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NEW TRANSFORMATIONS FOR HYPERGEOMETRIC FUNCTIONS DEDUCIBLE BY FRACTIONAL CALCULUS

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Abstract. Recently, many authors have obtained several hypergeometric identities involving hypergeometric functions of one and multi-variables such as the Appell's functions and Horn's functions. In this paper, we obtain several new transformations suitably by applying the fractional calculus operator to these hypergeometric identities, which was introduced recently by Tremblay.

1. Introduction and preliminaries

For nonnegative integers p and q, the generalized hypergeometric function in a variable (argument) z with p numerator parametres $\alpha_1, \ldots, \alpha_p$ and q denominators $\beta_1, \ldots, \beta_q$ is, defined by (see, *e.g.*, [11, 14, 15])

$$
\begin{aligned}
{}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p;\\ \beta_1,\ldots,\beta_q;\end{matrix}\, z\right] &= {}_pF_q\left[\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q\ ;\ z\right] \\
&= \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!},
\end{aligned} \tag{1.1}
$$

whenever this series converges and elsewhere by analytic continuation. Here Γ is the familiar Gamma function and $(\cdot)_m$ stands for the Pochhammer (or shifted factorial) symbol defined for any complex number α and nonnegative integers m by $(\alpha)_0 = 1$ and $(\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)$. The series defining ${}_pF_q$ converges for all values of z when $p \le q$. If $p = q+1$, then the series (1.1) converges when $|z| < 1$, and it is absolutely convergent on the unit circle if $\Re(\beta_1 + \cdots + \beta_q - \alpha_1 - \cdots - \alpha_p) > 0$ and it is convergent on the circle $|z| = 1$ except at $z = 1$ if $-1 < \Re(\beta_1 + \cdots + \beta_q - \alpha_1 - \cdots - \alpha_p) \le 0$.

Let us recall the general definition of the double hypergeometric function given by Srivastava and Panda (see [18], p. 423, Eq. (26)). Let (H_h) denotes the sequence of parameters $(H_1, H_2, \ldots, H_h)$, and let nonnegative integers

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define the Pochhammer symbol $\big((H_h)\big)_n = (H_1)_n(H_2)_n \cdots (H_h)_n$. Then the generalized version of the Kampé de Fériet function is defined as follows:

(1.2)

$$F^{h:a;b}_{g:c;d}\left[\begin{matrix}(H_h):\,(A_a);\,(B_b);\\(G_g):\,(C_c);\,(D_d);\end{matrix}\;x,\,y\right]=\sum_{m,n=0}^{\infty}\frac{((H_h))_{m+n}\,((A_a))_m\,((B_b))_n}{((G_g))_{m+n}\,((C_c))_m\,((D_d))_n}\frac{x^m}{m!}\frac{y^n}{n!}.$$

For the detailed convergence conditions for this function, the interested reader may refer to [19].

Some special cases of hypergeometric functions of two variables are the following Appell functions (see [1, 2, 4, 14, 16]):

$$F_1\,[a;b_1,\,b_2;c;x,\,y]=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}\quad(|x|<1,|y|<1); \tag{1.3}$$

$$F_2\,[a;b_1,\,b_2;c_1,\,c_2;x,\,y]=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c_1)_m(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\quad(|x|+|y|<1); \tag{1.4}$$

$$F_3\,[a_1,\,a_2;b_1,\,b_2;c;x,\,y]=\sum_{m,n=0}^{\infty}\frac{(a_1)_m(a_2)_n(b_1)_m(b_2)_n}{(c)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}\quad(|x|<1,|y|<1); \tag{1.5}$$

$$F_4\,[a;b;c_1,\,c_2;x,\,y]=\sum_{m,n=0}^{\infty}\frac{(a)_{m+n}(b)_{m+n}}{(c_1)_m(c_2)_n}\frac{x^m}{m!}\frac{y^n}{n!}\quad(|x|^{\frac12}+|y|^{\frac12}<1). \tag{1.6}$$

