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ON CONSTRUCTIBLE SETS AND DIMENSION OF FIBERS

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Abstract. In this paper, we compare the dimension of a constructible set with the dimension of its inverse image under morphisms of finite type of noetherian Jacobson schemes.

1. Introduction and the main results

The image of a morphism of algebraic varieties is, in general, neither closed nor open. However, by a well-known theorem of Chevalley, it is always a constructible set. In fact, Chevalley proved that the image of any constructible set under a morphism of finite type of noetherian schemes is constructible ([2, Théorèm 3]; see also [4, Theorem 10.20]).

Besides this fact, constructible sets also occur naturally in algebraic geometry, for instance, as loci of various local properties of schemes and morphisms. There is a long list of constructible sets arisen in this way in [4], which gathers results on constructibility from [5]. For more achievements in this direction, we refer to [8].

In connection with Chevalley's theorem on constructibility, the following fundamental theorem on dimension of fibers of morphisms, also due to Chevalley, is crucial in computing dimension of fibers and schemes. We cite here a general version of this result [4, Corollary 14.116] (see [6, Lecture 11] for numerous applications of the theorem for quasi-projective varieties).

Theorem 1.1 (Chevalley)**.** *Let $\varphi : X \to Y$ be a dominant morphism of finite type of irreducible schemes. Assume that Y is noetherian, universally catenary and of finite dimension. Then*:

(i) *There exists an open dense subset U of Y such that $U \subseteq \varphi(X)$ and $\dim \varphi^{-1}(y) = \dim X - \dim U$ for all $y \in U$.*
(ii) *For all $y \in \varphi(X)$, $\dim \varphi^{-1}(y) \geq \dim X - \dim U$.*

Note that by assumptions, the scheme X is also noetherian, universally catenary and of finite dimension.

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Because of the importance of the above results, it is natural and practical to study the behaviour of the dimension of constructible sets under morphisms of finite type of schemes. In fact, the motivation of this paper originated from several contexts of algebraic groups acting on varieties or schemes. We state here our main results, which show that the dimension behaves well as expected. We also note that they extend known results, which traditionally apply only for algebraic varieties or schemes over a field (compare, for instance, [4, Corollaries 14.118 and 14.119]).

Theorem 1.2. *Let $f : X \to Y$ be a morphism of finite type of noetherian Jacobson, universally catenary and finite-dimensional schemes. Let c be a non-negative integer and assume Y' is a constructible subset of Y such that $\dim f^{-1}(y) \leq c$ for all $y \in Y'$. Then $\dim f^{-1}(Y') \leq c + \dim Y'$.*

Theorem 1.3. *Let $f : X \to Y$ be a morphism of finite type of noetherian Jacobson, universally catenary and finite-dimensional schemes. Let c be a non-negative integer and assume Y' is a constructible subset of Y such that $\dim f^{-1}(y) \geq c$ for all non-empty fibers $f^{-1}(y)$, $y \in Y'$, and $Y' \subseteq \overline{f(f^{-1}(Y'))}$ (equivalently, $\overline{Y'} = \overline{f(f^{-1}(Y'))}$). Then $\dim f^{-1}(Y') \geq c + \dim Y'$.*

Remark 1.4. The assumptions of Theorem 1.3 are satisfied if Y' is a constructible subset of Y such that $\dim f^{-1}(y) \geq c$ for all $y \in Y'$. It is because in this case $f : f^{-1}(Y') \to Y'$ is a surjective map and thus $f(f^{-1}(Y')) = Y'$. Moreover, the condition $Y' \subseteq \overline{f(f^{-1}(Y'))}$ is essential; otherwise, let f be a constant map and $Y' = Y$, then $f^{-1}(Y') = X$ and obviously the inequality $\dim X \geq c + \dim Y$ would not hold in general.

Remark 1.5. Theorem 1.3 does not hold if the assumption on Jacobson property is removed. Indeed, let A be the localization of the ring of integers at the prime ideal (2). Then A is a one-dimensional non-Jacobson integral domain, whose fraction field is $\mathbb{Q} = A\left[\frac{1}{2}\right]$, the field of rational numbers. The inclusion map $A \to \mathbb{Q}$ corresponds to the morphism $f : X = \mathrm{Spec}(\mathbb{Q}) \to \mathrm{Spec}(A) = Y$ which sends the unique point of X to the generic point (0) of Y. Now let $Y' = Y$, then $f^{-1}(Y') = X$ has dimension $c = 0$. In this case, the conclusion of Theorem 1.3 does not follow.

