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DUAL SURFACES DEFINED BY $z = f(u) + g(v)$ IN SIMPLY ISOTROPIC 3-SPACE $\mathbb{I}_3^1$

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Abstract. In this study, we define the dual surfaces by $z = f(u) + g(v)$ and also classify these surfaces in $\mathbb{I}_3^1$ satisfying some algebraic equations in terms of the coordinate functions and the Laplace operators according to fundamental forms of the surface.

1. Introduction

A surface obtained by translating a curve $\alpha(u)$ over another curve $\beta(v)$ is called a translation surface. A translation surface can be defined as the sum of the two generating curves $\alpha(u)$ and $\beta(v)$. Therefore, translation surfaces are made up of quadrilateral, that is, four sided, facets. Because of this property, translation surfaces are used in architecture to design and construct free-form glass roofing structures. A translation surface in a Euclidean 3-space $\mathbb{E}^3$ formed by translating two curves lying in orthogonal planes is the graph of a function $z(u,v) = f(u) + g(v)$, where $f(u)$ and $g(v)$ are smooth functions on some interval of $\mathbb{R}$ ([1,9]).

In 1835, H. F. Scherk studied translation surfaces in $\mathbb{E}^3$ defined as graph of the function $z(u,v) = f(u) + g(v)$ and he proved that, besides the planes, the only minimal translation surfaces are the surfaces given by

$$z(u,v) = \frac{1}{a}\log\left|\frac{\cos(au)}{\cos(av)}\right| = \frac{1}{a}\log|\cos(au)| - \frac{1}{a}\log|\cos(av)|,$$

where a is a non-zero constant. These surfaces are now referred as Scherk's minimal surfaces ([21]).

Translation surfaces have been investigated from various viewpoints by many differential geometers. Liu described translation surfaces having constant Gaussian and mean curvature in the Euclidean and Minkowski space ([12]). Goemans proved classification theorems of Weingarten translation surfaces

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([9]). Baba-Hamed, Bekkar and Zoubir studied coordinate finite type translation surfaces in a 3-dimensional Minkowski space ([3]). Yoon classified coordinate finite type translation surfaces in a 3-dimensional Galilean space ([20]). Bekkar and Senoussi researched the translation surfaces in the 3-dimensional space satisfying the equation

$$\Delta^{\mathbf{III}}\mathbf{r}_i = \mu_i \mathbf{r}_i,$$

where $\mathbf{r}_i$ is the coordinate functions of the position vector and the Laplace operator $\Delta^{\mathbf{III}}$ with respect to the third fundamental form, respectively ([4]). Cakmak, Karacan, Kiziltug and Yoon studied the translation surfaces in the 3-dimensional Galilean space satisfying the equation

$$\Delta^{\mathbf{II}}\mathbf{x}_i = \lambda_i \mathbf{x}_i$$

([8]). Sipus described translation surfaces in a simply isotropic space having constant isotropic Gaussian or mean curvature ([17]). Aydin studied the translation surfaces generated by a space curve and a planar curve in the isotropic 3-space $\mathbb{I}_3$ ([2]). Bukcu, Karacan and Yoon classified translation surfaces of Type 1 and Type 2 that satisfy the condition

$$\Delta^{\mathbf{I},\mathbf{II},\mathbf{III}}\mathbf{x}_i = \lambda_i \mathbf{x}_i$$

in $\mathbb{I}_3^1$ ([6, 7, 11]).

In this study, we examine the dual surfaces defined by $z = f(u) + g(v)$ in $\mathbb{I}_3^1$ satisfying the condition $\Delta^J \mathbf{x}_i = \lambda_i \mathbf{x}_i$, $J = \mathbf{I}, \mathbf{II}, \mathbf{III}$, where $\lambda_i \in \mathbb{R}$ and Δ^J indicate the Laplace operator according to the first, second and third fundamental forms, respectively.

