Commun. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/CKMS.c170379
pISSN: 1225-1763 / eISSN: 2234-3024

AN ALTERNATIVE q-ANALOGUE OF THE RUCIŃSKI-VOIGT NUMBERS

Wardah M. Bent-Usman, Amerah M. Dibagulun, Mahid M. Mangontarum, and Charles B. Montero

Abstract. In this paper, we define an alternative q-analogue of the Ruciński-Voigt numbers. We obtain fundamental combinatorial properties such as recurrence relations, generating functions and explicit formulas which are shown to be q-deformations of similar properties for the Ruciński-Voigt numbers, and are generalizations of the results obtained by other authors. A combinatorial interpretation in the context of A-tableaux is also given where convolution-type identities are consequently obtained. Lastly, we establish the matrix decompositions of the Ruciński-Voigt and the q-Ruciński-Voigt numbers.

1. Introduction

Ruciński and Voigt [31] defined the numbers $S_k^n(\mathbf{a})$ as coefficients in the expansion of the relation

$$x^n = \sum_{k=0}^{n} S_k^n(\mathbf{a}) P_k^{\mathbf{a}}(x), \tag{1}$$

where $\mathbf{a} = (a, a+r, a+2r, a+3r, \ldots)$ and

$$P_k^{\mathbf{a}}(x) = (x-a)(x-(a+r))(x-(a+2r))\cdots(x-(a+(k-1)r)). \tag{2}$$

These numbers, often referred to as the "Ruciński-Voigt numbers" (see [14]), are also known to satisfy the following combinatorial properties (cf. [14, 31]):

- triangular recurrence relation

$$S_k^{n+1}(\mathbf{a}) = S_{k-1}^n(\mathbf{a}) + (kr+a) S_k^n(\mathbf{a}), \tag{3}$$

- exponential generating function

$$\sum_{n=k}^{\infty} S_k^n(\mathbf{a}) \frac{x^n}{n!} = \frac{1}{r^k k!} e^{ax} (e^{rx}-1)^k, \tag{4}$$

Received September 10, 2017; Accepted January 10, 2018.
2010 *Mathematics Subject Classification.* Primary 11B83, 11B73, 05A30, 15B36.
Key words and phrases. Stirling number, Ruciński-Voigt number, Whitney number, q-analogue, A-tabluea, matrix decomposition.
This research did not receive any financial assistance from any funding agency.

- rational generating function

$$\sum_{n=0}^{\infty} S_k^n(\mathbf{a})x^n = \frac{x^k}{\prod_{j=0}^{k}(1-(rj+a)x)}, \tag{5}$$

- explicit formulas

$$S_k^n(\mathbf{a}) = \frac{1}{r^k k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}(rj+a)^n, \tag{6}$$

$$S_k^n(\mathbf{a}) = \sum_{c_0+c_1+\cdots+c_k=n-k}\prod_{j=0}^{k}(rj+a)^{c_j}. \tag{7}$$

Evidently, the well-known Stirling numbers of the second kind [9, 32], denoted by $S(n,k)$, can be related to the Ruciński-Voigt numbers as follows:

$$S_k^n(\mathbf{m}) = S(n,k), \tag{8}$$

where $\mathbf{m} = (0,1,2,3,\dots)$ is the sequence of nonnegative integers. It can also be shown that several known generalizations of $S(n,k)$ are particular cases of the Ruciński-Voigt numbers. To be precise, we have

(i) the r-Stirling numbers of the second kind $\left\{ {n+r \atop k+r} \right\}_r$ of Broder [5] are given by

$$S_k^n(\mathbf{a}_1) = \left\{ {n+r \atop k+r} \right\}_r,$$

where $\mathbf{a}_1 = (r, r+1, r+2, r+3, \dots)$;

(ii) the Whitney numbers of the second kind of Dowling Lattices $W_m(n,k)$ of Benoumhani [3, 4] are given by

$$S_k^n(\mathbf{a}_2) = W_m(n,k),$$

where $\mathbf{a}_2 = (1, 1+m, 1+2m, 1+3m, \dots)$;