Combining Theorem 1.2 and Theorem 1.3, we obtain immediately the following consequences.

Corollary 1.6. *Let $f : X \to Y$ be a morphism of finite type of noetherian Jacobson, universally catenary and finite-dimensional schemes. Let c be a non-negative integer and assume Y' is a constructible subset of Y such that $\dim f^{-1}(y) = c$ for all non-empty fibers $f^{-1}(y)$, $y \in Y'$, and $Y' \subseteq \overline{f(f^{-1}(Y'))}$. Then $\dim f^{-1}(Y') = c + \dim Y'$.*

Corollary 1.7. *Let $f : X \to Y$ be a morphism of finite type of noetherian Jacobson, universally catenary and finite-dimensional schemes. Let c be a* *non-negative integer and assume* Y' *is a constructible subset of* Y *such that* $\dim f^{-1}(y) = c$ *for all* $y \in Y'$*. Then* $\dim f^{-1}(Y') = c + \dim Y'$*.*

This paper is organized as follows. In Section 2 we briefly recall the definitions of constructible sets and Jacobson schemes, and then prove a result of H. Schoutens (Proposition 2.1) which allows to reduce Theorem 1.1 to Proposition 2.2 in the case of Jacobson schemes. The proofs of Theorem 1.2 and Theorem 1.3 are presented in Section 3. The key idea is to apply Proposition 2.2 to the closures of irreducible components of the constructible sets in question. We conclude this paper with two examples illustrating our results in Section 4.

Throughout this paper we use a standard notation: if C is a subset of a topological space X, then $\overline{C}$ denotes the closure of C in the ambient space X.

2. Constructible sets and Jacobson schemes

Let X be a topological space. A locally closed subset of X is a subset which is open in its closure in X, or equivalently, it is the intersection of an open subset and a closed subset of X. A constructible subset of X is a finite union of locally closed subsets. It is clear that open subsets and closed subsets, as well as locally closed subsets, are constructible. Moreover, constructibility is closed under finite unions and intersections, under complements, and under inverse images via continuous maps.

Recall that the dimension of the space X, denoted by $\dim X$, is the supremum of the lengths of chains consisting of irreducible closed subsets of X. If X is a noetherian space, then it has only finitely many irreducible components and $\dim X$ is the maximum of the dimensions of the irreducible components. Now if C is a constructible subset of a noetherian space X, then C itself with the induced topology is a noetherian space and hence we may talk about its dimension and irreducible components.

The topological space X is called a Jacobson space if every non-empty locally closed subset of X contains a closed point, or equivalently, if the set of closed points is dense in every non-empty closed subset of X (see [5, Section 10.3]). A Jacobson scheme is a scheme whose underlying topological space is Jacobson. Examples of Jacobson schemes are schemes locally of finite type over a Jacobson ring A, such as A is a field or A is the ring of integers [5, Corollaire 10.4.7]. Recall that a commutative ring A is a Jacobson ring if every prime ideal of A is the intersection of the maximal ideals containing it.

On the page 1063 of [8], the following result is asserted (compare [5, Corollaire 10.6.2]).

Proposition 2.1 ([8])**.** *Let* X *be a noetherian scheme. Then the following are equivalent*:

(i) X *is Jacobson*;
(ii) *Every closed subset of* X *of dimension* $d > 0$ *contains infinitely many irreducible closed subsets of dimension* $d - 1$;

(iii) *Every locally closed subset of X has the same dimension as its closure*;
(iv) *Every constructible subset of X has the same dimension as its closure.*

However, to the best of our knowledge, the reference for the proof given therein has not been published yet. For completeness, we give a proof of the proposition.

Proof. (i) $\Rightarrow$ (ii): Let Z be a closed subset of X of dimension $d \geq 1$. Replacing Z by its irreducible component of maximal dimension if necessary, we may assume that Z is irreducible. Since Z can be covered by affine open subsets, we may assume further that $Z = \operatorname{Spec}(R)$ for a Jacobson noetherian domain R. In the case $d = 1$, the Jacobson property implies that Z contains infinitely many closed points. For $d \geq 2$, let

$$0 = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \cdots \subsetneq \mathfrak{p}_d$$

be a maximal chain of prime ideals of R. Then, by a property of prime ideals in noetherian rings (see, for instance, [7, Theorem 144]), there are infinitely many prime ideals $\mathfrak{q}$ of R such that $\mathfrak{p}_0 \subsetneq \mathfrak{q} \subsetneq \mathfrak{p}_2$. Such ideals $\mathfrak{q}$ give rise to infinitely many irreducible closed subsets $\operatorname{Spec}(R/\mathfrak{q})$ of Z of dimension $d-1$.

