By Remark 7.4 (see Appendix) then $h_\mu f - h_\mu S_\mu f$ are in $L^\infty(\mathbb{R}_+)$. By (1) of Lemma 2.11 we have that

$$(1 + y^2)|h_\mu f| \leq M,$$

so, $h_\mu f \in L^1(\mathbb{R}_+)$.

We have thus proved that for $f \in D\big[(S_\mu)_Y\big]$ then f and $h_\mu(f)$ are in $L^1(\mathbb{R}_+) \bigcap L^2(\mathbb{R}_+)$. Then, by Remark 7.3 (see Appendix), we obtain that

$$h_\mu(h_\mu(f))(x) = f(x), \quad \text{a.e.} \quad x \in \mathbb{R}_+.$$

Since $h_\mu(f) \in L^1(\mathbb{R}_+)$ then by Proposition 7.5 (see Appendix), $f = g$ a.e. in $\mathbb{R}_+$ with $g \in C_0(\mathbb{R}_+)$. □

Now, we consider the following space:

$$\mathcal{B} = \{f \in Y : (S_\mu)^k f \in Y \quad \text{for} \quad k = 0, 1, 2, \dots\} = \bigcap_{k=0}^\infty D\big[((S_\mu)_Y)^k\big],$$

with the seminorms

$$\rho_m(f) = \max_{0 \leq k \leq m} (\|(S_\mu)^k f\|_Y), \quad m = 0, 1, 2, \dots.$$

Remark 5.3. From Proposition 5.2 it is evident that $\mathcal{B} \subset C^\infty(\mathbb{R}_+) \bigcap C_0(\mathbb{R}_+)$. Moreover, it is clear from (11) that $\mathcal{B} \subset L^p(\mathbb{R}_+, sr^p)$ for all $1 \leq p < \infty$, and considering (1) of Proposition 2.8 we have that $\mathcal{H}_\mu \subset \mathcal{B}$ and the topology of

$\mathcal{H}_\mu$ induced by $\mathcal{B}$ is weaker than the usual topology given in Section 2. Indeed, from (15) and (16) we have that

$$\|\phi\|_Y \leq C\left[\gamma^\mu_{0,0}(\phi) + \gamma^\mu_{k,0}(\phi)\right], \quad \phi \in \mathcal{H}_\mu \tag{29}$$

for $k > 2\mu+2$, and by continuity of S_μ in $\mathcal{H}_\mu$, we deduce that given a seminorm ρ_m there exist a finite set of seminorms $\{\gamma^\mu_{m_i,k_i}\}_{i=1}^r$ and constants $c_1, \dots, c_r$ such that

$$\rho_m(\phi) \leq \sum c_i \gamma^\mu_{m_i,k_i}(\phi), \quad \phi \in \mathcal{H}_\mu.$$

Moreover, from the density of $\mathcal{D}(\mathbb{R}_+)$ in $\mathcal{B}$ we deduce the density of $\mathcal{H}_\mu$ in $\mathcal{B}$.

Proposition 5.4. *$\mathcal{B}$ is not normable.*

Proof. Suppose that $\mathcal{B}$ is normable. Then $\mathcal{B}$ is locally bounded and consequently 0 has a bounded neighborhood (see [9], Theorem 1.39). Since ρ_m is a increasing family of seminorms, then there exists an integer positive n such that the set

$$V_n = \left\{\phi \in \mathcal{B} : \rho_n(\phi) < \frac{1}{n}\right\},$$

are bounded. Consequently, there exists a constant $t_n > 0$ such that

$$V_n \subset t_n V_{n+1}. \tag{30}$$

Let $\phi \in \mathcal{B}$ and $\varphi = \big((n+1)\rho_n(\phi)\big)^{-1}\phi$. Then $\varphi \in V_n$ and by (30) $(t_n)^{-1}\varphi \in V_{n+1}$, and hence $\rho_{n+1}((t_n)^{-1}\varphi) < \frac{1}{n+1}$, so

$$\rho_{n+1}(\phi) \leq t_n \rho_n(\phi). \tag{31}$$

Given a constant $l > 0$ and $f, g \in C^{2k}(\mathbb{R}_+)$ related by $f(x) = g(lx)$, we have that