(iii) the noncentral Stirling numbers of the second kind $S_a(n,k)$ of Koutras' [21] (or Carlitz' [7] weighted Stirling numbers of the second kind) are given by

$$S_k^n(\mathbf{a}_3) = S_a(n,k),$$

where $\mathbf{a}_3 = (-a, -a+1, -a+2, -a+3, \dots)$; and

(iv) the translated Whitney numbers of the second kind $\widetilde{W}_{(\alpha)}(n,k)$ first defined by Belbachir and Bousbaa [2] and extensively studied in [25, 27] are given by

$$S_k^n(\mathbf{a}_4) = \widetilde{W}_{(\alpha)}(n,k),$$

where $\mathbf{a}_4 = (0, \alpha, 2\alpha, 3\alpha, \dots)$.

It is important to note that the Ruciński-Voigt numbers can be shown to be equivalent to the numbers defined by Corcino [10], Mező [29], and Mangontarum et al. [24]. That is,

$$S_k^n(\mathbf{c}) = \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle_{r,\beta}, \; S_k^n(\mathbf{d}) = W_{m,r}(n,k), \; S_k^n(\mathbf{e}) = \widetilde{W}_{a,m}(n,k),$$

where $\langle {n \atop k} \rangle_{r,\beta}$, $W_{m,r}(n,k)$ and $\widetilde{W}_{a,m}(n,k)$ denote the (r,β)-Stirling numbers, r-Whitney and noncentral Whitney numbers of the second kinds, respectively, and $\mathbf{c} = (r, r+\beta, r+2\beta, r+3\beta, \ldots)$, $\mathbf{d} = (r, r+m, r+2m, r+3m, \ldots)$, and $\mathbf{e} = (-a, -a+m, -a+2m, -a+3m, \ldots)$.

The study of q-analogues of classical identities has been popular to a number of mathematicians. This is, perhaps, due to its applications to diverse fields. For the case of special sequences, it can be traced back to the works of Carlitz [6] and Gould [17] on the q-analogues of the classical Stirling numbers, where q-deformations of fundamental combinatorial properties were obtained. Cigler [8], on the other hand, defined another q-analogue of the Stirling numbers using the concept of set partitions. Motivated by this, a q-analogue of the r-Stirling numbers was done by Corcino and Fernandez [12] using combinatorial interpretations in terms of set partitions. The q-analogue of the translated Whitney numbers was defined by Mangontarum et al. [23] by modification of the horizontal generating function seen in [2]. Also, distinct q-analogues of the multiparameter noncentral Stirling numbers were done by El-Desouky et al. [16] and Corcino and Mangontarum [13].

In an earlier paper, Corcino and Montero [14] defined a q-analogue for the Ruciński-Voigt numbers, denoted by $\sigma[n,k]_q^{\beta,r}$, via recurrence relation

$$\sigma[n,k]_q^{\beta,r} = \sigma[n-1,k-1]_q^{\beta,r} + ([k\beta]_q + [r]_q)\,\sigma[n-1,k-1]_q^{\beta,r}. \tag{9}$$

The said q-analogue is known to satisfy the explicit formula [14, Theorem 3.2]

$$\sigma[n,k]_q^{\beta,r} = \frac{1}{[k]_{q^\beta}![\beta]_q^k} \sum_{j=0}^{k} (-1)^{k-j} q^{\beta\left(\binom{k-j}{2} - \binom{k}{2}\right)} \binom{k}{j}_{q^\beta} ([j\beta]_q + [r]_q)^n, \tag{10}$$

where

$$\binom{n}{k}_q = \prod_{j=1}^{k} \frac{q^{n-j+1}-1}{q^j - 1} = \frac{[n]_q!}{[k]_q![n-k]_q!} \tag{11}$$