(ii) $\Rightarrow$ (iii): We first show that $\dim Z = \dim \overline{Z}$, where Z is an irreducible locally closed subset of X and $\dim \overline{Z} = d > 0$. We proceed by induction on d. Assume that $\overline{Z} \backslash Z$ is nonempty; then it is a closed subset of $\overline{Z}$ and $\dim \overline{Z} \backslash Z < \dim \overline{Z} = d$. By (ii), $\overline{Z}$ contains infinitely many irreducible closed subsets of dimension $d-1$. Observe that at most finitely many of these irreducible closed subsets are contained in $\overline{Z} \backslash Z$, as $\overline{Z} \backslash Z$ has only finitely many irreducible components of dimension less than d. Thus there exists an irreducible closed subset F of $\overline{Z}$ of dimension $d-1$ such that $F \cap Z \neq \emptyset$. Note that $F \cap Z$ is an open dense subset of F. Applying induction hypothesis to the irreducible locally closed subset $F \cap Z$, we obtain a chain of irreducible closed subsets of length $d-1$ in $F \cap Z$:

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_{d-1} = F \cap Z.$$

Thus

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_{d-1} \subsetneq Z$$

is a chain of length d in Z. It follows that $\dim Z \geq d = \dim \overline{Z}$, hence $\dim Z = \dim \overline{Z}$.

Now let V be a locally closed subset of X and let $V_1, \ldots, V_n$ be the irreducible components of V. Then $\overline{V_1}, \ldots, \overline{V_n}$ are the irreducible components of $\overline{V}$. Moreover, $\dim V_i = \dim \overline{V_i}$ by the previous claim, as V_i is open in $\overline{V_i}$ $(1 \leq i \leq n)$. Thus we get

$$\dim V = \max_{1 \leq i \leq n} \dim V_i = \max_{1 \leq i \leq n} \dim \overline{V_i} = \dim \overline{V}.$$

(iii) $\Rightarrow$ (iv): Let $C = C_1 \cup \cdots \cup C_r$ be a constructible subset of X, where $C_1, \ldots, C_r$ are locally closed subsets. Then $\overline{C} = \overline{C_1} \cup \cdots \cup \overline{C_r}$ (this need not be a decomposition of $\overline{C}$ into irreducible components). Using (iii), we obtain

$$\dim \overline{C} = \max_{1\leq j\leq r} \dim \overline{C_j} = \max_{1\leq j\leq r} \dim C_j \leq \dim C.$$

The opposite inequality $\dim C \leq \dim \overline{C}$ follows as $C \subseteq \overline{C}$.

(iv) ⇒ (i): Let Z be a non-empty locally closed subset of X of dimension d. We need to show that Z contains a closed point. Suppose this is not the case, and let

$$Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_d$$

be a maximal chain of irreducible closed subsets in Z. Then

$$\overline{Z_0} \subsetneq \overline{Z_1} \subsetneq \cdots \subsetneq \overline{Z_d}$$

be a chain of length d in $\overline{Z}$. By our assumption, $\overline{Z_0}$ is not a singleton and since $\overline{Z_0}$ is a noetherian scheme (with the reduced subscheme structure), $\overline{Z_0}$ contains a closed point. It follows that $\dim \overline{Z} \geq d+1 > d = \dim Z$, which contradicts (iv). This finishes the proof of the proposition. □

Combining Theorem 1.1 and Proposition 2.1, we obtain the following useful result.

Proposition 2.2. *Let $\varphi : X \to Y$ be a dominant morphism of finite type of irreducible schemes. Assume that Y is noetherian Jacobson, universally catenary and of finite dimension. Then:*

(i) *There exists an open dense subset U of Y such that $U \subseteq \varphi(X)$ and $\dim \varphi^{-1}(y) = \dim X - \dim Y$ for all $y \in U$.*
(ii) *For all $y \in \varphi(X)$, $\dim \varphi^{-1}(y) \geq \dim X - \dim Y$.*

We note that Proposition 2.2 does not hold without the Jacobson property; see the counterexample in Remark 1.5.

We also need the following well-known result. For a proof, we refer to [1, Chapter AG, Proposition 1.3].

