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FABER POLYNOMIAL COEFFICIENT ESTIMATES FOR CERTAIN SUBCLASS OF MEROMORPHIC BI-UNIVALENT FUNCTIONS

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Abstract. In this paper, we obtain the coefficient bounds for subclass of meromorphic bi-univalent functions by using the Faber polynomial expansions. The results presented in this paper would generalize and improve some recent works.

1. Introduction

Let Σ denote the class of meromorphic univalent functions f of the form

$$f(z) = z + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{z^n}, \tag{1.1}$$

defined on the domain $\Delta = \{z \in \mathbb{C} : 1 < |z| < \infty\}$. Since $f \in \Sigma$ is univalent, it has an inverse f^{-1}, that satisfy

$$f^{-1}(f(z)) = z \ (z \in \Delta)$$

and

$$f(f^{-1}(w)) = w \ \ (M < |w| < \infty, \ M > 0)\,.$$

A simple calculation shows that the function $g := f^{-1}$ is given by

$$\begin{aligned} g(w) &= w + B_0 + \sum_{n=1}^{\infty} \frac{B_n}{w^n} \\ &= w - b_0 - \frac{b_1}{w} - \frac{b_2 + b_0 b_1}{w^2} - \frac{b_3 + 2b_0 b_1 + b_0^2 b_1 + b_1^2}{w^3} + \cdots\,. \end{aligned} \tag{1.2}$$

A function $f \in \Sigma$ is said to be meromorphic bi-univalent in Δ if both f and f^{-1} are univalent in Δ. The family of all meromorphic bi-univalent functions is denoted by $\Sigma_{\mathfrak{B}}$. Estimates on the coefficient of meromorphic univalent functions were widely investigated in the literature; for example, Schiffer [15]

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obtained the estimate $|b_2| \leq 2/3$ for meromorphic univalent functions $f \in \Sigma$ with $b_0 = 0$ and Duren [6] proved that $|b_n| \leq 2/(n+1)$ for $f \in \Sigma$ with $b_k = 0$, $1 \leq k \leq n/2$.

For the coefficients of inverses of meromorphic univalent functions, Springer [17] proved that

$$|B_3| \leq 1 \quad \text{and} \quad |B_3 + \frac{1}{2}B_1^2| \leq \frac{1}{2}$$

and conjectured that

$$|B_{2n-1}| \leq \frac{(2n-2)!}{n!(n-1)!} \ (n = 1, 2, \ldots).$$

In 1977, Kubota [12] proved that the Springer conjecture is true for $n = 3, 4, 5$ and subsequently Schober [16] obtained a sharp bounds for the coefficients B_{2n-1}, $1 \leq n \leq 7$.

Several researchers (for example see [4, 5, 8–11, 19]) introduced and investigated new subclasses of meromorphically bi-univalent functions.

Recently, T. Panigrahi [13] introduced the following subclass $\mathcal{T}_{\Sigma_{\mathfrak{B}}}(\beta, \lambda)$ of meromorphic bi-univalent function and obtained non sharp estimates on the initial coefficients $|b_0|$, $|b_1|$ and $|b_2|$ for functions in this subclass. In this paper, we use the Faber polynomial expansion [7] to obtain not only improvement of estimates of coefficients $|b_0|$, $|b_1|$ and $|b_2|$ which obtained by Panigrahi [13], but also we find estimates of coefficients $|b_n|$ where $n \geq 3$.

Definition 1.1 ([13, Definition 3.1])**.** A function $f(z) \in \Sigma_{\mathfrak{B}}$ given by (1.1) is said to be in the class $\mathcal{T}_{\Sigma_{\mathfrak{B}}}(\beta, \lambda)$, if the following conditions are satisfied:

$$\operatorname{Re}\left\{\lambda \frac{zf'(z)}{f(z)} + (1-\lambda)\left(1 + \frac{zf''(z)}{f'(z)}\right)\right\} > \beta \ (0 \leq \beta < 1, \lambda \geq 1, z \in \Delta)$$

and

$$\operatorname{Re}\left\{\lambda \frac{wg'(w)}{g(w)} + (1-\lambda)\left(1 + \frac{wg''(w)}{g'(w)}\right)\right\} > \beta \ (0 \leq \beta < 1, \lambda \geq 1, w \in \Delta),$$

where the function g is the inverse of f given by (1.2).