2. Preliminaries

The simply isotropic space $\mathbb{I}_3^1$ is a Cayley–Klein space described from the projective 3-space $\mathcal{P}(\mathbb{R}^3)$ with an absolute figure consisting of a plane w and two complex-conjugate straight lines f_1, f_2 in w. The homogeneous coordinates in $\mathcal{P}(\mathbb{R}^3)$ are introduced in such a way that the absolute plane w is given by $x_0 = 0$ and the absolute lines f_1, f_2 by $x_0 = x_1 + ix_2 = 0$, $x_0 = x_1 - ix_2 = 0$. $\mathbb{F}(0:0:0:1)$ is described as intersection point of these two lines and called as the absolute point. The group of motions of $\mathbb{I}_3^1$ is a six-parameter group given in the affine coordinates $x = \frac{x_1}{x_0}, y = \frac{x_2}{x_0}, z = \frac{x_3}{x_0}$ by

$$\begin{cases} x' = c_1 + x\cos\alpha - y\sin\alpha \\ y' = c_2 + x\sin\alpha + y\cos\alpha \\ z' = c_3 + c_4 x + c_5 y + z, \end{cases} \tag{2.1}$$

where $c_1, c_2, c_3, c_4, c_5, \alpha \in \mathbb{R}$. These affine transformations are called isotropic congruence transformations ([13, 14]). The metric of $\mathbb{I}_3^1$ is given by

$$ds^2 = dx^2 + dy^2.$$

This metric is induced by the absolute figure. The lines parallel to the z-direction are defined as isotropic lines. In addition, if the planes containing an isotropic line are defined as isotropic planes. Otherwise, they are non-isotropic.

Let $\mathbf{M}$ be a surface immersed in $\mathbb{I}_3^1$. This surface is described as admissible if it has no isotropic tangent planes. In this case, the coefficients E, F, G of the first fundamental form I of $\mathbf{M}$ and the coefficients e, f, g of the second fundamental form II of $\mathbf{M}$ are easily determined according to the induced metric. Hence, the (isotropic) Gaussian curvature K and (isotropic) mean curvature H are described as

$$\mathbf{K} = k_1 k_2 = \frac{eg - f^2}{EG - F^2}, \quad 2\mathbf{H} = k_1 + k_2 = \frac{Eg - 2Ff + Ge}{EG - F^2}, \tag{2.2}$$

where k_1, k_2 are principal curvatures. In other words, k_1, k_2 are extrema of the normal curvature determined by the normal section of a surface. Here, if $K = 0$, the surface M is isotropic flat. If $H = 0$, the surface M is isotropic minimal [2, 15, 17, 18]. The Laplace operators $\Delta^{\mathbf{I}}$, $\Delta^{\mathbf{II}}$, $\Delta^{\mathbf{III}}$ of the I., the II. and the III. fundamental forms on $\mathbf{M}$ according to local coordinates $\{u, v\}$ of $\mathbf{M}$ are defined by ([3–5, 7, 8, 10, 11, 16])

$$\Delta^{\mathbf{I}}\mathbf{x} = -\frac{1}{\sqrt{|EG - F^2|}} \left[\frac{\partial}{\partial u} \left(\frac{G\mathbf{x}_u - F\mathbf{x}_v}{\sqrt{|EG - F^2|}} \right) - \frac{\partial}{\partial v} \left(\frac{F\mathbf{x}_u - E\mathbf{x}_v}{\sqrt{|EG - F^2|}} \right) \right], \tag{2.3}$$

and

$$\Delta^{\mathbf{II}}\mathbf{x} = -\frac{1}{\sqrt{|eg - f^2|}} \left[\frac{\partial}{\partial u} \left(\frac{g\mathbf{x}_u - f\mathbf{x}_v}{\sqrt{|eg - f^2|}} \right) - \frac{\partial}{\partial v} \left(\frac{f\mathbf{x}_u - e\mathbf{x}_v}{\sqrt{|eg - f^2|}} \right) \right], \tag{2.4}$$

$$\Delta^{\mathbf{III}}\mathbf{x} = -\frac{\sqrt{|EG - F^2|}}{eg - f^2} \left[\begin{array}{c} \frac{\partial}{\partial u} \left(\frac{Z\mathbf{x}_u - Y\mathbf{x}_v}{(eg - f^2)\sqrt{|EG - F^2|}} \right) - \\ \frac{\partial}{\partial v} \left(\frac{Y\mathbf{x}_u - X\mathbf{x}_v}{(eg - f^2)\sqrt{|EG - F^2|}} \right) \end{array} \right], \tag{2.5}$$

where

$$X = Ef^2 - 2Fef + Ge^2,$$
$$Y = Efg - Feg + Gef - Ff^2,$$
$$Z = Gf^2 - 2Fgf + Eg^2.$$