Other interesting special cases of hypergeometric functions of two variables are the following Horn's functions G_1, G_2 and H_3 (see [2, 4, 15]):

$$G_1\,(\alpha;\beta_1,\,\beta_2;x,\,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta_1)_{n-m}(\beta_2)_{m-n}}{m!n!}x^m\,y^n\quad(|x|<r,|y|<s,r+s=1); \tag{1.7}$$

$$G_2\,(\alpha_1,\alpha_2;\beta_1,\,\beta_2;x,\,y)=\sum_{m,n=0}^{\infty}\frac{(\alpha_1)_m(\alpha_2)_n(\beta_1)_{n-m}(\beta_2)_{m-n}}{m!n!}x^m\,y^n\quad(|x|<1,|y|<1); \tag{1.8}$$

$$H_3\,[\alpha,\,\beta;\gamma;x,\,y]=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{2m+n}(\beta)_n}{(\gamma)_{m+n}}\frac{x^m}{m!}\frac{y^n}{n!}\quad\left(|x|<r,|y|<r,r+\left(s-\frac12\right)^2=\frac14\right). \tag{1.9}$$

For our purpose, we need to introduce Srivastava's triple hypergeometric series $F^{(3)}[x,y,z]$ (see [16, p. 44]) defined by

$$\begin{aligned}F^{(3)}[x,y,z]&=F^{(3)}\left[\begin{matrix}(a)::\;\;(b);(b');(b''):\;\;(c);(c');(c'');\\(e)::\;\;(g);(g');(g''):\;\;(h);(h');(h'');\end{matrix}\;x,\,y,\,z\right]\\&=\sum_{m,n,p\geqslant 0}^{\infty}\Lambda\,(m,\,n,\,p)\,\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!},\end{aligned} \tag{1.10}$$

where, for convenience,

$$\Lambda\,(m,\,n,\,p)=\frac{\Pi_{j=1}^{A}(a_j)_{m+n+p}\Pi_{j=1}^{B}(b_j)_{m+n}\Pi_{j=1}^{B'}(b_j')_{n+p}\Pi_{j=1}^{B''}(b_j'')_{m+p}}{\Pi_{j=1}^{F}(e_j)_{m+n+p}\Pi_{j=1}^{G}(g_j)_{m+n}\Pi_{j=1}^{G'}(g_j')_{n+p}\Pi_{j=1}^{G''}(g_j'')_{m+p}} \cdot\frac{\Pi_{j=1}^{C}(c_j)_{m}\Pi_{j=1}^{C'}(c_j')_{n}\Pi_{j=1}^{C''}(c_j'')_{p}}{\Pi_{j=1}^{H}(h_j)_{m}\Pi_{j=1}^{H'}(h_j')_{n}\Pi_{j=1}^{H''}(h_j'')_{p}}. \tag{1.11}$$

The most familiar representation for fractional derivative of order α of $z^p f(z)$ is the Riemann-Liouville integral [13], that is,

$$D_z^\alpha z^p f(z)=\frac{1}{\Gamma(-\alpha)}\int_0^z f(\xi)\xi^p(\xi-z)^{-\alpha-1}\,d\xi \quad (\Re(\alpha)<0,\Re(p)>1), \tag{1.12}$$

where the integration is done along a straight line from 0 to z in the ξ-plane. By integrating by parts m times, we obtain

$$D_z^\alpha z^p f(z)=\frac{d^m}{dz^m}D_z^{\alpha-m}z^p f(z). \tag{1.13}$$

This allows us to modify the restriction $\Re(\alpha)<0$ to $\Re(\alpha)<m$ (see [9]). It is well known that (see [3], [8, p. 83, Eq. (2.4)])

$$D_z^\alpha z^p=\frac{\Gamma(1+p)}{\Gamma(1+p-\alpha)}z^{p-\alpha} \quad (\Re(p)>-1). \tag{1.14}$$

Adopting the Pochhammer based representation for the fractional derivative modifies the restriction to the case when p is not a negative integer.

