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ON U-GROUP RINGS

Emad Abu Osba, Hasan Al-Ezeh, and Manal Ghanem

Abstract. Let R be a commutative ring, G be an Abelian group, and let RG be the group ring. We say that RG is a U-group ring if a is a unit in RG if and only if $\epsilon(a)$ is a unit in R. We show that RG is a U-group ring if and only if G is a p-group and $p \in J(R)$. We give some properties of U-group rings and investigate some properties of well known rings, such as Hermite rings and rings with stable range, in the presence of U-group rings.

1. Introduction

Throughout this paper all rings considered are commutative with unity and all groups are assumed to be Abelian. Let $J(R)$, $Nil(R)$ and $U(R)$ denote the Jacobson radical of R, the nil radical of R and the set of units of R, respectively.

Let R be a ring and let G be a group. Define $RG = \{\sum_{i=1}^{n} a_i g_i : a_i \in R, g_i \in G, n \in \mathbb{N}\}$. Let $\epsilon : RG \to R$ be the ring homomorphism defined by $\epsilon\left(\sum_{g\in G} a_g g\right) = \sum_{g\in G} a_g$, see [3]. Let $\Delta(G) = \ker(\epsilon) = \{a \in RG : \epsilon(a) = 0\}$. If $g \in G$ is of order $n < \infty$, then let $\hat{g} = 1 + g + g^2 + \cdots + g^{n-1}$ and it is clear that if $f \in \langle g\rangle$, then $f\hat{g} = \epsilon(f)\hat{g}$. For each $a = \sum_{i=1}^{n} a_i g_i \in RG$, let $Supp(a) = \{g_i : a_i \neq 0\}$. For each $n \in \mathbb{N}$, let $C_n = \langle g\rangle$ be the multiplicative cyclic group of order n.

In this article we define U-group rings to be group rings RG such that $a \in U(RG)$ if and only if $\epsilon(a) \in U(R)$ and we investigate some basic properties of them. It was proved in [1] that if G is a p-group and $p \in J(R)$, then RG is a U-group ring and here we show that the converse is also true. For U-group rings many algebraic properties are shared by RG and R, as in [1] and [10]. This article is a continuation of the work done on these two articles. We show that if RG is a U-group ring, then RG has stable range d if and only if R has. We also show that if RG is a U-group ring, then RG is d-Hermite if and only if R is. Finally we use properties of U-group rings to show that if a positive integer

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n is not a power of a prime number, then the combinations: $\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1}$ are relatively prime.

2. U-group rings

While it is clear that if $a \in U(RG)$, then $\epsilon(a) \in U(R)$, the converse is not in general true. If $C_2 = \langle g \rangle$, then $\epsilon(1+g) = 2 \in U(\mathbb{R})$, where $\mathbb{R}$ is the field of real numbers, while $1 + g \notin U(\mathbb{R}C_2)$, because $0 = (1-g)(1+g)$. To ensure that the converse is true, we give the following definition.

Definition 1. Let R be a ring and let G be a group. We say that the group ring RG is a U-group ring provided that $a \in U(RG)$ if and only if $\epsilon(a) \in U(R)$.

We first investigate some basic properties of U-group rings.

Theorem 2. *The group ring RG is a U-group ring if and only if RH is a U-group ring for any (finitely generated) subgroup H of G.*

Proof. Assume that RG is a U-group ring, and H is any subgroup of G. Let $a = \sum a_i h_i \in RH$ such that $\sum a_i \in U(R)$. Since RG is a U-group ring, there exists $b \in RG$ such that $ab = 1$. But it follows from [3, Proposition 4(i)] that $b \in \langle Supp(b) \rangle \leq \langle Supp(a) \rangle \leq H$. Thus RH is a U-group ring.

For the converse, let $a = \sum a_i g_i \in RG$ such that $\sum a_i \in U(R)$. Let $H = \langle Supp(a) \rangle$. Then H is finitely generated and RH is a U-group ring and so there exists $b \in RH \subseteq RG$ such that $ab = 1$. Thus, RG is a U-group ring. $\square$

Theorem 3. *Let $R = R_1 \times R_2$ and let G be a group. Then RG is a U-group ring if and only if $R_i G$ is a U-group ring for $i = 1, 2$.*

Proof. We recall first that $(R_1 \times R_2)G \simeq R_1 G \times R_2 G$ with the isomorphism φ mapping $(a, b)g$ to (ag, bg).