$$(S_\mu)^k f(x) = (l^2)^k((S_\mu)^k g)(lx). \tag{32}$$

Now, let $\phi \in \mathcal{B}$ such that $(S_\mu)^{n+1}\phi$ is not an identically vanishing function and a constant $s > 1$. Then $\psi(x) = \phi(s^{-1}x)$ remains in $\mathcal{B}$ and verified that $(S_\mu)^{n+1}\psi$ is not an identically vanishing function and $\phi(x) = \psi(sx)$. Then,

$$\begin{aligned}\|(S_\mu)^{n+1}\psi\|_{L^\infty(\mathbb{R}_+,r)} &= s^{-\mu-\frac{1}{2}}s^{-2(n+1)}\|(S_\mu)^{n+1}\phi\|_{L^\infty(\mathbb{R}_+,r)} \\ &\leq s^{-\mu-\frac{1}{2}}s^{-2(n+1)}t_n\rho_n(\phi) \\ &\leq s^{-\mu-\frac{1}{2}}s^{-2(n+1)}s^{\mu+\frac{1}{2}}s^{2n}t_n\rho_n(\psi) \\ &= s^{-2}t_n\rho_n(\psi).\end{aligned} \tag{33}$$

Since (33) is verified for all $s > 1$, taking $s \to \infty$, we conclude that

$$\|(S_\mu)^{n+1}\psi\|_{L^\infty(\mathbb{R}_+,r)} = 0$$

which contradicts the assumption about ψ. Then the proposition follows. □

We denote with $(S_\mu)_\mathcal{B}$ the part of Bessel operator S_μ in $\mathcal{B}$. By definition of $\mathcal{B}$, it is evident that the domain of $(S_\mu)_\mathcal{B}$ is $\mathcal{B}$ and the following result holds.

Theorem 5.5. *$\mathcal{B}$ is a Fréchet space and $-(S_\mu)_\mathcal{B}$ is continuous and nonnegative operator on $\mathcal{B}$.*

Proof. The proof is immediate by Proposition 1.4.2 given in [5]. □

6. Nonnegativity of Bessel operator S_μ in the distributional space $\mathcal{B}'$

In this section we study the nonnegativity of Bessel operator in the topological dual space of $\mathcal{B}$ with the strong topology, i.e., the space $\mathcal{B}'$ with the seminorms $\{|.|_B\}$, where the sets B are in the family of bounded sets in $\mathcal{B}$, and are given by

$$|T|_B = \sup_{\phi\in B} |(T,\phi)|, \quad T \in \mathcal{B}'.$$

Remark 6.1. As in [6, Remark 3.4, p. 263], $\mathcal{B}'$ is sequentially complete because $\mathcal{B}$ is not normable. Moreover, for $1 \le p \le \infty$ we have $L^p(\mathbb{R}_+, sr^p) \subset \mathcal{B}'$. To prove this, we observe that given $f \in L^p(\mathbb{R}_+, sr^p)$ and $\phi \in \mathcal{B}$ and q the conjugate of p then

$$\Big|\int_0^\infty f\phi\Big| = \Big|\int_0^\infty f\phi s^{-1}r^{-p}sr^p\Big| \le \|f\|_{L^p(\mathbb{R}_+,sr^p)}\|\phi s^{-1}r^{-p}\|_{L^q(\mathbb{R}_+,sr^p)}, \tag{34}$$

and

$$\|\phi s^{-1}r^{-p}\|_{L^q(\mathbb{R}_+,sr^p)} = \Big\{\int_0^\infty |\phi s^{-1}r^{-p}|^q sr^p\Big\}^{\frac{1}{q}} = \Big\{\int_0^\infty |\phi|^q(c_\mu r^2 r^{-p})^q sr^p\Big\}^{\frac{1}{q}}$$

$$= c_\mu\Big\{\int_0^\infty |\phi|^q r^{2q-pq+p}s\Big\}^{\frac{1}{q}} = c_\mu\Big\{\int_0^\infty |\phi|^q sr^q\Big\}^{\frac{1}{q}}. \tag{35}$$

Moreover, by (14) we have that

$$\|\phi\|_{L^q(\mathbb{R}_+,sr^q)} \le \rho_0(\phi), \tag{36}$$

and from (34), (35) and (36) we obtain that $f \in \mathcal{B}'$.