The aim of this paper is to present several (presumably) new general transformations involving the generalized hypergeometric functions of one and multi-variables. These transformations can be viewed as generalizations for some of those obtained recently by Paris, Rathie-Pogany, Srivastava and Panda, and Wei *et al.* [23]. All these transformations are obtained by using a fractional calculus operator ${}_zO_\beta^\alpha$ introduced recently by Tremblay [20, 21].

2. The well poised fractional calculus operator ${}_zO_\beta^\alpha$

In this section, we recall some of the important properties of the fractional calculus operator ${}_zO_\beta^\alpha$ that was introduced by Tremblay [20] as follows:

$${}_zO_\beta^\alpha=\frac{\Gamma(\beta)}{\Gamma(\alpha)}z^{1-\beta}D_z^{\alpha-\beta}z^{\alpha-1} \quad \left(\beta\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}\right). \tag{2.1}$$

Here are some known properties of ${}_zO_\beta^\alpha$:

(1) Linearity:

$${}_zO_\beta^\alpha\left[\lambda_1 f(z)+\lambda_2 g(z)\right]=\lambda_1\,{}_zO_\beta^\alpha f(z)+\lambda_2\,{}_zO_\beta^\alpha g(z). \tag{2.2}$$

(2) Identity

$${}_zO_\alpha^\alpha=I. \tag{2.3}$$

(3) Reduction

$$ {}_zO^{\alpha}_{\beta}\,{}_zO^{\beta}_{\gamma} = {}_zO^{\alpha}_{\gamma}, \tag{2.4}$$

$$ {}_zO^{\alpha}_{\beta}\,{}_zO^{\gamma}_{\alpha} = {}_zO^{\gamma}_{\beta}. \tag{2.5}$$

(4) Elementary cases

$$ {}_zO^{\alpha}_{\beta}1 = 1, \tag{2.6}$$

$$ {}_zO^{\alpha}_{\beta}z^n = \frac{(\alpha)_n}{(\beta)_n}z^n, \tag{2.7}$$

$$ {}_zO^{\alpha}_{\beta}(1-z)^{-\gamma} = {}_2F_1\left[\begin{matrix}\gamma,\,\alpha \\ \beta\end{matrix};z\right]. \tag{2.8}$$

(5) Useful cases

$$ {}_zO^{\alpha}_{\beta}\left[z^{\lambda}f(z)\right] = \frac{\Gamma(\beta)\Gamma(\alpha+\lambda)}{\Gamma(\alpha)\Gamma(\beta+\lambda)}z^{\lambda}\,{}_zO^{\alpha+\lambda}_{\beta+\lambda}\left[f(z)\right], \tag{2.9}$$

$$ {}_zO^{\alpha}_{\beta}\left[(w-z)^{\theta}f(z)\right]_{w=z} = \frac{\Gamma(\beta)\Gamma(\beta-\alpha+\theta)}{\Gamma(\beta-\alpha)\Gamma(\beta+\theta)}z^{\theta}\,{}_zO^{\alpha}_{\beta+\theta}\left[f(z)\right]. \tag{2.10}$$

It is worthy to mention that the operator ${}_zO^{\alpha}_{\beta}$ has a lot more interesting properties and applications. Tremblay introduced this operator in order to deal with special functions more efficiently and to facilitate the obtention of new relations such as hypergeometric transformations.

For this work, the most important property of the operator ${}_zO^{\alpha}_{\beta}$ is given by the following relation:

$$ B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta+\gamma)}{\Gamma(\alpha+\beta+\gamma)}\,{}_zO^{\alpha+\beta}_{\beta}z^{\gamma}|_{z=1}. \tag{2.11}$$

This relation shows, in fact, that the so-called beta integral method consists in a fractional derivative evaluated at the point $z=1$.

3. Main results

In this section, we apply the fractional calculus operator ${}_zO^{\alpha}_{\beta}$ to certain transformations involving the generalized hypergeometric functions of one and multi-variables in order to obtain new transformations more general than those obtained by means of the beta integral method.