Assume that RG is a U-group ring and let $\sum_{i=1}^n a_i g_i \in R_1 G$ such that $\sum_{i=1}^n a_i \in U(R_1)$. Then $(\sum_{i=1}^n a_i, 1) \in U(R)$ and so, $(a_1, 1)g_1 + (a_2, 0)g_2 + \cdots + (a_n, 0)g_n \in U(RG)$, which implies that $(\sum_{i=1}^n a_i g_i, g_1) \in U((R_1 \times R_2)G)$ and hence $\sum_{i=1}^n a_i g_i \in U(R_1 G)$.

Assume now that $R_1 G$ and $R_2 G$ are U-group rings and assume that

$$\sum_{i=1}^{n} (a_i, b_i) g_i \in RG,$$

with $\sum_{i=1}^n (a_i, b_i) = (\sum_{i=1}^n a_i, \sum_{i=1}^n b_i) \in U(R)$. Then $\sum_{i=1}^n a_i \in U(R_1)$ and $\sum_{i=1}^n b_i \in U(R_2)$. Thus we have $\sum_{i=1}^n a_i g_i \in U(R_1 G)$ and $\sum_{i=1}^n b_i g_i \in U(R_2 G)$ and hence, $\sum_{i=1}^n (a_i, b_i) g_i = \varphi^{-1}(\sum_{i=1}^n a_i g_i, \sum_{i=1}^n b_i g_i) \in U(RG)$. $\square$

Looking for examples of U-group rings, we first investigate groups containing elements of infinite order.

Theorem 4. *Let G be a group containing elements of infinite order. Then RG cannot be a U-group ring for any ring R.*

Proof. Let R be a ring and let $g \in G$ such that $|g| = \infty$.

If $2 \notin Nil(R)$, then $(1-2g)^{-1} = \sum_{k=0}^{\infty}(2g)^k \in R(\langle g\rangle) \setminus RG$, where $R(\langle g\rangle) = \{\sum_{n=r}^{\infty} a_n g^n : a_n \in R, r \in \mathbb{Z}\}$, since $(2g)^n = (2g)^m$ if and only if $n = m$. But $\epsilon(1-2g) = -1 \in U(R)$. So, RG cannot be a U-group ring.

If $2 \in Nil(R)$, then $\epsilon(1+g+g^2) = 3 = 1+2 \in 1+J(R) \subseteq U(R)$. To show that $1+g+g^2 \notin U(RG)$, note that $1+g \notin Nil(RG)$, since $|g| = \infty$ and if $g^n(1+g)^n = g^m(1+g)^m$, then

$$g^n + \binom{n}{1} g^{n+1} + \binom{n}{2} g^{n+2} + \cdots + g^{2n} = g^m + \binom{m}{1} g^{m+1} + \binom{m}{2} g^{m+2} + \cdots + g^{2m},$$

and so, by uniqueness of representation in RG, we must have $n = m$.

Thus, $(1+g+g^2)^{-1} = (1+g(1+g))^{-1} = \sum_{k=0}^{\infty}(-1)^k(g(1+g))^k \in R(\langle g\rangle) \setminus RG$. So, RG cannot be a U-group ring. □

The result of Theorem 4 restricts our investigation of U-group rings to torsion groups only.

Theorem 5. *Let R be a ring and let G be a torsion group. Then the following are equivalent*:

(1) *G is a p-group such that $p \in J(R)$.*
(2) *$\Delta(G) \subseteq J(RG)$.*
(3) *RG is a U-group ring.*

Proof. For the equivalence of (1) and (2), see [10] and for (1) ⇒ (3), see [1].

(3) ⇒ (1) Assume that G contains elements g_1 and g_2 such that $|g_1| = p$ and $|g_2| = q$, where p and q are distinct primes. Then there exist $n, m \in \mathbb{N}$ such that $1 = np + mq$. Let $a = n\hat{g}_1 + m\hat{g}_2$. Then $\epsilon(a) = np + mq = 1 \in U(R)$. But $(1-g_1)(1-g_2)a = 0$ and $(1-g_1)(1-g_2) = 1 - g_2 - g_1 + g_1g_2 \neq 0$. So, $a \notin U(RG)$ and RG is not a U-group ring.