is the q-binomial coefficient, $[n]_q! = \prod_{i=1}^{n} [i]_q$ is the q-factorial of n and $[n]_q = \frac{q^n-1}{q-1}$ is the q-integer (n and k are nonnegative integers). On the other hand, a q-analogue of the r-Whitney numbers of the second kind, denoted by $W_{m,r,q}(n,k)$, was introduced by Mangontarum and Katriel [26] as coefficients in

$$(ma^\dagger a + r)^n = \sum_{k=0}^{n} m^k W_{m,r,q}(n,k) (a^\dagger)^k a^k, \tag{12}$$

where $a^\dagger$ and a are the q-Boson operators [1] satisfying the commutation relation

$$[a, a^\dagger]_q \equiv aa^\dagger - qa^\dagger a = 1. \tag{13}$$

By comparing (10) with the explicit formula of $W_{m,r,q}(n,k)$ [26, Theorem 16] given by

$$W_{m,r,q}(n,\ell) = \frac{1}{m^\ell [\ell]_q!} \sum_{k=0}^{\ell} (-1)^{\ell-k} q^{\binom{\ell-k}{2}} \binom{\ell}{k}_q (m[k]_q + r)^n, \tag{14}$$

it is obvious that the q-analogues $\sigma[n,k]_q^{\beta,r}$ and $W_{m,r,q}(n,k)$ are different from one another. In fact, the former was motivated by Carlitz' [6] definition of the q-Stirling numbers of the second kind $S_q[n,k]$ which is in terms of the recurrence relation

$$S_q[n,k] = S_q[n-1,k-1] + [k]_q S_q[n-1,k], \tag{15}$$

while the latter was motivated by the horizontal generating function (see [19, 26])

$$(a^\dagger a)^n = \sum_{k=1}^{n} S_q[n,k](a^\dagger)^k a^k. \tag{16}$$

Another q-analogue that is distinctly motivated is the q-noncentral Stirling numbers of the second kind $S_\alpha[n,k]_q$ defined by Corcino et al. [11] as follows:

$$S_\alpha[n,k]_q = q^{(k-1)-\alpha} S_\alpha[n-1,k-1]_q + [k-\alpha]_q S_\alpha[n-1,k]_q. \tag{17}$$

This type of q-analogue was said to be adapted in the work of Ehrenborg [15]. In [11], some combinatorial properties were obtained. These include convolution-type formulas which were derived using the combinatorics of the A-tablaux. Lastly, the Hankel transform of the sum of the numbers $S_a[n,k]_q$, called q-noncentral Bell numbers, is presented in the same paper.

The main concern of this paper is to define an alternative q-analogue of the Ruciński-Voigt numbers that is consistent with (1) (makes use of the sequence $\mathbf{a}$), not motivated by the works of Carlitz' [6] and Katriel [19], and generalizes identities such recurrence relations, explicit formulas and generating functions obtained by Corcino et al. [11] and Mangontarum et al. [23]. A combinatorial interpetation of this q-analogue is presented and some formulas including convolution-type identities are obtained. Finally, matrix decompositions of the Ruciński-Voigt numbers and the newly-defined q-analogue are established.

2. Definition and combinatorial properties

Let

$$Q_q^{k,\mathbf{a}}(x) = \prod_{i=0}^{k-1} [x - (a + ir)]_q, \tag{18}$$

where $\mathbf{a} = (a, a+r, a+2r, a+3r, \ldots)$. For $x > 0$, nonnegative integers n and k, and complex numbers a and r, we define the q-Ruciński-Voigt numbers (an alternative q-analogue of the Ruciński-Voigt numbers), denoted by $S_q^{n,k}(\mathbf{a})$, as coefficients of $Q_q^{k,\mathbf{a}}(x)$ in the expansion of

$$[x]_q^n = \sum_{k=0}^{n} S_q^{n,k}(\mathbf{a}) Q_q^{k,\mathbf{a}}(x). \tag{19}$$

By convention, we set $S_q^{n,k}(\mathbf{a}) = 0$ for $n < k$ or $n, k < 0$.