Theorem 1. *The following transformation holds*:

$$ {}_1F_1\left[\begin{matrix}\alpha; \\ \beta;\end{matrix} -x\right] = \frac{(b)_n\,(f)_n}{(b-a-1)_n\,(f+1)_n}\,{}_3F_3\left[\begin{matrix}\alpha,\,b-a-1,\,f+1; \\ \beta,\,b,\,f;\end{matrix} \quad -x\right] \tag{3.1}$$

provided $\Re(b-a+n) > 1$ *and* $f = \frac{c(1+a-b)}{(a-c)}$.

Proof. Recently, Paris [10] deduced a Kummer-type I transformation formula for the generalized hypergeometric function ${}_2F_2(x)$, namely

$$ {}_2F_2\left[\begin{matrix} a,\, 1+c; \\ b,\, c; \end{matrix}\; x\right] = e^x\, {}_2F_2\left[\begin{matrix} b-a-1,\, 1+f; \\ b,\, f; \end{matrix}\; -x\right], \tag{3.2} $$

where $f \equiv \frac{c(1+a-b)}{(a-c)}$.

Next, we apply the fractional calculus operator ${}_xO_\beta^\alpha$ on both sides of (3.2). We just have

$$ {}_1F_1\left[\begin{matrix} \alpha; \\ \beta; \end{matrix}\; -x\right] {}_3F_2\left[\begin{matrix} -n,\, a,\, 1+c; \\ b,\, c; \end{matrix}\; 1\right] = {}_3F_3\left[\begin{matrix} \alpha,\, b-a-1,\, 1+f; \\ \beta,\, b,\, f; \end{matrix}\; -x\right]. \tag{3.3} $$

Miller [7] showed that

$$ {}_3F_2\left[\begin{matrix} -n,\, a,\, 1+c; \\ b,\, c; \end{matrix}\; 1\right] = \frac{(b-a-1)_n\,(f+1)_n}{(b)_n\,(f)_n}\; (n \in \mathbb{N}_0)\,. \tag{3.4} $$

Applying (3.4) to (3.3), we obtain the desired result. □

Theorem 2. *The following transformation holds*:

$$ \begin{aligned} & {}_1F_1\left[\begin{matrix} \alpha; \\ \beta; \end{matrix}\; -\tfrac{1}{2}x\right] \cdot {}_3F_2\left[\begin{matrix} -n,\, a,\, d+1; \\ 2a+1,\, d; \end{matrix}\; 2\right] \\ &= {}_2F_3\left[\begin{matrix} \tfrac{1}{2}\alpha,\, \tfrac{1}{2}\alpha+\tfrac{1}{2}; \\ a+\tfrac{1}{2},\, \tfrac{1}{2}\beta,\, \tfrac{1}{2}\beta+\tfrac{1}{2}; \end{matrix}\; \frac{x^2}{16}\right] \\ &\quad - \frac{\alpha(1-\frac{2a}{d})x}{2\beta(2a+1)}\, {}_2F_3\left[\begin{matrix} \tfrac{1}{2}\alpha+\tfrac{1}{2},\, \tfrac{1}{2}\alpha+1; \\ a+\tfrac{3}{2},\, \tfrac{1}{2}\beta+\tfrac{1}{2},\, \tfrac{1}{2}\beta+1; \end{matrix}\; \frac{x^2}{16}\right], \end{aligned} \tag{3.5} $$

where $d \neq 0, -1, -2, \ldots$.