3. Curvatures of the dual surfaces defined by $z = f(u) + g(v)$ in $\mathbb{I}_3^1$

In this section, we define the dual surfaces defined by $z = f(u) + g(v)$ in the three dimensional simply Isotropic space. Consider a surface in $\mathbb{I}_3^1$ as the graph of a function $z = h(u, v)$ of two variables, which is itself the sum of two functions f and g of one variable. A surface can be defined via the surface patch $z = f(u) + g(v)$.

Here, we restrict our topic to regular surfaces $\mathbf{x}$ without isotropic tangent planes. Thus, we can express in open form as

$$\mathbf{x}: \; z = h(u,v). \tag{3.1}$$

A surface $\mathbf{x}: z = h(u,v)$ is a set of contact elements. This surface corresponds to a surface $\mathbf{x}^*$, given by

$$\begin{cases} x^* = h_u(u,v), \\ y^* = h_v(u,v), \\ z^* = uh_u(u,v) + vh_v(u,v) - h(u,v). \end{cases} \tag{3.2}$$

So, using the equations (3.1) and (3.2), we can define the dual surfaces defined by $z = f(u) + g(v)$ as

$$\mathbf{x}^*(u,v) = \left(f'(u), g'(v), uf'(u) + vg'(v) - f(u) - g(v)\right). \tag{3.3}$$

Let $(\mathbf{M}, \mathbf{M}^*)$ be a dual surface pair. In this case, the relationship between the curvatures of these surfaces is as follows:

$$\mathbf{K}^* = \frac{1}{\mathbf{K}}, \;\; \mathbf{H}^* = \frac{\mathbf{H}}{\mathbf{K}}. \tag{3.4}$$

As it can be seen, if $\mathbf{K} = 0$, $\mathbf{M}^*$ may have singularities. In addition, the dual isotropic minimal surface is also isotropic minimal ([13, 14, 19]). Using the equation (3.3), the coefficients of the first and the second fundamental forms are given by

$$E = f''^2(u), \; G = g''^2(v), \; F = 0, \tag{3.5}$$

and

$$e = f''(u), \; g = g''(v), \; f = 0, \tag{3.6}$$

respectively. The dual Gaussian curvature $\mathbf{K}^*$ and the mean curvature $\mathbf{H}^*$ of the dual surfaces defined by $z = f(u) + g(v)$ are given by

$$\mathbf{K}^* = \frac{1}{f''(u)g''(v)} \tag{3.7}$$

and

$$\mathbf{H}^* = \frac{f''(u) + g''(v)}{2f''(u)g''(v)}, \tag{3.8}$$

respectively.

Let's assume that the dual surface has the constant Gaussian curvature. Then

$$\frac{1}{f''(u)g''(v)} = A, \tag{3.9}$$

where $A \in \mathbb{R}$. If we use separation of variables method, the Gaussian curvature $\mathbf{K}^* = const. \neq 0$ if and only if

$$\frac{1}{f''(u)} = const. = A_1 \neq 0 \tag{3.10}$$

and

$$\frac{1}{g''(v)} = const. = A_2 \neq 0. \tag{3.11}$$

We can get easily

$$\begin{cases} f(u) = c_1 + uc_2 + \frac{u^2}{2A_1}, \\ g(v) = c_3 + vc_4 + \frac{v^2}{2A_2}, \end{cases} \tag{3.12}$$

where $c_i, A_1, A_2 \in \mathbb{R}$. Thus, we have the following results:

Corollary 3.1. *Let $\mathbf{M}^*$ be the dual surface defined by $z = f(u) + g(v)$ with the constant Gaussian curvature $\mathbf{K}^* \neq 0$ in $\mathbb{I}_3^1$. Then, z can be written as* (3.12).