Theorem 2.1. *The q-Ruciński-Voigt numbers $S_q^{n,k}(\mathbf{a})$ have the following recurrence relations*:

(i) *triangular*:

$$S_q^{n+1,k}(\mathbf{a}) = q^{a+r(k-1)} S_q^{n,k-1}(\mathbf{a}) + [a+rk]_q S_q^{n,k}(\mathbf{a}), \tag{20}$$

(ii) *vertical*:

$$S_q^{n+1,k+1}(\mathbf{a}) = q^{a+rk} \sum_{j=k}^{n-k} [a+r(k+1)]_q^{n-j} S_q^{j,k}(\mathbf{a}), \tag{21}$$

(iii) *horizontal*:

$$S_q^{n,k}(\mathbf{a}) = \sum_{j=0}^{n-k} (-1)^j \frac{\langle a|r\rangle_{q,k+j+1} \, S_q^{n+1,k+j+1}(\mathbf{a})}{\langle a|r\rangle_{q,k+1} \, q^{(a+rk)(j+1)+r\binom{j+1}{2}}}, \tag{22}$$

where $\langle a|r\rangle_{q,n} = \prod_{i=0}^{n-1} [a+ri]_q$.

Proof. Since

$$[x-a-rk]_q = ([x]_q - [a+rk]_q) \frac{1}{q^{a+rk}},$$

then

$$\begin{aligned}
\sum_{k=0}^{n+1} S_q^{n+1,k}(\mathbf{a}) Q_q^{k,\mathbf{a}}(x) &= [x]_q^n [x]_q \\
&= \left(\sum_{k=0}^{n} S_q^{n,k}(\mathbf{a}) Q_q^{k,\mathbf{a}}(x) \right) \left(q^{a+rk}[x-a-rk]_q + [a+rk]_q \right) \\
&= \sum_{k=0}^{n+1} \left\{ q^{a+r(k-1)} S_q^{n,k-1}(\mathbf{a}) + [a+rk]_q S_q^{n,k}(\mathbf{a}) \right\} Q_q^{k,\mathbf{a}}(x).
\end{aligned}$$

The triangular recurrence relation is obtained by comparing the coefficients of $Q_q^{k,\mathbf{a}}(x)$. The vertical recurrence relation can be derived by repeated application of (20). That is,

$$\begin{aligned}
S_q^{n+1,k+1}(\mathbf{a}) = q^{a+rk} S_q^{n,k}(\mathbf{a}) &+ q^{a+rk}[a+r(k+1)]_q S_q^{n-1,k}(\mathbf{a}) \\
&+ q^{a+rk}[a+r(k+1)]_q^2 S_q^{n-2,k}(\mathbf{a})
\end{aligned}$$

$$
\begin{aligned}
&+ q^{a+rk}[a+r(k+1)]_q^3 S_q^{n-3,k}(\mathbf{a})\\
&+\cdots + q^{a+rk}[a+r(k+1)]_q^{n-k} S_q^{k,k}(\mathbf{a})\\
&= q^{a+rk}\sum_{j=k}^{n-k}[a+r(k+1)]_q^{n-j} S_q^{j,k}(\mathbf{a}).
\end{aligned}
$$