Hence, if RG is a U-group ring, G must be a p-group.

Now, we show that $p \in J(R)$. Let $a \in R, g \in G$ such that $|g| = p$ and let $f = (1+ap) - a\hat{g}$. Then $\epsilon(f) = 1$ and there exists $h \in RG$ such that $1 = fh$. One can write $h = \sum_{i=0}^{p-1} a_i g^i$, by [3, Proposition 4(i)]. Thus we have

$$1 = (1+ap)\sum_{i=0}^{p-1} a_i g^i - a\left(\sum_{i=0}^{p-1} a_i\right)\hat{g},$$

and so,

$$1 = (1+ap)a_0 - a\left(\sum_{i=0}^{p-1} a_i\right),$$

$$0 = (1+ap)a_1 - a\left(\sum_{i=0}^{p-1} a_i\right).$$

Subtracting the two equations we get

$$1 = (1+ap)(a_0 - a_1).$$

Therefore, $\epsilon(1+ap) = (1+ap) \in U(R)$ and $p \in J(R)$. □

Using the equivalent conditions in Theorem 5, we can deduce the following proposition.

Proposition 6. *Let RG be a U-group ring. Then*

(1) [9, Page 138] *RG is a local ring if and only if R is.*
(2) [10, Theorem 2.3] *RG is a clean ring (every element is a sum of a unit and an idempotent) if and only if R is.*
(3) [1, Theorem 3.8] *RG is a présimplifiable ring (the zero-divisors are contained in the Jacobson radical) if and only if R is.*
(4) [1, Theorem 3.1 (ii)] *$a \in J(RG)$ if and only if $\epsilon(a) \in J(R)$.*
(5) [10, page 542] *$RG/J(RG) \simeq R/J(R)$.*
(6) [1, Theorem 3.1 (iv)] *If $p \in Nil(R)$, then $a \in Nil(RG)$ if and only if $\epsilon(a) \in Nil(R)$.*

Using Theorem 5 together with Theorem 2, we have the following corollary.

Corollary 7. *Let R be a ring, H and K be groups and let $G = H \times K$. Then RG is a U-group ring if and only if RH and RK are U-group rings.*

3. Some applications

In this section, we investigate some properties of well known rings, such as Hermite rings and stable range rings, in the presence of U-group rings.

3.1. Stable range of a ring

A sequence $\{a_1, a_2, \ldots, a_n\}$ in a ring R is said to be unimodular if $a_1R + a_2R + \cdots + a_nR = R$. In case $n \geq 2$, such a sequence is said to be reducible if there exist $r_1, r_2, \ldots, r_{n-1} \in R$ such that $(a_1 + r_1a_n)R + (a_2 + r_2a_n)R + \cdots + (a_{n-1} + r_{n-1}a_n)R = R$. A ring R is said to have stable range $\leq d$ if every unimodular sequence of length greater than d is reducible. The smallest such d is said to be the stable range of R. We write simply $sr(R) = d$. If no such d exists, then we say $sr(R) = \infty$, see [6].

If a ring S is a homomorphic image of a ring R, then $sr(S) \leq sr(R)$, in particular, $sr(R) \leq sr(RG)$. Note that $sr(\mathbb{Z}) = 2 < sr(\mathbb{Z}C_2)$, since $\{3, 5, g\}$ is unimodular in $\mathbb{Z}C_2$, but it is not reducible.

Now, we show that if RG is a U-group ring, then we will get equality.

Theorem 8. *Let RG be a U-group ring. Then $sr(R) = sr(RG)$.*

Proof. According to [6, Proposition 1.5], $sr(R) = sr(R/J(R))$ and since RG is a U-group ring, we have $R/J(R) \simeq R(G)/J(R(G))$. Thus, $sr(R) = sr(R/J(R)) = sr(R(G)/J(R(G))) = sr(RG)$. □

It is shown in [9, Proposition 4] that if R is a Boolean ring and G is a locally finite group, then RG is clean and hence $sr(RG) = 1$. Thus one can see that $sr(RG) = 1 = sr(R)$, although RG is not necessarily a U-group ring.