Now, let B be a bounded set in $\mathcal{B}$ then

$$\sup_{\phi\in\mathcal{B}}\Big|\int_0^\infty f\phi\Big| \le c_\mu\|f\|_{L^p(\mathbb{R}_+,sr^p)} \sup_{\phi\in\mathcal{B}} \|\phi\|_{L^q(\mathbb{R}_+,sr^q)} \le c_\mu\|f\|_{L^p(\mathbb{R}_+,sr^p)} \sup_{\phi\in\mathcal{B}} \rho_0(\phi).$$

Consequently, the topology in $L^p(\mathbb{R}_+, sr^p)$ induced by $\mathcal{B}'$ with strong topology is weaker than the usual topology.

Remark 6.2. By Remark 5.3, $\mathcal{B}' \subset \mathcal{H}'_\mu$. Moreover, from the continuity of the Bessel operator in $\mathcal{B}$, we can consider S_μ in $\mathcal{B}'$ as adjoint operator of S_μ in $\mathcal{B}$, that is

$$(S_\mu T, \phi) = (T, S_\mu\phi), \quad T \in \mathcal{B}', \phi \in \mathcal{B},$$

and we denote with $(S_\mu)_{\mathcal{B}'}$ the part of Bessel operator in $\mathcal{B}'$.

Theorem 6.3. *The operator $-(S_\mu)_{\mathcal{B}'}$ is continuous and nonnegative considering the strong topology in $\mathcal{B}'$.*

Proof. The proof of continuity is identical to the proof given in [6, Theorem 3.5, p. 264] for the Laplacean operator and the nonnegativity is a consequence of theory of fractional powers in distributional spaces (see [5, p. 24]). □

Remark 6.4. The operator $(S_\mu)_{\mathcal{B}'}$ is not injective because the function $x^{\mu+\frac{1}{2}}$ is solution of $S_\mu = 0$ and belongs to $\mathcal{B}'$, in fact

$$|(x^{\mu+\frac{1}{2}}, \phi)| \leq c_\mu \|\phi\|_{L^1(\mathbb{R}_+, sr)} \leq c_\mu \rho_0(\phi), \ \ (\phi \in \mathcal{B}).$$

According to representation of fractional powers of operators in locally convex spaces given in [5], for $\operatorname{Re}\alpha > 0$, $n > \operatorname{Re}\alpha$, $T \in \mathcal{B}'$, $(-(S_\mu)_{\mathcal{B}'})^\alpha$ is given by

$$(-(S_\mu)_{\mathcal{B}'})^\alpha T = \frac{\Gamma(n)}{\Gamma(\alpha)\Gamma(n-\alpha)} \int_0^\infty \lambda^{\alpha-1} \Big[-(S_\mu)_{\mathcal{B}'}(\lambda - (S_\mu)_{\mathcal{B}'})^{-1}\Big]^n T \, d\lambda.$$

From the general theory of fractional power in sequentially complete locally convex spaces (see [5, p. 134]), we deduce immediately some properties of powers such as multiplicativity, spectral mapping theorem, and

1) If $\operatorname{Re}\alpha > 0$, then

$$\Big((-(S_\mu)_{\mathcal{B}})^\alpha\Big)^* = \Big((-(S_\mu)_{\mathcal{B}})^*\Big)^\alpha. \tag{37}$$

Since $(-(S_\mu)_{\mathcal{B}})^* = -(S_\mu)_{\mathcal{B}'}$ then from (37) we obtain the following duality formula

$$((-(S_\mu)_{\mathcal{B}'})^\alpha T, \phi) = (T, (-(S_\mu)_{\mathcal{B}})^\alpha \phi), \ \ (\phi \in \mathcal{B}, T \in \mathcal{B}').$$

2) Since the usual topology in $L^p(\mathbb{R}_+, sr^p)$ is stronger than the topology induced by $\mathcal{B}'$ then we can deduce that

$$\big[(-(S_\mu)_{\mathcal{B}'})^\alpha\big]_{L^p(\mathbb{R}_+, sr^p)} = ((-(S_{\mu,p}))^\alpha,$$

if $\operatorname{Re}\alpha > 0$, (see [5, Theorem 12.1.6, p. 284]).