Finally, by evaluating the right-hand side of (22) using (20), we get

$$
\begin{aligned}
&\sum_{j=0}^{n-k}(-1)^j \frac{\langle a|r\rangle_{q,k+j+1}\, S_q^{n+1,k+j+1}(\mathbf{a})}{\langle a|r\rangle_{q,k+1}\, q^{(a+rk)(j+1)+r\binom{j+1}{2}}}\\
&= \sum_{j=0}^{n-k}(-1)^j \frac{\langle a|r\rangle_{q,k+j+1}\, q^{a+r(k+j)} S_q^{n,k+j}(\mathbf{a})}{\langle a|r\rangle_{q,k+1}\, q^{(a+rk)(j+1)+r\binom{j+1}{2}}}\\
&\quad + \sum_{j=0}^{n-k}(-1)^j \frac{\langle a|r\rangle_{q,k+j+1}\, S_q^{n,k+j+1}(\mathbf{a})}{\langle a|r\rangle_{q,k+1}\, q^{(a+rk)(j+1)+r\binom{j+1}{2}}}\\
&= \sum_{j=1}^{n-k}(-1)^j \frac{\langle a|r\rangle_{q,k+j+1}\, q^{a+r(k+j)} S_q^{n,k+j}(\mathbf{a})}{\langle a|r\rangle_{q,k+1}\, q^{(a+rk)(j+1)+r\binom{j+1}{2}}}\\
&\quad + S_q^{n,k}(\mathbf{a}) + \sum_{j=1}^{n-k}(-1)^{j-1} \frac{\langle a|r\rangle_{q,k+j+1}\, S_q^{n,k+j}(\mathbf{a})}{\langle a|r\rangle_{q,k+1}\, q^{(a+rk)j+r\binom{j}{2}}}\\
&= S_q^{n,k}(\mathbf{a}).
\end{aligned}
$$

These prove the theorem. □

Remark 2.2. The following observations are significant:

(i) From (20), we have

$$S_q^{n,0}(\mathbf{a}) = [a]_q^n \tag{23}$$

and

$$S_q^{n,n}(\mathbf{a}) = q^{r\binom{n}{2}+an}. \tag{24}$$

(ii) By taking the limits as $q \to 1$, the results in Theorem 2.1 reduce back to the triangular, vertical and horizontal recurrence relations for the classical Ruciński-Voigt numbers presented in [14].

(iii) When $a = -\alpha$ and $r = 1$ in Theorem 2.1, we obtain the q-noncentral Stirling numbers of the second kind [11, Definition 1 and Theorem 4]. That is,

$$S_q^{n,k}(\mathbf{a}_5) = S_\alpha[n,k]_q, \tag{25}$$

where $\mathbf{a}_5 = (-\alpha, 1-\alpha, 2-\alpha, 3-\alpha, \ldots)$.

(iv) When $a = 0$ and $r = \alpha$ in Theorem 2.1, we obtain the q-analogue of the translated Whitney numbers of the second kind, denoted by $w^2_{(\alpha)}n, k]_q$ [23, Equations 30, 34 and 41]. That is,

$$S_q^{n,k}(\mathbf{a}_4) = w^2_{(\alpha)}n, k]_q. \tag{26}$$

The defining relation in (19) may be expressed as

$$\begin{aligned}[a + rk]_q &= \sum_{j=0}^{n} S_q^{n,j}(\mathbf{a}) \prod_{i=0}^{j-1} [kr - ir]_q \\ &= \sum_{j=0}^{k} \binom{k}{j}_{q^r} \left\{ \frac{S_q^{n,j}(\mathbf{a}) \prod_{i=0}^{j-1} [kr - ir]_q}{\binom{k}{j}_{q^r}} \right\}.\end{aligned}$$

Applying the q-binomial inversion formula (see [9]) and since $\prod_{i=0}^{k-1} [kr - ir]_q = [k]_{q^r}! [r]_q^k$, we get

$$S_q^{n,k}(\mathbf{a}) = \frac{1}{[k]_{q^r}! [r]_q^k} \sum_{j=0}^{k} (-1)^{k-j} q^{r\binom{k-j}{2}} \binom{k}{j}_{q^r} [a + rj]_q^n. \tag{27}$$