3.2. Hermite group rings

For any integer $d \geq 0$, a ring R is called d-Hermite if any unimodular row over R of length $\geq d+2$ can be completed to a square invertible matrix over R. A 0-Hermite ring is simply called Hermite. Also, since if $ax+by=1$, then the matrix $\left[\begin{smallmatrix} a & b \\ -y & x \end{smallmatrix}\right]$ is invertible, so 1-Hermite is still synonymous with Hermite, see [6].

It is shown in [8] that if $sr(R)=d$, then R is a d-Hermite ring. However, the converse is not in general true, since $sr(\mathbb{Z}) \neq 1$ because $2\mathbb{Z}+5\mathbb{Z}=\mathbb{Z}$, but $(2+5r)\mathbb{Z} \neq \mathbb{Z}$, for any $r \in \mathbb{Z}$, while $\mathbb{Z}$ is Hermite.

Theorem 9. *Let R be a ring and let G be a group. If RG is a d-Hermite ring, then so is R.*

Proof. Assume that RG is a d-Hermite ring and let $m \geq d+2$, $(a_1, a_2, \ldots, a_m)$, $(r_1, r_2, \ldots, r_m) \in R^m$ such that $\sum_{i=1}^{m} a_i r_i = 1$. Since RG is a d-Hermite ring, there exists an $m \times m$ matrix M over RG with first row $(a_1, a_2, \ldots, a_m)$ and $det(M) \in U(RG)$.

$$\text{Assume } M = \begin{bmatrix} a_1 & a_2 & \cdots & a_m \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{bmatrix}.$$

Then $\det(M) = \sum_{\sigma \in S_m} (sgn\sigma) a_{\sigma 1} a_{2\sigma 2} \cdots a_{m\sigma m} \in U(RG)$. Since $a_{kj} \in R(G)$ for $2 \leq k \leq m$, $1 \leq j \leq m$, we have $a_{kj} = \sum_{g \in G} a_{kj,g} g$ where $a_{kj,g} \in R$. So, $T = \epsilon(\det(M)) = \sum_{\sigma \in S_n} (sgn\sigma) a_{\sigma 1} (\sum_{g \in G} a_{2\sigma 2,g}) \cdots (\sum_{g \in G} a_{m\sigma m,g}) \in U(R)$.

$$\text{Now, consider } L = \begin{bmatrix} a_1 & a_2 & \cdots & a_m \\ \sum_{g \in G} a_{21,g} & \sum_{g \in G} a_{22,g} & \cdots & \sum_{g \in G} a_{2m,g} \\ \vdots & \vdots & & \vdots \\ \sum_{g \in G} a_{m1,g} & \sum_{g \in G} a_{m2,g} & \cdots & \sum_{g \in G} a_{mm,g} \end{bmatrix}.$$

Then, L is an $m \times m$ matrix defined over R with $\det(L) = T \in U(R)$. Hence R is a d-Hermite ring. □

We don't know yet if the converse of Theorem 9 is true, but to have a partial answer, we need to add an extra condition.

Theorem 10. *Let RG be a U-group ring. Then R is a d-Hermite ring if and only if RG is.*

Proof. Suppose that R is a d-Hermite ring. Let $r_k = \sum_{g \in G} r_{kg} g$, $a_k = \sum_{g \in G} a_{kg} g \in RG$ for $1 \leq k \leq m$ and $m \geq d+2$. If $\sum_{i=1}^{m} r_k a_k = 1$, then $1 = \epsilon(\sum_{i=1}^{m} r_k a_k) = \sum_{i=1}^{m} (\sum_{g \in G} r_{kg} \sum_{g \in G} a_{kg})$. Then there exists an $m \times m$

matrix M over R such that

$$M = \begin{bmatrix} \sum_{g\in G} a_{1g} & \sum_{g\in G} a_{2g} & \cdots & \sum_{g\in G} a_{mg} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mm} \\ \end{bmatrix},$$

and $\det(M) = \sum_{\sigma\in S_m} (sgn\sigma)(\sum_{g\in G} a_{\sigma ig})b_{2\sigma 2}\cdots b_{m\sigma m} \in U(R)$.

By assumption we would have

$$S = \sum_{\sigma\in S_n} (sgn\sigma)(\sum_{g\in G} a_{\sigma ig}g)b_{2\sigma 2}\cdots b_{m\sigma m} \in U(RG).$$

Now, let

$$B = \begin{bmatrix} \sum_{g\in G} a_{1g}g & \sum_{g\in G} a_{2g}g & \cdots & \sum_{g\in G} a_{mg}g \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mm} \\ \end{bmatrix}.$$

Then $\det(B) = S \in U(RG)$ and RG is a d-Hermite ring.