7. Appendix

7.1. Some properties of Hankel transform in Lebesgue spaces

Proposition 7.1. *Let $f, g \in L^1(\mathbb{R}_+, sr)$. Then*

(1) $h_\mu f \in L^\infty(r)$.

(2)

$$\int_0^\infty h_\mu f g = \int_0^\infty f h_\mu g. \tag{38}$$

Proof. The proof is immediate. □

In [3] is studied a version of Hankel transform given by:

$$H_\mu(f)(x) = c_\mu \int_0^\infty (xy)^{-\mu} J_\mu(xy) f(y) s(y) dy \tag{39}$$

for $f \in L^1(\mathbb{R}_+, s)$. H_μ is related whit h_μ by

$$h_\mu(f) = r^{-1} H_\mu(rf)$$

for $f \in L^1(\mathbb{R}_+, sr)$ (r and s like as in Section 2). From this relation and the inversion theorem for H_μ (see [3, Corollary 2e, p. 316]), we obtain the following inversion theorem for h_μ.

Proposition 7.2. *If $f \in L^1(\mathbb{R}_+, sr)$ and $h_\mu(f) \in L^1(\mathbb{R}_+, sr)$ then f may be redefined on a set of measure zero so that it is continuous in $\mathbb{R}_+$ and*

$$f(x) = \int_0^\infty \sqrt{xy} J_\mu(xy) h_\mu(f)(y) dy = h_\mu(h_\mu(f))(x). \tag{40}$$

Remark 7.3. From the above proposition we deduce immediately the validity of equality $h_\mu h_\mu f = f$ in $\mathcal{H}_\mu$ and $\mathcal{H}'_\mu$.

With $L^p(\mathbb{R}_+)$ we denote the usual Lebesgue space of functions defined in $\mathbb{R}_+$ and with norm:

$$\|f\|_p = \left\{ \int_0^\infty |f(x)|^p \, dx \right\}^{\frac{1}{p}}.$$

Remark 7.4. Since the function $(z)^{\frac{1}{2}} J_\mu(z)$ is bounded in $\mathbb{R}_+$ for $\mu > -\frac{1}{2}$, then for $f \in L^1(\mathbb{R}_+)$ we have that $h_\mu f$ is continuous and $\|h_\mu f\|_\infty \le C\|f\|_1$.

As usual, we denote with $C_0(\mathbb{R}_+)$ the set of continuous functions defined in $\mathbb{R}_+$ and vanishes at infinity. We have the following proposition:

Proposition 7.5. $h_\mu(L^1(\mathbb{R}_+)) \subset C_0(\mathbb{R}_+)$.

Proof. First, we observe that

$$L^1(\mathbb{R}_+, sr) \cap L^\infty(\mathbb{R}_+, r) \subset L^1(\mathbb{R}_+). \tag{41}$$

Indeed,

$$\begin{aligned}\int_0^\infty |f| \, dx = \int_0^\infty |f| r r^{-1} \, dx &= \int_0^1 |f| r r^{-1} \, dx + \int_1^\infty |f| r r^{-1} \, dx \\ &\le \|f\|_{L^\infty(\mathbb{R}_+, r)} \int_0^1 r^{-1} \, dx + \int_1^\infty |f| r^{-1} \, dx \\ &= C\|f\|_{L^\infty(\mathbb{R}_+, r)} + c_\mu \|f\|_{L^1(\mathbb{R}_+, rs)},\end{aligned}$$

because $r < 1$ in $[1, \infty)$, $\mu + \frac{1}{2} > 0$ and $rs = c_\mu^{-1} r^{-1}$.

By (41) and (11) we deduce that $\mathcal{H}_\mu \subset L^1(\mathbb{R}_+)$. Since $\mathcal{D}(\mathbb{R}_+) \subset \mathcal{H}_\mu$ then $\mathcal{H}_\mu$ is dense in $L^1(\mathbb{R}_+)$. Given $f \in L^1(\mathbb{R}_+)$ and $\{\phi_n\} \subset \mathcal{H}_\mu$ such that $\phi_n \to f$ in $L^1(\mathbb{R}_+)$ then by Remark 7.4 $h_\mu(\phi_n) \to h_\mu(f)$ uniformly. Since $h_\mu(\phi_n) \in C_0(\mathbb{R}_+)$ then $h_\mu(f) \in C_0(\mathbb{R}_+)$. □

Remark 7.6. For $\mu > -\frac{1}{2}$, $\mathcal{H}_\mu$ is a dense subset of $L^2(\mathbb{R}_+)$ and for $\phi \in \mathcal{H}_\mu$ we have that

$$\|h_\mu \phi\|_2 = \|\phi\|_2.$$

So, we can consider the extension to $L^2(\mathbb{R}_+)$ of h_μ and

$$\|h_\mu f\|_2 = \|f\|_2$$

for $f \in L^2(\mathbb{R}_+)$.