Corollary 3.2. *There is no dual surface $\mathbf{M}^*$ defined by $z = f(u) + g(v)$ with the zero Gaussian curvature $\mathbf{K}^* = 0$ (flat) in $\mathbb{I}_3^1$.*

Let's assume that the dual surface has the constant mean curvature, so

$$\frac{f''(u) + g''(v)}{f''(u)g''(v)} = 2C, \tag{3.13}$$

where $C \in \mathbb{R}$. If we use seperation of variables method, the mean curvature $\mathbf{H}^* = const. \neq 0$ if and only if

$$\frac{1}{f''(u)} = C_1, \tag{3.14}$$

and

$$\frac{1}{g''(v)} = C_2, \tag{3.15}$$

where $C_1, C_2 \in \mathbb{R}$. Thus, we get

$$\begin{cases} f(u) = c_1 + uc_2 + \frac{u^2}{2C_1}, \\ g(v) = c_3 + vc_4 + \frac{v^2}{2C_2}, \end{cases} \tag{3.16}$$

where $c_i \in \mathbb{R}$.

Theorem 3.3. *Let $\mathbf{M}^*$ be the dual surface defined by $z = f(u) + g(v)$ with the constant mean curvature $\mathbf{H}^* = C \neq 0$ in $\mathbb{I}_3^1$. Then z can be written as* (3.16).

Suppose that $\mathbf{H}^*$ satisfies the condition $\mathbf{H}^* = 0$. In this case, we define as a surface satisfying that condition dual isotropic minimal. Then, from the equation (3.8) we can write

$$f''(u) + g''(v) = 0, \tag{3.17}$$

where u, v are independent variables and both sides of the equation (3.17) are constant. If we show that this constant is equal to p, we get

$$f''(u) = p = -g''(v). \tag{3.18}$$

Hence, we can write

$$\left\{\begin{array}{l} f(u)=c_1+uc_2+\frac{pu^2}{2}, \\ g(v)=c_3+vc_4-\frac{pv^2}{2}, \end{array}\right. \tag{3.19}$$

where $p, c_i \in \mathbb{R}$. Here, if $p=0$, we obtain

$$\left\{\begin{array}{l} f(u)=c_1+uc_2, \\ g(v)=c_3+vc_4, \end{array}\right. \tag{3.20}$$

where $c_i \in \mathbb{R}$.

Theorem 3.4. *Let* $\mathbf{M}^*$ *be the dual surface defined by* $z=f(u)+g(v)$ *with zero mean curvature (dual isotropic minimal,* $\mathbf{H}^*=0$*) in* $\mathbb{I}_3^1$*. Then* z *can be written as* (3.19) *or* (3.20)*.*

4. The dual surfaces defined by $z=f(u)+g(v)$ satisfying $\Delta^{\mathbf{I}}\mathbf{x}_i^*=\lambda_i\mathbf{x}_i^*$

In this section, we classify dual surface defined by $z=f(u)+g(v)$ in $\mathbb{I}_3^1$ under the condition

$$\Delta^{\mathbf{I}}\mathbf{x}_i^*=\lambda_i\mathbf{x}_i^*, \tag{4.1}$$

where $\lambda_i \in \mathbb{R}$, $i=1,2,3$ and

$$\Delta^{\mathbf{I}}\mathbf{x}^*=\left(\Delta^{\mathbf{I}}\mathbf{x}_1^*,\Delta^{\mathbf{I}}\mathbf{x}_2^*,\Delta^{\mathbf{I}}\mathbf{x}_3^*\right), \tag{4.2}$$

with

$$\mathbf{x}_1^*=f'(u),\ \mathbf{x}_2^*=g'(v),\ \mathbf{x}_3^*=uf'(u)+vg'(v)-f(u)-g(v). \tag{4.3}$$

From the equations (4.2), (4.3) and (2.7), we obtain

$$\Delta^{\mathbf{I}}\mathbf{x}_i^*=\left(0,0,\frac{-f''(u)-g''(v)}{f''(u)g''(v)}\right). \tag{4.4}$$

If $\mathbf{M}^*$ satisfies the equation (4.1), from the equations (4.3) and (4.4), we get

$$\frac{-f''(u)-g''(v)}{f''(u)g''(v)}=\lambda\left(uf'(u)+vg'(v)-f(u)-g(v)\right), \tag{4.5}$$

where $\lambda \in \mathbb{R}$. Then, $\mathbf{M}^*$ is of 1-type. In this case, if $\mathbf{M}^*$ satisfies the condition $\Delta^{\mathbf{I}}\mathbf{x}_i^*=0$, this surface is defined as a harmonic surface or dual isotropic minimal. As a result of the equation (4.5), we obtain

$$f''(u)+g''(v)=0. \tag{4.6}$$

Thus we have the solutions of the equations (3.19) and (3.20).