Lemma 2.3. *Let C be a constructible subset of a noetherian space X. Then C contains an open dense subset of $\overline{C}$.*

3. Proofs of the main results

Proof of Theorem 1.2. Let $Y_1, \ldots, Y_n$ be the irreducible components of Y'. Observe that Y_i for all $1 \leq i \leq n$ are constructible subsets of Y. Then $f^{-1}(Y') = \bigcup_{i=1}^n f^{-1}(Y_i)$, where the sets $f^{-1}(Y_i)$ are closed in $f^{-1}(Y')$. We further decompose each non-empty constructible set $f^{-1}(Y_i)$ into irreducible components

$$f^{-1}(Y_i) = \bigcup_{j=1}^{r_i} X_{ij}.$$

Since $f(X_{ij}) \subseteq Y_i$, it follows that $f(\overline{X_{ij}}) \subseteq \overline{f(X_{ij})} \subseteq \overline{Y_i}$. Thus $\overline{f(\overline{X_{ij}})} \subseteq \overline{Y_i}$. The restriction of f to $\overline{X_{ij}}$ induces a dominant morphism

$$g : \overline{X_{ij}} \to \overline{f(\overline{X_{ij}})}$$

of irreducible schemes, endowed with the corresponding reduced subscheme structures.

By Proposition 2.2, we have

$$\dim \overline{X_{ij}} \leq \dim g^{-1}(y) + \dim \overline{f(\overline{X_{ij}})}$$

for all $y \in f(\overline{X_{ij}})$. By assumptions, $\dim g^{-1}(y) \leq \dim f^{-1}(y) \leq c$ for all $y \in f(X_{ij})$. Thus

$$\dim \overline{X_{ij}} \leq c + \dim \overline{f(\overline{X_{ij}})} \leq c + \dim \overline{Y_i}.$$

In view of Proposition 2.1, this implies that

$$\dim X_{ij} \leq c + \dim Y_i \leq c + \dim Y'.$$

Since X_{ij} are the irreducible components of $f^{-1}(Y_i)$, we obtain

$$\dim f^{-1}(Y_i) \leq c + \dim Y'.$$

Hence $\dim f^{-1}(Y') \leq c + \dim Y'$, as $\bigcup_i f^{-1}(Y_i)$ is a closed covering of $f^{-1}(Y')$. □

Proof of Theorem 1.3. Let

$$Y' = \bigcup_{i=1}^{n} Y_i$$

and

$$f^{-1}(Y_i) = \bigcup_{j=1}^{r_i} X_{ij}, \quad 1 \leq i \leq n,$$

be the decompositions of Y' and $f^{-1}(Y_i)$ into irreducible components, respectively. Then

$$f^{-1}(Y') = \bigcup_{i=1}^{n} f^{-1}(Y_i) = \bigcup_{i=1}^{n} \bigcup_{j=1}^{r_i} X_{ij}.$$

By assumptions, we get

$$\overline{Y'} = \overline{f(f^{-1}(Y'))} = \bigcup_i \bigcup_j \overline{f(X_{ij})}.$$

Choose an index i such that $\dim Y' = \dim Y_i$. The irreducible set $\overline{Y_i}$ is contained in $\bigcup_{i,j} \overline{f(X_{ij})}$, thus $\overline{Y_i} \subseteq \overline{f(X_{kl})}$ for some $1 \leq k \leq n$ and $1 \leq l \leq r_k$. On the other hand, we have $\overline{f(X_{kl})} \subseteq \overline{Y_k}$. Since $\overline{Y_1}, \ldots, \overline{Y_n}$ are the irreducible components of $\overline{Y'}$, we conclude that $k = i$ and so $\overline{f(X_{il})} = \overline{Y_i}$ for some $1 \leq l \leq r_i$. As a consequence, we obtain $\overline{f(\overline{X_{il}})} = \overline{Y_i}$.

For each $1 \leq j \leq r_i$, the restriction of f to $\overline{X_{ij}}$ induces a morphism

$$f_j : \overline{X_{ij}} \to \overline{Y}_i$$

of irreducible schemes, provided with the reduced subscheme structures.

The above observation shows that at least one map f_j is dominant. Moreover, for the purpose of calculating dimension of fibers, we may assume that all the maps f_j are dominant. Indeed, for all those f_j that are not dominant, the sets $V_j = \overline{Y_i} \backslash \overline{f_j(\overline{X_{ij}})}$ are open and dense in $\overline{Y_i}$. On the other hand, for each map f_j that is dominant, $f_j(X_{ij}) = f(X_{ij})$ is a constructible subset of Y. Combining Proposition 2.2 and Lemma 2.3, it follows that $f_j(X_{ij})$ contains an open dense subset U_j of its closure $\overline{Y_i}$, such that every fiber over U_j has the least dimension as possible. Now intersecting the U_j's with the V_j's, we obtain an open dense subset U of $\overline{Y_i}$ such that $U \subseteq Y_i$ and for all $y \in U$, the fibers $f_j^{-1}(y)$ are empty for all the non-dominant maps f_j but $f^{-1}(y)$ itself is non-empty. The latter implies, by assumptions, that $\dim f^{-1}(y) \geq c$.