Proof. Rathie and Pogany (see [12, Eq. 11]) presented a Kummer-type II transformation for the generalized hypergeometric function ${}_2F_2$ in the form

$$ e^{-\frac{1}{2}x}{}_2F_2\left[\begin{matrix} a,\, 1+d; \\ 2a+1,\, d; \end{matrix}\; x\right] = {}_0F_1\left[\begin{matrix} -; \\ a+\tfrac{1}{2}; \end{matrix}\; \frac{x^2}{16}\right] - \frac{x(1-\frac{2a}{d})}{2(2a+1)}\, {}_0F_1\left[\begin{matrix} -; \\ a+\tfrac{3}{2}; \end{matrix}\; \frac{x^2}{16}\right]. \tag{3.6} $$

Next, we apply the fractional calculus operator ${}_xO_\beta^\alpha$ on both sides of (3.6). We immediately have

$$ \begin{aligned} & {}_1F_1\left[\begin{matrix} \alpha; \\ \beta; \end{matrix}\; -\tfrac{1}{2}x\right] {}_3F_2\left[\begin{matrix} -n,\, a,\, 1+d; \\ 2a+1,\, d; \end{matrix}\; 2\right] \\ &= {}_2F_3\left[\begin{matrix} \tfrac{1}{2}\alpha,\, \tfrac{1}{2}\alpha+\tfrac{1}{2}; \\ a+\tfrac{1}{2},\, \tfrac{1}{2}\beta,\, \tfrac{1}{2}\beta+\tfrac{1}{2}; \end{matrix}\; \frac{x^2}{16}\right] \\ &\quad - \frac{\alpha(1-\frac{2a}{d})x}{2\beta(2a+1)}\, {}_2F_3\left[\begin{matrix} \tfrac{1}{2}\alpha+\tfrac{1}{2},\, \tfrac{1}{2}\alpha+1; \\ a+\tfrac{3}{2},\, \tfrac{1}{2}\beta+\tfrac{1}{2},\, \tfrac{1}{2}\beta+1; \end{matrix}\; \frac{x^2}{16}\right]. \end{aligned} \tag{3.7} $$

□

Moreover, Kim *et al.* (see [6, Theorem 2]) derived the result (3.6) by an elementary way and deduced the following two elegant results for the terminating ${}_3F_2$:

$$ {}_3F_2\left[\begin{matrix} -2n,\ a,\ 1+d; \\ 2a+1,\ d; \end{matrix}\ 2\right] = \frac{\left(\frac{1}{2}\right)_n}{\left(a+\frac{1}{2}\right)_n}, \tag{3.8} $$

and

$$ {}_3F_2\left[\begin{matrix} -2n-1,\ a,\ 1+d; \\ 2a+1,\ d; \end{matrix}\ 2\right] = \frac{\left(1-\frac{2a}{d}\right)\left(\frac{3}{2}\right)_n}{(2a+1)\left(a+\frac{3}{2}\right)_n}\quad (n\in\mathbb{N}_0). \tag{3.9} $$

Applying (3.8) and (3.9) to (3.7), we have an another expression as follows:

$$ \begin{aligned} &\frac{\left(\frac{1}{2}\right)_n}{\left(a+\frac{1}{2}\right)_n}\,{}_1F_1\left[\begin{matrix}\alpha; \\ \beta;\end{matrix}\ -\tfrac{1}{2}x\right] + \frac{\left(1-\frac{2a}{d}\right)\left(\frac{3}{2}\right)_n}{(2a+1)\left(a+\frac{3}{2}\right)_n}\,{}_1F_1\left[\begin{matrix}\alpha; \\ \beta;\end{matrix}\ -\tfrac{1}{2}x\right] \\ &= {}_2F_3\left[\begin{matrix} \frac{1}{2}\alpha,\ \frac{1}{2}\alpha+\frac{1}{2}; \\ a+\frac{1}{2},\ \frac{1}{2}\beta,\ \frac{1}{2}\beta+\frac{1}{2}; \end{matrix}\ \frac{x^2}{16}\right] \\ &\quad - \frac{\alpha(1-\frac{2a}{d})x}{2\beta(2a+1)}\,{}_2F_3\left[\begin{matrix} \frac{1}{2}\alpha+\frac{1}{2},\ \frac{1}{2}\alpha+1; \\ a+\frac{3}{2},\ \frac{1}{2}\beta+\frac{1}{2},\ \frac{1}{2}\beta+1; \end{matrix}\ \frac{x^2}{16}\right]. \end{aligned} \tag{3.10} $$