Theorem 1.2 ([13, Thoerem 3.2])**.** *Let $f(z)$ given by (1.1) be in the class $\mathcal{T}_{\Sigma_{\mathfrak{B}}}(\beta, \lambda)$. Then*

$$|b_0| \leq \frac{2(1-\beta)}{\lambda},$$

$$|b_1| \leq \frac{(1-\beta)}{2\lambda - 1}\sqrt{1 + \frac{4(1-\beta)^2}{\lambda^2}}$$

and

$$|b_2| \leq \frac{2(1-\beta)}{3(3\lambda - 2)}\left[1 + \frac{4(1-\beta)^2}{\lambda^2}\right].$$

2. Preliminary results

In the present paper by using the Faber polynomial expansions we obtain estimates of coefficients $|a_n|$ where $n \geq 3$, of functions in the class $\mathcal{T}_{\Sigma_{\mathfrak{B}}}(\beta, \lambda)$. The Faber polynomials introduced by Faber [7] play an important role in various areas of mathematical sciences, especially in geometric function theory. Several authors worked on using Faber polynomial expansions to find coefficient estimates for classes bi-univalent functions, see for example [3,5,8–11,18]. For this purpose we need the following lemmas.

Lemma 2.1 ([1,2])**.** *Let $f(z) = z + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{z^n}$, be meromorphic univalent function in $\Delta = \{z \in \mathbb{C} : 1 < |z| < \infty\}$. Then we can write,*

$$\frac{zf'(z)}{f(z)} = 1 + \sum_{k=1}^{\infty} F_k(b_0, b_1, \dots, b_{k-1}) \frac{1}{z^k}, \tag{2.1}$$

where $F_k(b_0, b_1, \dots, b_{k-1})$ is a Faber polynomial of degree k,

$$F_k(b_0, b_2, \dots, b_{k-1}) = \sum_{i_1+2i_2+\cdots+ki_k=k} A_{(i_1,i_2,\dots,i_k)} b_0^{i_1} b_1^{i_1} \cdots b_{k-1}^{i_k}$$

and

$$A_{(i_1,i_2,\dots,i_k)} := (-1)^{k+2i_1+3i_2+\cdots+(k+1)i_k} \frac{(i_1 + i_2 + \cdots + i_k - 1)!k}{i_1! i_2! \cdots i_k!}.$$

The first Faber polynomials $F_k(b_0, b_1, \dots, b_k)$ are given by:

$$F_1(b_0) = -b_0, \; F_2(b_0, b_1) = b_0^2 - 2b_1 \quad \text{and} \quad F_3(b_0, b_1, b_2) = -b_0^3 + 3b_0 b_1 - 3b_2.$$

Lemma 2.2 ([3, page 52])**.** *Let $f(z) = z + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{z^n}$, be meromorphic bi-univalent in $\Delta = \{z \in \mathbb{C} : 1 < |z| < \infty\}$. Then the coefficients of function $g := f^{-1}$ are given*

$$g(w) = w + B_0 + \sum_{n=1}^{\infty} \frac{B_n}{w^n} = w - b_0 - \sum_{n=1}^{\infty} \frac{1}{n} K_{n+1}^n \frac{1}{w^n}; \; M < |w| < \infty, \tag{2.2}$$

where

$$K_{n+1}^n = n b_0^{n-1} b_1 + n(n-1) b_0^{n-2} b_2 + \frac{1}{2} n(n-1)(n-2) b_0^{n-3}(b_3 + b_1^2) + \frac{n(n-1)(n-2)(n-3)}{3!} b_0^{n-4}(b_4 + 3b_1 b_2) + \sum_{j \geq 5} b_0^{n-j} V_j$$

and V_j with $5 \leq j \leq n$ is a homogeneous polynomial of degree j in the variables $b_1, b_2, \dots, b_n$. In particular $K_2^1 = b_1$, $\frac{1}{2} K_3^2 = b_0 b_1 + b_2$ and $\frac{1}{3} K_4^3 = b_0^2 b_1 + 2b_0 b_1 + b_3 + b_1^2$.

By applying Lemma 2.1 for a function $\frac{(zf'(z))'}{f'(z)} = 1 + \frac{zf''(z)}{f'(z)}$ we can obtain the following lemma.