Furthermore, let

$$f_k(t) := \sum_{n=0}^{\infty} S_q^{n,k}(\mathbf{a}) \frac{t^n}{[n]_q!}$$

be the exponential generating function of $S_q^{n,k}(\mathbf{a})$. Then multiplying both sides of (27) by $\frac{t^n}{[n]_q!}$ and summing over n gives

$$f_k(t) = \frac{1}{[k]_{q^r}! [r]_q^k} \sum_{j=0}^{k} (-1)^{k-j} q^{r\binom{k-j}{2}} \binom{k}{j}_{q^r} e_q\left(t\, [a + jr]_q\right), \tag{28}$$

where $e_q\left(t\, [jr + a]_q\right)$ is the q-exponential function defined by

$$e_q(x) = \sum_{j=0}^{\infty} \frac{x^j}{[j]_q!}. \tag{29}$$

Making use of the explicit formula of the known q-difference operator (see the work of Kim and Son [20]) given by

$$\Delta_q^k f(x) = \sum_{j=0}^{k} (-1)^{k-j} q^{\binom{k-j}{2}} \binom{k}{j}_q f(x + j), \tag{30}$$

gives

$$f_k(t) = \left\{ \Delta_q^k \left(\frac{e_q\left(t [a + rx]_q\right)}{[k]_{q^r}! [r]_q^k} \right) \right\}_{x=0}. \tag{31}$$

Hence, we have proved the results in the next theorem.

Theorem 2.3. *The q-Ruciński-Voigt numbers $S_q^{n,k}(\mathbf{a})$ satisfy the explicit formula*

$$S_q^{n,k}(\mathbf{a}) = \frac{1}{[k]_{q^r}![r]_q^k} \sum_{j=0}^{k} (-1)^{k-j} q^{r\binom{k-j}{2}} \binom{k}{j}_{q^r} [a+rj]_q^n \tag{32}$$

and the exponential generating function

$$f_k(t) := \sum_{n=0}^{\infty} S_q^{n,k}(\mathbf{a}) \frac{t^n}{[n]_q!} = \left\{ \Delta_q^k \left(\frac{e_q\left(t[a+rx]_q\right)}{[k]_{q^r}![r]_q^k} \right) \right\}_{x=0}. \tag{33}$$

Remark 2.4. Observe that if we take the limits of (32) and (33) as $q \to 1$, we get

$$\lim_{q\to 1} S_q^{n,k}(\mathbf{a}) = \frac{1}{k!r^k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (jr+a)^n = S_k^n(\mathbf{a})$$

and

$$\lim_{q\to 1} f_k(t) = \frac{1}{r^k k!} e^{ax}(e^{rx}-1)^k = \sum_{n=k}^{\infty} S_k^n(\mathbf{a}) \frac{x^n}{n!},$$

respectively. The first limit implies that $S_q^{n,k}(\mathbf{a})$ is a proper q-analogue of the numbers $S_k^n(\mathbf{a})$. We note that the exponential generating function in (33) still holds when t is replaced with $[t]_q$. That is,

$$\sum_{n=0}^{\infty} S_q^{n,k}(\mathbf{a}) \frac{[t]_q^n}{[n]_q!} = \left\{ \Delta_q^k \left(\frac{e_q\left([t]_q[a+rx]_q\right)}{[k]_{q^r}![r]_q^k} \right) \right\}_{x=0}. \tag{34}$$

And when $a = -\alpha$ and $r = 1$, (32) and (34) reduce to similar formulas for the q-noncentral Stirling numbers of the second kind (cf. [11, Theorems 5 and 8]). Similarly, when $a = 0$ and $r = \alpha$, (32) and (34) reduce to similar formulas for the q-analogue of the translated Whitney numbers of the second kind (cf. [23, Theorem 2.11]).

Theorem 2.5. *The q-Ruciński-Voigt numbers $S_q^{n,k}(\mathbf{a})$ satisfy the rational generating function given by*

$$g_k(t) := \sum_{n=k}^{\infty} S_q^{n,k}(\mathbf{a}) t^{n-k} = \frac{q^{r\binom{k}{2}+ka}}{\prod_{j=0}^{k}\left(1 - t\,[a+rj]_q\right)}, \tag{35}$$

and the explicit formula in complete symmetric polynomial form given by

$$S_q^{n,k}(\mathbf{a}) = q^{r\binom{k}{2}+ka} \sum_{0\le j_1\le j_2\le\cdots\le j_{n-k}\le k} \prod_{i=1}^{n-k} [a+rj_i]_q\,. \tag{36}$$