The other implication follows from Theorem 9. □

A ring R is called semilocal if R has only finitely many maximal ideals. Fields and Artinian rings are examples of semilocal rings. In [7] it is proved that a semilocal ring is Hermite. In [2] it is proved that a group ring RG of a semilocal ring R and a finite group G is semilocal. So, the ring $\mathbb{R}C_7$ is Hermite even though $7 \notin J(\mathbb{R})$. This gives an example of a Hermite group ring RG, which is not a U-group ring. Moreover, the torsionness of the group is not a necessary condition as was proved in [5] that if R is semilocal with $J(R)$ is nil, then $R\mathbb{Z}$ is Hermite. In fact it is proved in [4] that if G is a finitely generated Abelian group, then $\mathbb{Z}G$ is a Hermite ring, although $\mathbb{Z}G$ is not a U-group ring for any group G.

Recall that an R-module M is a stably free module if there exist $m, n \in \mathbb{N}$ such that $M \oplus R^m = R^n$. If M is finitely generated, then we set $\text{rank}(M) = n - m$. Clearly any free module is stably free. Since R is a d-Hermite ring if and only if every finitely generated stably free R-module of rank $\geq d$ is free, see [7], we get the following:

Corollary 11. *Let RG be a U-group ring. Then every finitely generated stably free R-module of* rank $\geq d$ *is free if and only if every finitely generated stably free RG-module of* rank $\geq d$ *is free.*

3.3. Divisors of $\binom{n}{k}$

It is well known that if p is a prime integer and $n \in \mathbb{N}$, then $p \mid \binom{p^n}{k}$ for all $0 < k < p^n$. Now, if n is an integer divisible by more than one prime, is there a common divisor of $\binom{n}{k}$ for all $0 < k < n$? We first give the following simple lemma.

Lemma 12. *Let n be an integer greater than 1 and let $0 < k < n$. Then* $\gcd(n, \binom{n}{k}) \neq 1$.

Proof. It is clear that $n\binom{n-1}{k-1} = k\binom{n}{k}$ and so, $1 \neq \gcd(n, \frac{n}{\gcd(n,k)}\binom{n-1}{k-1}) = \gcd(n, \frac{k}{\gcd(n,k)}\binom{n}{k}) = \gcd(n, \binom{n}{k})$. □

Theorem 13. *Let n be a positive integer divisible by at least two distinct primes. Then the integers $\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1}$ are relatively prime.*

Proof. Assume that there exists a prime p such that $p \mid \binom{n}{i}$ for all $0 < i < n$.

Let $a = \sum_{i=1}^{k} a_i g_i \in \mathbb{Z}_p C_n$ such that $\sum_{i=1}^{k} a_i \in U(\mathbb{Z}_p)$. Then $a^n = \sum_{i=1}^{k} a_i^n g_i^n + \sum_{i=1}^{n-1} \binom{n}{i} y_i = \sum_{i=1}^{k} a_i^n$, where $y_i \in \mathbb{Z}_p C_n$ for each i. Applying the augmentation homomorphism we get:

$$\left(\sum_{i=1}^{k} a_i\right)^n = \sum_{i=1}^{k} a_i^n,$$

and so we have:

$$a^n = \sum_{i=1}^{k} a_i^n = \left(\sum_{i=1}^{k} a_i\right)^n \in U(\mathbb{Z}_p) \subseteq U(\mathbb{Z}_p C_n).$$

Hence we have $a \in U(\mathbb{Z}_p C_n)$ and $\mathbb{Z}_p C_n$ is a U-group ring, contradicting Theorem 5, since C_n is not a p-group. Thus the integers $\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1}$ are relatively prime. □

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Emad Abu Osba
Department of Mathematics
School of Science
The University of Jordan
Amman 11942, Jordan
Email address: eabuosba@ju.edu.jo

Hasan Al-Ezeh
Department of Mathematics
School of Science
The University of Jordan
Amman 11942, Jordan
Email address: alezehh@ju.edu.jo

Manal Ghanem
Department of Mathematics
School of Science
The University of Jordan
Amman 11942, Jordan
Email address: m.ghanem@ju.edu.jo