Lemma 2.3. *Let $f(z) = z + b_0 + \sum_{n=1}^{\infty} \frac{b_n}{z^n}$, be meromorphic univalent function in $\Delta = \{z \in \mathbb{C} : 1 < |z| < \infty\}$. Then we can write,*

$$\begin{aligned}
&\lambda\left(\frac{zf'(z)}{f(z)}\right) + (1-\lambda)\left(1 + \frac{zf''(z)}{f'(z)}\right)\\
&= \lambda\left(\frac{zf'(z)}{f(z)}\right) + (1-\lambda)\left(\frac{z(zf'(z))'}{zf'(z)}\right)\\
&= \lambda\left[1 + \sum_{n=1}^{\infty} F_n(b_0, b_1, \dots, b_{n-1})\frac{1}{z^n}\right] \qquad (2.3)\\
&\quad + (1-\lambda)\left[1 + \sum_{n=1}^{\infty} F_n(0, -b_1, \dots, -(n-1)b_{n-1})\frac{1}{z^n}\right]\\
&= 1 + \sum_{n=0}^{\infty}\left[\lambda F_{n+1}(b_0, b_1, \dots, b_n) + (1-\lambda)F_{n+1}(0, -b_1, \dots, -nb_n)\right]\frac{1}{z^{n+1}}.
\end{aligned}$$

Lemma 2.4 ([14])**.** *If $h \in \mathcal{P}$, then $|c_k| \le 2$ for each k, where $\mathcal{P}$ is the family of all functions h analytic in $\Delta = \{z \in \mathbb{C} : 1 < |z| < \infty\}$ for which $\mathrm{Re}(h(z)) > 0$ where $h(z) = 1 + \frac{c_1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \cdots$.*

3. Coefficient estimates

Theorem 3.1. *Let $f(z) \in \Sigma_{\mathfrak{B}}$ given by (1.1) be in the class $\mathcal{T}_{\Sigma_{\mathfrak{B}}}(\beta, \lambda)$ ($\lambda \ge 1$, $0 \le \beta < 1$). If $b_1 = b_2 = \cdots = b_{n-1} = 0$ for n being odd or if $b_0 = b_1 = \cdots = b_{n-1} = 0$ for n being even, then*

$$|b_n| \le \frac{2(1-\beta)}{(n+1)((n+1)\lambda - n)}, \quad n \ge 1. \qquad (3.1)$$

Proof. For meromorphic bi-univalent function f of the form (1.1) by applying Lemma 2.3, we have:

$$\begin{aligned}
&\lambda\left(\frac{zf'(z)}{f(z)}\right) + (1-\lambda)\left(1 + \frac{zf''(z)}{f'(z)}\right) \qquad (3.2)\\
&= 1 + \sum_{n=0}^{\infty}\left[(\lambda F_{n+1}(b_0, \dots, b_n) + (1-\lambda)F_{n+1}(0, -b_1, \dots, -nb_n)\right]\frac{1}{z^{n+1}}
\end{aligned}$$

and again by applying Lemma 2.3 for its inverse map $g = f^{-1}$, we have:

$$\begin{aligned}
&\lambda\left(\frac{wg'(w)}{g(w)}\right) + (1-\lambda)\left(1 + \frac{wg''(w)}{g'(w)}\right) \qquad (3.3)\\
&= 1 + \sum_{n=0}^{\infty}\left[\lambda F_{n+1}(B_0, \dots, B_n) + (1-\lambda)F_{n+1}(0, -B_1, \dots, -nB_n)\right]\frac{1}{w^{n+1}}.
\end{aligned}$$

Since $f \in \mathcal{T}_{\Sigma_{\mathfrak{B}}}(\beta, \lambda)$, by definition, there exist two positive real-part functions $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^{-n}$ and $q(w) = 1 + \sum_{n=1}^{\infty} d_n w^{-n}$, where $\mathrm{Re}\{p(z)\} > 0$ and

$\mathrm{Re}\{q(w)\} > 0$ in Δ so that:

$$\lambda\left(\frac{zf'(z)}{f(z)}\right)+(1-\lambda)\left(1+\frac{zf''(z)}{f'(z)}\right) = 1+(1-\beta)\sum_{n=0}^{\infty}K^1_{n+1}(c_1,c_2,\ldots,c_{n+1})\frac{1}{z^{n+1}} \tag{3.4}$$

and

$$\lambda\left(\frac{wg'(w)}{g(w)}\right)+(1-\lambda)\left(1+\frac{wg''(w)}{g'(w)}\right) = 1+(1-\beta)\sum_{n=0}^{\infty}K^1_{n+1}(d_1,d_2,\ldots,d_{n+1})\frac{1}{w^{n+1}}. \tag{3.5}$$