Theorem 4.1. *Suppose that* $\mathbf{M}^*$ *is a dual surface which satisfies the condition* (3.1) *in* $\mathbb{I}_3^1$*. If* $\mathbf{M}^*$ *is harmonic or dual isotropic minimal, then* z *can be written as* (3.19) *or* (3.20)*.*

If $\lambda \neq 0$, from the equation (4.5), we get

$$-\frac{1}{f''(u)} - \lambda u f'(u) + \lambda f(u) = \frac{1}{g''(v)} + \lambda v g'(v) - \lambda g(v), \tag{4.7}$$

which implies there exists a real number p such that

$$-\frac{1}{f''(u)} - \lambda u f'(u) + \lambda f(u) = p = \frac{1}{g''(v)} + \lambda v g'(v) - \lambda g(v). \tag{4.8}$$

The second order nonlinear differential equation (4.8) can not be solved analytically. If we differentiate both sides of the equation (4.8) with respect to u and v, we obtain the following

$$-\lambda u f'' + \frac{f'''}{f''^2} = 0, \tag{4.9}$$

$$\lambda v g'' - \frac{g'''}{g''^2} = 0. \tag{4.10}$$

We deal with two cases with respect to constant λ.

Case 1: If $\lambda > 0$, the general solutions of the equations (4.9) and (4.10) are given by

$$\begin{cases} f(u) = c_1 + uc_2 \pm \dfrac{u \arctan\left(\frac{u\sqrt{\lambda}}{\sqrt{-\lambda u^2 - 2c_3}}\right) + \frac{\sqrt{-\lambda u^2 - 2c_3}}{\sqrt{\lambda}}}{\sqrt{\lambda}}, \\ g(v) = c_3 + vc_4 \pm \dfrac{v \arctan\left(\frac{v\sqrt{\lambda}}{\sqrt{-\lambda v^2 - 2c_5}}\right) + \frac{\sqrt{-\lambda v^2 - 2c_5}}{\sqrt{\lambda}}}{\sqrt{\lambda}} \end{cases} \tag{4.11}$$

where $\lambda, c_i \neq 0 \in \mathbb{R}$.

Case 2: If $\lambda < 0$, the general solutions of the equations (4.9) and (4.10) are given by

$$\begin{cases} f(u) = c_1 + uc_2 \pm \dfrac{-\frac{\sqrt{\lambda u^2 - 2c_3}}{\sqrt{\lambda}} + u \log\left(u\lambda + \sqrt{\lambda^2 u^2 - 2\lambda c_3}\right)}{\sqrt{\lambda}}, \\ g(v) = c_3 + vc_4 \pm \dfrac{-\frac{\sqrt{\lambda v^2 - 2c_5}}{\sqrt{\lambda}} + v \log\left(v\lambda + \sqrt{\lambda^2 u^2 - 2\lambda c_5}\right)}{\sqrt{\lambda}} \end{cases} \tag{4.12}$$

where $\lambda, c_i \neq 0 \in \mathbb{R}$.

Theorem 4.2. *Suppose that* $\mathbf{M}^*$ *is a non harmonic dual surface which satisfies the condition* (3.1) *in* $\mathbb{I}_3^1$. *If the surface* $\mathbf{M}^*$ *satisfies the equation* $\Delta^{\mathbf{I}}\mathbf{x}_i^* = \lambda \mathbf{x}_i^*$, *where* $\lambda \in \mathbb{R}$, $i = 1, 2, 3$, *then* $z(u,v)$ *can be written as* (4.11) *or* (4.12).