7.2. Hankel convolution

In this section, we prove Lemma 2.7. Before this we observe that the kernel $D_\mu(x,y,z)$ of Hankel convolution satisfies that $D_\mu(x,y,z) \geq 0$ and

$$\int_0^\infty z^{\mu+\frac{1}{2}} D_\mu(x,y,z)\,dz = c_\mu^{-1} x^{\mu+\frac{1}{2}} y^{\mu+\frac{1}{2}} \tag{42}$$

for $x,y,z \in (0,\infty)$.

Proof. By hypothesis and (42) we have that

$$\int_0^\infty \int_0^\infty x_0^{-\mu-\frac{1}{2}} y^{\mu+\frac{1}{2}} D_\mu(x_0,y,z)\phi_n(z)\,dydz = 1,$$

then

$$\begin{aligned} &f\sharp\phi_n(x_0) - f(x_0) \\ &= \int_0^\infty \int_0^\infty D_\mu(x_0,y,z)\phi_n(z) y^{\mu+\frac{1}{2}} \big(y^{-\mu-\frac{1}{2}} f(y) - x_0^{-\mu-\frac{1}{2}} f(x_0)\big)\,dydz. \end{aligned}$$

By continuity of f in x_0 let $\delta > 0$ such that $|y^{-\mu-\frac{1}{2}} f(y) - x_0^{-\mu-\frac{1}{2}} f(x_0)| < \varepsilon$ if $|y - x_0| < \delta$, and we consider

$$|f\sharp\phi_n(x_0) - f(x_0)| \leq |I_1| + |I_2|,$$

where

$$|I_1| = \left|\int_0^\delta\int_0^\infty D_\mu(x_0,y,z)\phi_n(z) y^{\mu+\frac{1}{2}} \big(y^{-\mu-\frac{1}{2}} f(y) - x_0^{-\mu-\frac{1}{2}} f(x_0)\big)\,dydz\right|, \tag{43}$$

$$|I_2| = \left|\int_\delta^\infty\int_0^\infty D_\mu(x_0,y,z)\phi_n(z) y^{\mu+\frac{1}{2}} \big(y^{-\mu-\frac{1}{2}} f(y) - x_0^{-\mu-\frac{1}{2}} f(x_0)\big)\,dydz\right|. \tag{44}$$

Since $D_\mu(x_0,y,z) \neq 0$ only if $|x_0 - z| < y < x_0 + z$, and if $0 < z < \delta$, then $(|x_0 - z|, x_0 + z) \subset (x_0 - \delta, x_0 + \delta)$, then we obtain in (43) that

$$|I_1| \leq \varepsilon \int_0^\delta \int_0^\infty D_\mu(x_0,y,z)\phi_n(z) y^{\mu+\frac{1}{2}}\,dydz \leq \varepsilon x_0^{\mu+\frac{1}{2}}.$$

On the other hand

$$\begin{aligned} |I_2| &\leq 2\|f\|_{L^\infty(\mathbb{R}_+,r)} \int_\delta^\infty \int_0^\infty D_\mu(x_0,y,z)\phi_n(z) y^{\mu+\frac{1}{2}}\,dydz \\ &= 2\|f\|_{L^\infty(\mathbb{R}_+,r)}\, x_0^{\mu+\frac{1}{2}} \int_\delta^\infty c_\mu^{-1} z^{\mu+\frac{1}{2}} \phi_n(z)\,dz \end{aligned}$$

so, $|I_2| \to 0$ when $n \to \infty$ and the first assertion has been proven. The second affirmation follows from the previous proof and the uniformly continuity of rf. □