By equating the corresponding coefficients of (3.2) and (3.4), we have:

$$\lambda F_{n+1}(b_0,b_1,\ldots,b_n)+(1-\lambda)F_{n+1}(0,-b_1,\ldots,-nb_n) = (1-\beta)K^1_{n+1}(c_1,c_2,\ldots,c_{n+1}) \tag{3.6}$$

and, similarly, from (3.3) and (3.5), we obtain:

$$\lambda F_{n+1}(B_0,B_1,\ldots,B_n)+(1-\lambda)F_{n+1}(0,-B_1,\ldots,-nB_n) = (1-\beta)K^1_{n+1}(d_1,d_2,\ldots,d_{n+1}). \tag{3.7}$$

Note that for $b_k=0;\ 1\le k\le n-1$, we have $B_0=-b_0$, $B_n=-b_n$, then

$$F_{n+1}(b_0,0,\ldots,0,b_n)=(-1)^{n+1}b_0^{n+1}-(n+1)b_n. \tag{3.8}$$

Hence, when n is odd, by using equation (3.8) and $B_0=-b_0$, $B_n=-b_n$, the equalities (3.6) and (3.7) can be written as follow:

$$\lambda b_0^{n+1}+(n+1)\left[n-\lambda(n+1)\right]b_n=(1-\beta)c_{n+1},$$
$$\lambda b_0^{n+1}-(n+1)\left[n-\lambda(n+1)\right]b_n=(1-\beta)d_{n+1}.$$

Subtract two above equation, we have

$$2(n+1)\left[n-\lambda(n+1)\right]b_n=(1-\beta)(c_{n+1}-d_{n+1}).$$

Now using Lemma 2.4, we immediately have:

$$|b_n|=\frac{(1-\beta)|c_{n+1}-d_{n+1}|}{2(n+1)((n+1)\lambda-n)}\le\frac{2(1-\beta)}{(n+1)((n+1)\lambda-n)}.$$

When n is even, if ($b_0=\cdots=b_{n-1}=0$) again using equation (3.8), the equalities (3.6) and (3.7) can be written as a follow:

$$(n+1)\left[n-\lambda(n+1)\right]b_n=(1-\beta)c_{n+1},$$
$$-(n+1)\left[n-\lambda(n+1)\right]b_n=(1-\beta)d_{n+1}.$$

Now getting the absolute values of either of the above two equalities and using Lemma 2.4, we obtain:

$$|b_n| = \frac{(1-\beta)|c_{n+1}|}{(n+1)((n+1)\lambda - n)} \leq \frac{2(1-\beta)}{(n+1)((n+1)\lambda - n)}.$$

This evidently completes the proof of Theorem 3.1. □

Theorem 3.2. *Let* $f(z) \in \mathcal{T}_{\Sigma_{\mathfrak{B}}}(\beta, \lambda)$, *where* $(\lambda \geq 1,\ 0 \leq \beta < 1)$. *Then*

$$|b_0| \leq \begin{cases} \sqrt{\frac{2(1-\beta)}{\lambda}};\ \lambda + 2\beta \leq 2 \\ \frac{2(1-\beta)}{\lambda};\ \lambda + 2\beta \geq 2, \end{cases}$$

$$|b_1| \leq \frac{1-\beta}{|2\lambda - 1|},$$

and

$$|b_2| \leq \begin{cases} \frac{2(1-\beta)}{3(3\lambda-2)}[1 + \sqrt{\frac{2(1-\beta)}{\lambda}}];\ \lambda + 2\beta \leq 2 \\ \frac{2(1-\beta)}{3(3\lambda-2)}[1 + \frac{4(1-\beta)^2}{\lambda^2}];\ \lambda + 2\beta \geq 2. \end{cases}$$

Proof. Comparing corresponding coefficients of (3.2) and (3.4) for $n = 0, 1, 2$, we obtain:

$$-\lambda b_0 = (1-\beta)c_1, \tag{3.9}$$

$$\lambda b_0^2 + 2(1-2\lambda)b_1 = (1-\beta)c_2, \tag{3.10}$$

and

$$-\lambda b_0^3 + 3\lambda b_0 b_1 + 3(2-3\lambda)b_2 = (1-\beta)c_3. \tag{3.11}$$