5. The dual surfaces defined by $z = f(u) + g(v)$ satisfying $\Delta^{\mathbf{II}}\mathbf{x}_i^* = \lambda_i \mathbf{x}_i^*$

In this section, we consider dual surfaces with non-degenerate II. fundamental form in $\mathbb{I}_3^1$ under the condition

$$\Delta^{\mathbf{II}}\mathbf{x}_i^* = \lambda_i \mathbf{x}_i^*, \tag{5.1}$$

where $\lambda_i \in \mathbb{R}$, $i=1,2,3$ and

$$\Delta^{\mathbf{II}}\mathbf{x}^{*}=\left(\Delta^{\mathbf{II}}\mathbf{x}_1^{*},\Delta^{\mathbf{II}}\mathbf{x}_2^{*},\Delta^{\mathbf{II}}\mathbf{x}_3^{*}\right).$$

If dual surface $\mathbf{M}^*$ is constructed with component functions which are eigenfunctions of its Laplace operator $\boldsymbol{\Delta}^{\mathbf{II}}$, then we shall have

$$-\frac{f'''}{2f''} = \lambda_1 f' \tag{5.2}$$

$$-\frac{g'''}{2g''} = \lambda_2 g', \tag{5.3}$$

$$-2 - u\frac{f'''}{2f''} - v\frac{g'''}{2g''} = \lambda_3 \left(uf' + vg' - f - g\right), \tag{5.4}$$

where $\lambda_i \in \mathbb{R}$ and $\mathbf{M}^*$ is at least 3-type. From the equations (5.2), (5.3) and (5.4), we can write

$$-2 + u\lambda_1 f' - \lambda_3 uf' + \lambda_3 f = -\lambda_2 vg' + \lambda_3 vg' - \lambda_3 g. \tag{5.5}$$

The solutions of differential equation (5.5) are given by

$$\begin{cases} f(u) = c_1\left(u\lambda_1 - u\lambda_3\right)^{-\frac{\lambda_3}{\lambda_1-\lambda_3}} + \frac{(2+p)(u(\lambda_1-\lambda_3))^{\frac{\lambda_3}{\lambda_1-\lambda_3}}(u\lambda_1-u\lambda_3)^{-\frac{\lambda_3}{\lambda_1-\lambda_3}}}{\lambda_3}, \\ g(v) = c_2\left(v\lambda_2 - v\lambda_3\right)^{-\frac{\lambda_3}{\lambda_2-\lambda_3}} - \frac{p(v(\lambda_2-\lambda_3))^{\frac{\lambda_3}{\lambda_2-\lambda_3}}(v\lambda_1-v\lambda_3)^{-\frac{\lambda_3}{\lambda_2-\lambda_3}}}{\lambda_3}, \end{cases} \tag{5.6}$$

where for some constants $c_i \neq 0$ and $\lambda_i \neq 0$.

We discuss seven cases according to constants $\lambda_1, \lambda_2, \lambda_3$.

Case 1: If $\lambda_1 = 0$, $\lambda_2 \neq 0$, $\lambda_3 \neq 0$, from the equation (5.5), we obtain

$$-2 - \lambda_3 uf' + \lambda_3 f = -\lambda_2 vg' + \lambda_3 vg' - \lambda_3 g. \tag{5.7}$$

This differential equations admit the solutions

$$\begin{cases} f(u) = c_1 u + \frac{2+p}{\lambda_3}, \\ g(v) = c_2\left(v\lambda_2 - v\lambda_3\right)^{-\frac{\lambda_3}{\lambda_2-\lambda_3}} - \frac{p(v(\lambda_2-\lambda_3))^{\frac{\lambda_3}{\lambda_2-\lambda_3}}(v\lambda_1-v\lambda_3)^{-\frac{\lambda_3}{\lambda_2-\lambda_3}}}{\lambda_3}, \end{cases} \tag{5.8}$$

where $p, c_i \neq 0 \in \mathbb{R}$.

Case 2: If $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 \neq 0$, from the equation (5.5), we obtain

$$-2 - \lambda_3 uf' + \lambda_3 f = \lambda_3 vg' - \lambda_3 g. \tag{5.9}$$

We can get easily

$$\begin{cases} f(u) = c_1 u + \frac{2+p}{\lambda_3}, \\ g(v) = c_2 v - \frac{p}{\lambda_3}, \end{cases} \tag{5.10}$$

where p, $c_1 \in \mathbb{R}$.