Proof. We will prove the results by induction on k. Let $g_k(t)$ be the rational generating function of $S_q^{n,k}(\mathbf{a})$. When $k = 0$, we have

$$g_0(t) = \sum_{n=0}^{\infty} S_q^{n,0}(\mathbf{a}) t^n = \sum_{n=0}^{\infty} [a]_q^n t^n = \frac{1}{1-[a]_q t}.$$

Furthermore, with $k>0$ and (20) we obtain

$$\begin{aligned} g_k(t) = & \sum_{n=k}^{\infty} q^{a+r(k-1)} S_q^{n-1,k-1}(\mathbf{a}) t^{(n-1)-(k-1)} \\ & + t[a+rk]_q \sum_{n=k}^{\infty} S_q^{n-1,k}(\mathbf{a}) t^{(n-1)-k} \\ = & \, q^{a+r(k-1)} g_{k-1}(t) + t[a+rk]_q g_k(t). \end{aligned}$$

Hence,

$$\begin{aligned} g_k(t) &= \frac{q^{a+r(k-1)}}{1-t[a+rk]_q} g_{k-1}(t) \\ &= \frac{q^{r\binom{k}{2}+ka}}{\prod_{j=0}^{k}\left(1-t\,[a+rj]_q\right)}. \end{aligned}$$

Now, we note that (36) yields $S_q^{0,0}(\mathbf{a})=1$, which is in agreement with the initial value of $S_q^{n,k}(\mathbf{a})$. We suppose that (36) holds up to n for $k=0,1,2,\ldots,n$. Then by (20),

$$\begin{aligned} S_q^{n+1,k}(\mathbf{a}) = & \, q^{a+r(k-1)} \left(q^{r\binom{k-1}{2}+a(k-1)} \sum_{0\le j_1\le j_2\le\cdots\le j_{n-(k-1)}\le k-1} \prod_{i=1}^{n-(k-1)} [a+rj_i]_q \right) \\ & + [a+rk]_q \left(q^{r\binom{k}{2}+ka} \sum_{0\le j_1\le j_2\le\cdots\le j_{n-k}\le k} \prod_{i=1}^{n-k} [a+rj_i]_q \right) \\ = & \, q^{r\binom{k}{2}+ka} \sum_{0\le j_1\le j_2\le\cdots\le j_{n+1-k}\le k} \prod_{i=1}^{n+1-k} [a+rj_i]_q \,. \end{aligned}$$

Finally, (36) yields $S_q^{n+1,n+1}(\mathbf{a}) = q^{r\binom{n+1}{2}+a(n+1)}$ which is in agreement with (24). This completes the proof. □

Remark 2.6. Apart from $q^{r\binom{k}{2}+ka}$, the right-hand side of (36) is in complete symmetric polynomial form. We also observe that as $q\to 1$, the generating function and explicit formula obtained in the previous theorem reduce back to similar identities for the classical Ruciński-Voigt numbers. Now, if we replace t with $[t]_q$ in (35), we get

$$\sum_{n=k}^{\infty} S_q^{n,k}(\mathbf{a})[t]_q^{n-k} = \frac{q^{r\binom{k}{2}+ka}}{\prod_{j=0}^{k}\left(1-[t]_q\,[a+rj]_q\right)}. \tag{37}$$

The results of Corcino et al. [11, Theorems 10 and 11] can be obtained from this when $a=-\alpha$ and $r=1$ in (37) and (36), while the explicit formula for the q-analogue of Mangontarum et al. [23, Equation 57] is the case when $a = 0$ and $r = \alpha$ in (36).