Getting the absolute values of (3.9) and using Lemma 2.4, we have:

$$|b_0| \leq \frac{2(1-\beta)}{\lambda}. \tag{3.12}$$

Similarly, comparing corresponding coefficients of (3.3) and (3.5) for $n = 1$, we obtain

$$\lambda b_0^2 - 2(1-2\lambda)b_1 = (1-\beta)d_2. \tag{3.13}$$

Adding (3.10) and (3.13) yields:

$$2\lambda b_0^2 = (1-\beta)(c_2 + d_2).$$

Getting the absolute values of the above equality and using Lemma 2.4, we get:

$$|b_0| = \sqrt{\frac{(1-\beta)|c_2 + d_2|}{2\lambda}} \leq \sqrt{\frac{2(1-\beta)}{\lambda}}. \tag{3.14}$$

From (3.12) and (3.14), we obtain the first part of theorem.

To show the second part of the theorem, subtracting (3.13) from (3.10) we obtain:

$$4(1-2\lambda)b_1 = (1-\beta)(c_2-d_2).$$

Getting the absolute values of the above equality and using Lemma 2.4, we get:

$$|b_1| = \frac{(1-\beta)|c_2-d_2|}{4|1-2\lambda|} \leq \frac{1-\beta}{|2\lambda-1|}.$$

Finally, to determine the bound on $|b_2|$, comparing corresponding coefficients of (3.3) and (3.5) for $n=2$, we have

$$\lambda {b_0}^3 - 6(1-2\lambda)b_0b_1 - 3(2-3\lambda)b_2 = (1-\beta)d_3. \tag{3.15}$$

Similarly, consider the sum of (3.11) and (3.15), we have

$$3(5\lambda-2)b_0b_1 = (1-\beta)(c_3+d_3). \tag{3.16}$$

Subtracting (3.15) from (3.11) and using (3.16), we obtain

$$6(2-3\lambda)b_2 = (1-\beta)(c_3-d_3) - \frac{2-3\lambda}{5\lambda-2}(1-\beta)(c_3+d_3) + 2\lambda {b_0}^3, \tag{3.17}$$

i.e.,

$$6(2-3\lambda)b_2 = \frac{8\lambda-4}{5\lambda-2}(1-\beta)c_3 - \frac{2\lambda}{5\lambda-2}(1-\beta)d_3 + 2\lambda {b_0}^3. \tag{3.18}$$

By using Lemma 2.4 and (3.12), (3.14) we have the result. □

4. Corollaries and consequences

Remark 4.1. Trivially the estimates of $|b_0|$, $|b_1|$ and $|b_2|$ which obtained in Theorem 3.2 are better than the corresponding estimates in Theorem 1.2.

By putting $\lambda = 1$ in Theorem 3.1 and Theorem 3.2, we conclude the following results.

Corollary 4.2. *Let $f(z) \in \mathcal{T}_{\Sigma_{\mathfrak{B}}}(\beta)$ $(0 \leq \beta < 1)$. If $b_1 = b_2 = \cdots = b_{n-1} = 0$ for n being odd or if $b_0 = b_1 = \cdots = b_{n-1} = 0$ for n being even, then*

$$|b_n| \leq \frac{2(1-\beta)}{n+1}.$$

Corollary 4.3. *Let $f(z) \in \mathcal{T}_{\Sigma_{\mathfrak{B}}}(\beta)$ $(0 \leq \beta < 1)$, then*

$$|b_0| \leq \begin{cases} \sqrt{2(1-\beta)}; & 0 \leq \beta \leq \frac{1}{2} \\ 2(1-\beta); & \frac{1}{2} \leq \beta < 1, \end{cases}$$

$$|b_1| \leq 1-\beta,$$

and

$$|b_2| \leq \begin{cases} \frac{2(1-\beta)}{3}[1+\sqrt{2(1-\beta)}]; & 0 \leq \beta \leq \frac{1}{2} \\ \frac{2(1-\beta)}{3}[1+4(1-\beta)^2]; & \frac{1}{2} \leq \beta < 1. \end{cases}$$

Remark 4.4. The estimates of $|b_0|$ and $|b_1|$ which obtained in Corollary 4.3 are better than the corresponding estimates in [10, Theorem 2].

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