Case 3: If $\lambda_1 = 0, \lambda_2 \neq 0$, $\lambda_3 = 0$, from the equation (5.5), we obtain

$$-2 = -\lambda_2 vg'. \tag{5.11}$$

We can get easily

$$g(v) = c_1 + \frac{2 \log v}{\lambda_2}, \tag{5.12}$$

where $c_1 \in \mathbb{R}$ and $f(u)$ free of choice of $g(v)$. We can choose the function $f(u)$ as below

$$f(u) = c_3 u + c_4, \tag{5.13}$$

where $c_i \in \mathbb{R}$.

Case 4: If $\lambda_1 \neq 0$, $\lambda_2 = 0$, $\lambda_3 = 0$, from the equation (5.5), we obtain

$$-2 + u\lambda_1 f' = 0. \tag{5.14}$$

Also, the general solution of the equation (5.14) can be given by

$$f(u) = c_1 + \frac{2 \log u}{\lambda_1},$$

where $c_1 \in \mathbb{R}$ and the function $g(v)$ is independent of choice of $f(u)$. In this case, if we choose $g(v) = c_2 \in \mathbb{R}$, we get the following general solutions:

$$\begin{cases} f(u) = c_1 + \frac{2 \log u}{\lambda_1}, \\ g(v) = c_2. \end{cases} \tag{5.15}$$

Case 5: If $\lambda_1 \neq 0$, $\lambda_2 \neq 0$, $\lambda_3 = 0$, from the equation (5.5), we obtain

$$-2 + u\lambda_1 f' = -\lambda_2 v g'. \tag{5.16}$$

Then, the general solutions of the equation (5.16) are given by

$$\begin{cases} f(u) = c_1 + \frac{(2+p) \log u}{\lambda_1}, \\ g(v) = c_2 - \frac{p \log u}{\lambda_2}, \end{cases} \tag{5.17}$$

where $c_1, c_2, p \in \mathbb{R}$.

Case 6: If $\lambda_1 \neq 0$, $\lambda_2 = 0$, $\lambda_3 \neq 0$, from the equation (5.5), we obtain

$$-2 + u\lambda_1 f' - \lambda_3 u f' + \lambda_3 f = \lambda_3 v g' - \lambda_3 g. \tag{5.18}$$

and its general solutions are

$$\begin{cases} f(u) = c_1 \left(u\lambda_1 - u\lambda_3\right)^{-\frac{\lambda_3}{\lambda_1 - \lambda_3}} \\ \qquad + \dfrac{(2+p)\left(u\left(\lambda_1 - \lambda_3\right)\right)^{\frac{\lambda_3}{\lambda_1 - \lambda_3}} \left(u\lambda_1 - u\lambda_3\right)^{-\frac{\lambda_3}{\lambda_1 - \lambda_3}}}{\lambda_3}, \\ g(v) = c_2 v - \dfrac{p}{\lambda_3}, \end{cases} \tag{5.19}$$

where $c_1, c_2, p \in \mathbb{R}$.

Case 7: If $\lambda_1 = \lambda_2 = \lambda_3 = 0$, from (5.5), we obtain $-2 = 0$, this is a contradiction.

The solutions (5.6), (5.8), (5.15) and (5.19) do not satisfy (5.2) and (5.3) simultaneously. The solutions (5.10), (5.12), (5.13), (5.15) and (5.17) satisfy (5.2) and (5.3) simultaneously.

Definition. A dual surface in $\mathbb{I}_3^1$ is define as **II**-harmonic under the condition that $\Delta^{\mathbf{II}}\mathbf{x}^*=\mathbf{0}$.

Corollary 5.1. *There is no* **II**-*harmonic dual surface satisfying the equation* $\Delta^{\mathbf{II}}\mathbf{x}^*=\mathbf{0}$ *in* $\mathbb{I}_3^1$.

Theorem 5.2. *Suppose that* $\mathbf{M}^*$ *is a non* **II**-*harmonic dual surface in* $\mathbb{I}_3^1$. *If* $\mathbf{M}^*$ *satisfies the condition* $\Delta^{\mathbf{II}}\mathbf{x}_i^*=\lambda_i\mathbf{x}_i^*$, *where* $\lambda_i\in\mathbb{R}$, $i=1,2,3$, *then* $z(u,v)$ *can be written as* (5.10), (5.12), (5,13) *and* (5.17).