Since Y is a Jacobson scheme, U contains a closed point z of Y. Then the sets $f_j^{-1}(z)$ form a closed covering of $f^{-1}(z)$. Hence $\dim f^{-1}(z) = \dim f_j^{-1}(z)$ for some j. Applying Proposition 2.2 to the dominant morphism f_j and by the construction of U, we obtain

$$\dim \overline{X_{ij}} = \dim f_j^{-1}(z) + \dim \overline{Y_i}.$$

Thus

$$\dim X_{ij} = \dim f^{-1}(z) + \dim Y_i,$$

by Proposition 2.1. It follows that

$$\dim f^{-1}(Y') \geq \dim f^{-1}(Y_i) \geq \dim X_{ij} \geq c + \dim Y_i = c + \dim Y'.$$

Therefore $\dim f^{-1}(Y') \geq c + \dim Y'$, as required. □

4. Examples

We illustrate the obtained results with two examples.

Example 4.1. Let k be a field. Consider the morphism $f : k^3 \to k^2$ given by

$$f(x, y, z) = (xy, xz).$$

Let

$$C = \{(u, v) \in k^2 \mid u = 0 \Rightarrow v = 0\} = k^2 \backslash \{v\text{-axis}\} \cup \{(0, 0)\},$$

then C is a constructible subset of k^2. Observe that $\dim C = 2$, as $\overline{C} = k^2$. Now for any $(u, v) \in C$ with $u \neq 0$, we have $f^{-1}(u, v) = \{(x, x^{-1}u, x^{-1}v) \mid x \neq 0\}$, which is a one-dimensional variety. On the other hand, the fiber over the origin is easily seen to be of dimension 2. Thus all the fibers of f over C are of dimension at least 1. Applying Theorem 1.3, we get

$$\dim f^{-1}(C) \geq 1 + \dim C = 3, \text{ hence } \dim f^{-1}(C) = 3.$$

As a consequence, we obtain $\overline{f^{-1}(C)} = k^3$.

Example 4.2. This is a generalized version of a proposition in [3]. Let G be a connected algebraic group acting on an algebraic variety X (we say that X is a G-variety) and Y be a G-stable constructible subset of X. For each non-negative integer s, we define

$$Y_{(s)} = \{y \in Y \mid \dim Gy = s\}.$$

Here $Gy = \{gy \mid g \in G\}$ is the orbit of y with respect to the action of G. It follows from Chevalley's theorem on upper semi-continuity of fiber dimensions that $Y_{(s)}$ is constructible (see, for instance, [3]); moreover, it is also G-stable. Then the number of parameters of G on Y is defined to be

$$\mu_G(Y) = \max_s \{\dim Y_{(s)} - s\}.$$

Now let $f : Z \to X$ be a G-equivariant morphism between G-varieties. Assume that the inverse image of each G-orbit in Y has dimension at most d. We claim that $\dim f^{-1}(Y) \leq d + \mu_G(Y)$.

Indeed, for any $y \in Y_{(s)}$ such that $f^{-1}(Gy) \neq \emptyset$, the fibers of the morphism $f^{-1}(Gy) \to Gy$ induced by f are all isomorphic, as f is G-equivariant. Thus they have the same dimension. By Corollary 1.7, this dimension equals

$$\dim f^{-1}(Gy) - \dim Gy,$$

which is at most $d - s$ by assumptions. Now applying Theorem 1.2 to each constructible subset $Y_{(s)}$ of X, we obtain

$$\dim \overline{f^{-1}(Y_{(s)})} = \dim f^{-1}(Y_{(s)}) \leq d - s + \dim Y_{(s)} \leq d + \mu_G(Y).$$

Since $f^{-1}(Y) = \bigcup_s f^{-1}(Y_{(s)})$, taking closure of both sides yields

$$\dim f^{-1}(Y) = \dim \overline{f^{-1}(Y)} = \max_s \{\dim \overline{f^{-1}(Y_{(s)})}\}.$$

Therefore

$$\dim f^{-1}(Y) \leq d + \mu_G(Y),$$

as claimed.

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