7.3. Properties of N_λ

Proof of Lemma 4.2. a)

$$\begin{aligned}\|N_\lambda\|_{L^1(\mathbb{R}_+,sr)} &= \frac{1}{c_\mu}\int_0^\infty \lambda^{\frac{\mu}{2}} x^{\frac12}\mathcal{K}_\mu(\sqrt{\lambda}\,x)x^{\mu+\frac12}dx \\ &= \frac{1}{c_\mu}\Big(\frac12\Big)^{\mu+1}\lambda^\mu\int_0^\infty\left[\int_0^\infty x^{2\mu+1}e^{-\frac{\lambda x^2}{4t}}dx\right]e^{-t}\frac{dt}{t^{\mu+1}} \\ &= \frac{1}{c_\mu}2^\mu\lambda^{-1}\Gamma(\mu+1)\int_0^\infty e^{-t}dt = \lambda^{-1}.\end{aligned}$$

For b), in virtue of the following equality

$$\int_0^\infty x^{\mu+\frac12}e^{-\frac{x^2}{2}}\sqrt{xy}\,J_\mu(xy)\,dx = y^{\mu+\frac12}e^{-\frac{y^2}{2}}, \quad y>0, \tag{45}$$

(see [8, (5.9), p. 46]), setting $y = (\sqrt{a})^{-1}r$ with $a, r > 0$, and considering the change of variable $s = \frac{x}{\sqrt{a}}$, then we obtain that

$$\int_0^\infty (\sqrt{a}\,s)^{\mu+\frac12}e^{-\frac{as^2}{2}}\sqrt{sr}\,J_\mu(sr)\sqrt{a}\,ds = \left(\frac{r}{\sqrt{a}}\right)^{\mu+\frac12}e^{-\frac{r^2}{2a}}$$

so,

$$\int_0^\infty s^{\mu+1}e^{-\frac{as^2}{2}}\,J_\mu(sr)\,ds = a^{-\mu-1}r^\mu e^{-\frac{r^2}{2a}} \tag{46}$$

for all $a > 0$. Then,

$$\begin{aligned}h_\mu N_\lambda(y) &= \int_0^\infty \lambda^{\frac{\mu}{2}}x^{\frac12}\mathcal{K}_\mu(\sqrt{\lambda}\,x)\sqrt{xy}\,J_\mu(xy)\,dx \\ &= \lambda^\mu\,y^{\frac12}\Big(\frac12\Big)^{\mu+1}\int_0^\infty x^{\mu+1}\left[\int_0^\infty e^{-t-\frac{\lambda x^2}{4t}}\frac{dt}{t^{\mu+1}}\right]J_\mu(xy)\,dx.\end{aligned} \tag{47}$$

Since that

$$\begin{aligned}&\int_0^\infty x^{\mu+1}\left[\int_0^\infty e^{-t-\frac{\lambda x^2}{4t}}\frac{dt}{t^{\mu+1}}\right]|J_\mu(xy)|\,dx \\ &= y^\mu\int_0^\infty\left[\int_0^\infty e^{-\frac{\lambda x^2}{4t}}x^{2\mu+1}\big|(xy)^{-\mu}J_\mu(xy)\big|\,dx\right]e^{-t}\frac{dt}{t^{\mu+1}} < \infty,\end{aligned}$$

we can reverse the order of integration in (47) and applying (46) we obtain that

$$\begin{aligned}h_\mu N_\lambda(y) &= \lambda^\mu\,y^{\frac12}\Big(\frac12\Big)^{\mu+1}\int_0^\infty\left[\int_0^\infty x^{\mu+1}e^{-(\frac{\lambda}{2t})\frac{x^2}{2}}J_\mu(xy)\,dx\right]e^{-t}\frac{dt}{t^{\mu+1}} \\ &= \lambda^\mu\,y^{\frac12}\Big(\frac12\Big)^{\mu+1}\int_0^\infty\Big(\frac{\lambda}{2t}\Big)^{-\mu-1}y^\mu e^{-\frac{ty^2}{\lambda}}\,e^{-t}\frac{dt}{t^{\mu+1}}\end{aligned}$$

$$= \lambda^{-1}\, y^{\mu+\frac{1}{2}} \int_0^\infty e^{-t(1+\frac{y^2}{\lambda})}\, dt.$$