6. The dual surfaces defined by $z=f(u)+g(v)$ satisfying $\Delta^{\mathbf{III}}\mathbf{x}_i^*=\lambda_i\mathbf{x}_i^*$

In this section, we consider the dual surface with non-degenerate II fundamental form in $\mathbb{I}_3^1$ under the condition

$$\Delta^{\mathbf{III}}\mathbf{x}_i^*=\lambda_i\mathbf{x}_i^*, \tag{6.1}$$

where $\lambda_i\in\mathbb{R}$, $i=1,2,3$ and

$$\Delta^{\mathbf{III}}\mathbf{x}^*=\left(\Delta^{\mathbf{III}}\mathbf{x}_1^*,\Delta^{\mathbf{III}}\mathbf{x}_2^*,\Delta^{\mathbf{III}}\mathbf{x}_3^*\right). \tag{6.2}$$

Using the equation (6.2), the Laplacian of $\mathbf{M}^*$ can be expressed as follows

$$\Delta^{\mathbf{III}}\mathbf{x}^*=\left(-f''',-g''',-f''-g''-uf'''-vg'''\right). \tag{6.3}$$

By using the equations (6.1) and (6.3), we have the following equations

$$-f'''=\lambda_1 f', \tag{6.4}$$

$$-g'''=\lambda_2 g', \tag{6.5}$$

$$-f''-g''-uf'''-vg'''=\lambda_3\left(uf'+vg'-f-g\right), \tag{6.6}$$

where λ_1, λ_2 and $\lambda_3\in\mathbb{R}$. Therefore, $\mathbf{M}^*$ is at least 3- type. Using the equations (6.4), (6.5) and (6.6), we have

$$-f''+\lambda_1 uf'-\lambda_3 uf'+\lambda_3 f=g''-\lambda_2 vg'+\lambda_3 vg'+\lambda_3 g, \tag{6.7}$$

where u, v are independent variables and both sides of the equation (6.7) are constant. If we show that this constant is equal to p, we have

$$-f''+\lambda_1 uf'-\lambda_3 uf'+\lambda_3 f=p=g''-\lambda_2 vg'+\lambda_3 vg'+\lambda_3 g. \tag{6.8}$$

In order to solve the above system we have to distinguish seven cases depending on the constants $\lambda_1,\lambda_2,\lambda_3$. But there are no suitable solutions for the functions $f(u)$ and $g(v)$ for constants $\lambda_i\in\mathbb{R}$.

If $\lambda_1=\lambda_2=\lambda_3=0$, from the equation (6.8), we obtain

$$-f''=p=g''. \tag{6.9}$$

Hence, the general solutions of the equation (6.8) are given by

$$\begin{cases} f(u)=c_1+uc_2-p\frac{u^2}{2}, \\ g(v)=c_3+c_4 v+p\frac{u^2}{2}, \end{cases} \tag{6.10}$$

where $p, c_i \neq 0 \in \mathbb{R}$. Here, if $p = 0$, we write

$$\left\{ \begin{array}{l} f(u) = c_1 + uc_2, \\ g(v) = c_3 + c_4 v, \end{array} \right. \tag{6.11}$$

where $c_i \in \mathbb{R}$. In this case, the solutions (6.10) and (6.11) satisfy (6.4) and (6.5) simultaneously.

Definition. A dual surface in $\mathbb{I}_3^1$ is defined as **III**-harmonic under the condition that $\Delta^{\mathbf{III}}\mathbf{x}^* = \mathbf{0}$.

Theorem 6.1. *Suppose that* $\mathbf{M}^*$ *is a dual surface which satisfies the condition* (3.1) *in* $\mathbb{I}_3^1$. *If* $\mathbf{M}^*$ *is* **III***-harmonic, then* $z(u, v)$ *can be written as* (6.10) *or* (6.11).

Theorem 6.2 (Classification)**.** *There is no dual surface* $\mathbf{M}^*$ *satisfying the condition* $\Delta^{\mathbf{III}}\mathbf{x}_i^* = \lambda_i \mathbf{x}_i^*$, *where* $\lambda_i \in \mathbb{R}$ *in* $\mathbb{I}_3^1$.

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