An interesting results involving Appell's F_4 and Horn's functions H_3 were given by Srivastava and Panda (see [17, p. 57, (33)-(35)] and [18]):

$$ F_4\left[a,b;c,\ b;x,\ y\right] = (1-x-y)^{-a}H_3\left[a,c-b;\ c;\ \tfrac{xy}{(x+y-1)^2},\ \tfrac{x}{(x+y-1)}\right]; \tag{3.11} $$

$$ F_4\left[a,b;a,\ c;x,\ y\right] = (1-x-y)^{-b}H_3\left[b,c-a;\ c;\ \tfrac{xy}{(x+y-1)^2},\ \tfrac{x}{(x+y-1)}\right]; \tag{3.12} $$

or, equivalently,

$$ H_3\left[a,b;\ c;xy,\ y\right] = (1-x-y)^{-b}F_4\left[a,\ c-b;\ c-b,\ c;\ \tfrac{x}{(x+y-1)},\ \tfrac{y}{(x+y-1)}\right], \tag{3.13} $$

which exhibit that the Horn function H_3 can always be expressed in terms of Appell's function F_4.

Theorem 3. *Let β, b and $c \neq 0, -1, -2, \ldots$. Then the following transformation holds*:

$$ \begin{aligned} &F^{2:1;1}_{1:1;2}\left[\begin{matrix} a,\ b:\ \alpha;\ \beta-\alpha; \\ \beta:\ c;\ b,\beta; \end{matrix}\ z,\ z\right] \\ &= (1-z)^{-a}\sum_{m,n=0}^{\infty}\frac{(a)_{2m+n}(\alpha)_{m+n}(\beta-\alpha)_m}{(\beta)_{2m+n}(c)_{m+n}\,m!\,n!}\left(\frac{z}{(z-1)}\right)^{2m}\left(\frac{z}{(z-1)}\right)^{n}. \end{aligned} \tag{3.14} $$

Proof. By making the substitutions $x \mapsto z$ and $y \mapsto w-z$ in (3.11), we obtain

$$ F_4\left[a,b;c,\ b;z,\ w-z\right] = (1-w)^{-a}H_3\left[a,c-b;\ c;\ \tfrac{z(w-z)}{(w-1)^2},\ \tfrac{z}{(w-1)}\right]. \tag{3.15} $$

Applying the fractional calculus operator ${}_zO_\beta^\alpha$ on both sides of (3.15) with $w = z$ after operation, we have, for the left-hand side:

$$ {}_zO_\beta^\alpha F_4\left[a, b; c, b; z, w - z\right]|_{w=z} = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{(c)_m(b)_n\ m!\ n!}\, {}_zO_\beta^\alpha z^m (w-z)^n|_{w=z}. \tag{3.16} $$

Using (2.9) and (2.10), we have

$$ \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}(\alpha)_m(\beta-\alpha)_n}{(\beta)_{m+n}(c)_m(b)_n(\beta)_m} \frac{z^m}{m!} \frac{z^n}{n!}. $$

We obtain the right-hand side:

$$ \begin{aligned} & {}_zO_\beta^\alpha (1-w)^{-a} H_3\left[a, c-b;\ c;\ \tfrac{z(w-z)}{(w-1)^2}, \tfrac{z}{(w-1)}\right]|_{w=z} \\ &= (1-w)^{-a} \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n}(c-b)_n}{(c)_{m+n}(w-1)^{2m+n}\ m!\ n!}\, {}_zO_\beta^\alpha z^{m+n}(w-z)^m|_{w=z}. \end{aligned} \tag{3.17} $$

Using (2.9) and (2.10), we have

$$ (1-z)^{-a} \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n}(\alpha)_{m+n}(\beta-\alpha)_m}{(\beta)_{2m+n}(c)_{m+n} m\,!n!} \left(\frac{z}{(z-1)}\right)^{2m} \left(\frac{z}{(z-1)}\right)^{n}. $$

This completes the proof. □

Applying the fractional calculus operator ${}_zO_\beta^\alpha$ on both sides of (3.12) with $w = z$ after operation, a similar result can be also obtained.