Considering the change of variable $s = t(1 + \frac{y^2}{\lambda})$ in the last integral we obtain finally

$$\begin{aligned} h_\mu N_\lambda(y) &= \lambda^{-1}\, y^{\mu+\frac{1}{2}} \int_0^\infty e^{-s}\Big(1 + \frac{y^2}{\lambda}\Big)^{-1} ds \\ &= \lambda^{-1}\, y^{\mu+\frac{1}{2}}\Big(1 + \frac{y^2}{\lambda}\Big)^{-1} = \frac{y^{\mu+\frac{1}{2}}}{\lambda + y^2}. \end{aligned}$$

□

Proof of Lemma 4.3. Suppose that $f \in L^p(\mathbb{R}_+, sr^p)$ and $\psi \in \mathcal{H}_\mu$, we claim that

$$\int_0^\infty (N_\lambda \sharp f)(x)\psi(x)dx = \int_0^\infty f(z)\,(N_\lambda \sharp \psi)(z)dz. \tag{48}$$

Indeed, we first observe that the following integral is finite

$$\int_0^\infty |f(z)| \left[\int_0^\infty \int_0^\infty |N_\lambda(y)|\, |\psi(x)|\, D_\mu(x,y,z)dxdy\right] dz.$$

In fact, given a integer q such that $\frac{1}{p} + \frac{1}{q} = 1$, the function

$$G(z) = \int_0^\infty \int_0^\infty |N_\lambda(y)|\, |\psi(x)|\, D_\mu(x,y,z)dxdy$$

is in $L^q(\mathbb{R}_+, sr^q)$ because it is the convolution of $|N_\lambda(y)| \in L^1(\mathbb{R}_+, sr)$ and $|\psi(x)| \in L^q(\mathbb{R}_+, sr^q)$. Since $f \in L^p(\mathbb{R}_+, sr^p)$ then

$$\int_0^\infty |f(z)|\, G(z)dz = \int_0^\infty (r\,|f(z)|)(s^{-1}r^{-1}G(z))s\, dz < \infty$$

because $r\,|f| \in L^p(\mathbb{R}_+, s)$ and $s^{-1}r^{-1}G = c_\mu r G \in L^q(\mathbb{R}_+, s)$. Then

$$\begin{aligned} \int_0^\infty f(z)\,(N_\lambda \sharp \psi)(z)\, dz &= \int_0^\infty \left[\int_0^\infty \int_0^\infty f(z)N_\lambda(y)D_\mu(x,y,z)dz\, dy\right]\psi(x)dx \\ &= \int_0^\infty (N_\lambda \sharp\, f)(x)\psi(x)dx \end{aligned}$$

and we thus get (48). The proof for $f \in L^\infty(\mathbb{R}_+, r)$ is similar.

Now, given $\phi \in \mathcal{H}_\mu$ and $f \in L^p(\mathbb{R}_+, sr^p)$ or $L^\infty(\mathbb{R}_+, r)$, by (48), we have that

$$(h_\mu(N_\lambda \sharp\, f), \phi) = ((N_\lambda \sharp\, f), h_\mu \phi) = \int_0^\infty f(x)\, N_\lambda \sharp\, h_\mu \phi(x)\, dx. \tag{49}$$

By Lemma 2.6, Proposition 7.2 and item b) of Lemma 4.2 we obtain that

$$h_\mu(N_\lambda \sharp h_\mu \phi)(y) = r h_\mu(N_\lambda) h_\mu(h_\mu \phi)(y) = \frac{\phi(y)}{\lambda + y^2}.$$

So,

$$N_\lambda \sharp\, h_\mu \phi = h_\mu\Big(\frac{\phi}{\lambda + y^2} \Big). \tag{50}$$

Finally, from (49) and (50) we obtain that for $\phi \in \mathcal{H}_\mu$ that

$$\begin{aligned}(h_\mu(N_\lambda \sharp\, f), \phi) &= \int_0^\infty f(x)\, N_\lambda \sharp\, h_\mu \phi(x)\, dx = \int_0^\infty f(x)\, h_\mu\Big(\frac{\phi}{\lambda + y^2} \Big)(x)\, dx \\ &= \int_0^\infty \frac{1}{\lambda + x^2}\, h_\mu(f)(x)\phi(x)\, dx = \Big(\frac{1}{\lambda + x^2}\, h_\mu(f), \phi \Big). \qquad \square\end{aligned}$$

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