Theorem 4. *Let β, a and $c \neq 0, -1, -2, \ldots$. Then the following transformation holds*:

$$ \begin{aligned} & F_{1:1;2}^{2:1;1}\left[\begin{matrix} a,\ b:\ \alpha;\ \beta-\alpha; \\ \beta:\ a;\ c, \beta; \end{matrix}\ z,\ z\right] \\ &= (1-z)^{-b} \sum_{m,n=0}^{\infty} \frac{(b)_{2m+n}(\beta-\alpha)_{m+n}(\alpha)_m(c-a)_n}{(\beta)_{2m+n}(c)_{m+n} m!\, n!} \left(\frac{z}{(z-1)}\right)^{2m} \left(\frac{z}{(z-1)}\right)^{n}. \end{aligned} \tag{3.18} $$

Applying the fractional calculus operator ${}_zO_\beta^\alpha$ on both sides of (3.13) with $w = z$ after operation, we have:

Theorem 5. *Let β, c and $c - b \neq 0, -1, -2, \ldots$. Then the following transformation holds*:

$$ \begin{aligned} & F_{1:1;1}^{3:0;0}\left[\begin{matrix} a,\ c-b,\ \alpha:\ -;\ -; \\ \beta:\ c-b;\ c; \end{matrix}\ \tfrac{z}{z-1}, \tfrac{z}{z-1}\right] \\ &= (1-z)^{a} \sum_{m,n=0}^{\infty} \frac{(a)_{2m+n}(\beta-\alpha)_{m+n}(\alpha)_m(b)_n}{(\beta)_{2m+n}(c)_{m+n}} \frac{z^{2m}}{m!} \frac{z^n}{n!}. \end{aligned} \tag{3.19} $$

Theorem 6. *Let $\beta, c,\ a-b+1,\ \frac{\lambda}{2}$ and $\frac{\lambda+1}{2} \neq 0, -1, -2, \ldots$. Then the following transformation holds*:

$$
\begin{aligned}
&F^{2:1;2}_{0:2;3}\left[\begin{matrix} a,\, b:\ \alpha;\ \frac{\gamma}{2}\ \frac{\gamma+1}{2}; \\ -:\ c,\ \beta;\ a-b+1,\ \frac{\lambda}{2},\ \frac{\lambda+1}{2}; \end{matrix}\ x,\ y^2\right] \\
&= \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_m(\alpha)_m(a-b+\frac{1}{2})_n(\gamma)_n}{(c)_m(\beta)_m(2a-2b+1)_n(\lambda)_n} \frac{x^m}{m\,!} \frac{(4y)^n}{n\,!} \\
&\quad \cdot {}_2F_1\left[\begin{matrix} 2m+2n+2a,\ \gamma+n \\ \lambda+n; \end{matrix}\ ; -y\right].
\end{aligned}
\tag{3.20}
$$

Proof. We start with the formula (see [22, Eq. (54)]):

$$
\begin{aligned}
&F_4\left[a; b; c,\ a-b+1; x,\ y^2\right] \\
&= (1+y)^{-2a} F_2\left[a; b,\ a-b+\tfrac{1}{2}; c,\ 2a-2b+1; \tfrac{x}{(1+y)^2},\ \tfrac{4y}{(1+y)^2}\right],
\end{aligned}
\tag{3.21}
$$

provided a is a non-positive integer.

Applying successively the operator ${}_xO^{\alpha}_{\beta}$ and the operator ${}_yO^{\gamma}_{\lambda}$ on both sides (3.21), and making use of the identity

$$
(\alpha)_{2n} = 2^{2n}\left(\frac{1}{2}\alpha\right)_n \left(\frac{1}{2}\alpha + \frac{1}{2}\right)_n,
$$

we get the result. □

Hasanov and Turaev (see [5, Eq. (5.7)]) obtained a decomposition formulas regarding Appell's functions:

$$
\begin{aligned}
&G_1\left(a\,;\ b_1,\ b_2; x,\ y\right) \\
&= (1+x+y)^{-a} t F_4\left[a,\ 1-b_1-b_2; 1-b_1,\ 1-b_2; \tfrac{x}{1+x+y},\ \tfrac{y}{1+x+y}\right].
\end{aligned}
\tag{3.22}
$$

If we apply the operator ${}_xO^{\alpha}_{\beta}$ on the both sides (3.22) with $y = -2x$, the result follows easily after simple calculations.

Theorem 7. *The following transformation holds*:

$$
\begin{aligned}
&F^{(3)}\left[\begin{matrix} \alpha:\ a:\ 1-b_1-b_2;\ -,\ -:\ -,\ -;-; \\ \beta:\ -;\ -;\ -:1-b_1;\ 1-b_2;\ -; \end{matrix}\ x,\ -2x,\ x\right] \\
&= \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(a)_{m+n}(b_1)_{n-m}(b_2)_{m-n}(-2)^n}{(\beta)_{m+n}} \frac{x^m}{m\,!} \frac{x^n}{n\,!}.
\end{aligned}
\tag{3.23}
$$

Letting $x = 1$ in Theorem 7 gives

$$
\begin{aligned}
&F^{(3)}\left[\begin{matrix} \alpha:\ a;\ 1-b_1-b_2;\ -,\ -:\ -,\ -;-; \\ \beta:\ -;\ -;\ -:1-b_1;\ 1-b_2;\ -; \end{matrix}\ 1,\ -2,\ 1\right] \\
&= \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(a)_{m+n}(b_1)_{n-m}(b_2)_{m-n}(-2)^n}{(\beta)_{m+n}\, m!\, n!}.
\end{aligned}
\tag{3.24}
$$

Using the Gauss summation theorem ${}_2F_1(1)$ and $\frac{\Gamma(\alpha-n)}{\Gamma(\alpha)} = \frac{(-1)^n}{(1-\alpha)_n}$ on the left-hand side (3.24), we have, after simple manipulations,

$$\begin{aligned}&F^{(3)}\left[\begin{matrix}\alpha : a;\, 1-b_1-b_2;\, -,\, - : -,\, -; - ;\\ \beta : -; -; - : 1-b_1;\, 1-b_2;\, -;\end{matrix}\; 1,\, -2,\, 1\right]\\ &= \frac{\Gamma(\beta)\Gamma(\beta-a-\alpha)}{\Gamma(\beta-a)\Gamma(\beta-\alpha)} F^{4:0;0}_{2:1;1}\left[\begin{matrix}\alpha,\, \beta,\, a,\, 1-b_1-b_2 : -; -;\\ \beta,\, 1-\beta+a+\alpha : 1-b_1;\, 1-b_2;\end{matrix}\; -1,\, 2\right].\end{aligned} \tag{3.25}$$

Therefore, we obtain the following corollary.

Corollary 8. *Let α and β be two non-positive integers, or a be a non-positive integer. Then the following transformation holds*:

$$\begin{aligned}&F^{4:0;0}_{2:1;1}\left[\begin{matrix}\alpha,\, \beta,\, a,\, 1-b_1-b_2 : -; -;\\ \\ \beta,\, 1-\beta+a+\alpha : 1-b_1;\, 1-b_2;\end{matrix}\; -1,\, 2\right]\\ &= \frac{\Gamma(\beta-a)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\beta-a-\alpha)} \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(a)_{m+n}(b_1)_{n-m}(b_2)_{m-n}(-2)^n}{(\beta)_{m+n}\, m!\, n!}.\end{aligned} \tag{3.26}$$

4. Concluding remarks

The fractional calculus operator ${}_zO^{\alpha}_{\beta}$ used in this paper can provide several unknown transformation formulas involving single and multiple variable hypergeometric functions. These results may be useful in theoretical physics, engineering and applied mathematics.

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