3. In the context of A-tableaux

A 0-1 tableau is a pair $\varphi = (\lambda, f)$, where $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$ is a partition of an integer m and $f = (f_{ij})_{1\leq j\leq \lambda_i}$ is a "filling" of the cells of the corresponding Ferrers diagram of shape λ with 0's and 1's in such a way that there is exactly one 1 in each column. In line with this, an A-tableau is defined to be a list Φ of column c of a Ferrers diagram of λ (by decreasing order of length) such that the length $|c|$ is part of a sequence $A = (a_i)_{i\geq 0}$, a strictly increasing sequence of nonnegative integers. These tableaux were first introduced in the paper of de Médicis and Leroux [28]. Combinatorial interpretations of the Stirling numbers and their different extensions and generalizations can be seen in the same paper and in subsequent works of others (see [11, 14, 22, 23]).

Let ω be a function from the set of nonnegative integers N to a ring K, and suppose that Φ is an A-tableau with r columns of length $|c|$. It is known that Φ might contain a finite number of columns whose lengths are zero since $0 \in A$ and if $\omega(0) \neq 0$ (cf. [28]). Let $T^A(x, y)$ be the set of all A-tableaux with $A = \{0, 1, 2, 3, \ldots, x\}$ and exactly y columns, some of which are possibly of length zero. The next theorem expresses the q-Ruciński-Voigt numbers in terms of a sum of weights of A-tableaux.

Theorem 3.1. *Let $\omega : N \longrightarrow K$ be a function from the set of positive integers N to a ring K (column weights according to length) defined by $\omega(|c|) = [a + r|c|]_q$, where a and r are complex numbers, and $|c|$ is the length of column c of an A-tableau in $T^A(k, n-k)$. Then*

$$q^{-r\binom{k}{2}-ka} S_q^{n,k}(\mathbf{a}) = \sum_{\Phi\in T^A(k,n-k)} \prod_{c\in\Phi} \omega(|c|), \tag{38}$$

where $\mathbf{a} = (a, a + r, a + 2r, a + 3r, \ldots)$.

Proof. Let Φ in $T^A(k, n-k)$. This implies that Φ has exactly $n-k$ columns, say $c_1, c_2, \ldots, c_{n-k}$, whose lengths are $j_1, j_2, \ldots, j_{n-k}$, respectively. Moreover, for each column $c_i \in \Phi$, $i = 1, 2, \ldots, n-k$, we have $|c_i| = j_i$ and $\omega(|c_i|) = [a+rj_i]_q$. Hence, we get

$$\prod_{c\in\Phi} \omega(|c|) = \prod_{i=1}^{n-k} \omega(|c_i|) = \prod_{i=1}^{n-k} [a + rj_i]_q.$$

Since $\Phi \in T^A(k, n-k)$, then

$$\begin{aligned}\sum_{\Phi\in T^A(k,n-k)} \prod_{c\in\Phi} \omega(|c|) &= \sum_{0\leq j_1\leq j_2\leq\cdots\leq j_{n-k}\leq k} \prod_{i=1}^{n-k} \omega(|c_i|)\\ &= \sum_{0\leq j_1\leq j_2\leq\cdots\leq j_{n-k}\leq k} \prod_{i=1}^{n-k} [a + rj_i]_q.\end{aligned}$$

$$= q^{-r\binom{k}{2}-ka} S_q^{n,k}(\mathbf{a}).$$

This completes the proof. □

3.1. Combinatorics of A-tableaux and convolution-type identities

Our aim is to demonstrate a simple combinatorics of A-tableaux. Through this, convolution-type identities are obtained. To start, we first write (38) as

$$q^{-r\binom{k}{2}-ka} S_q^{n,k}(\mathbf{a}) = \sum_{\Phi \in T^A(k,n-k)} \omega_A(\Phi), \tag{39}$$

where

$$\omega_A(\Phi) = \prod_{c\in\Phi} \omega(|c|) = \prod_{c\in\Phi} [a + r|c|]_q, \ \ |c| \in \{0, 1, 2, \ldots, k\}. \tag{40}$$

The following theorem shows how an additive constant affects the recurrence formula for $S_q^{n,k}(\mathbf{a})$:

Theorem 3.2. *For nonnegative integers n and k, and complex numbers a and r, the q-Ruciński-Voigt numbers satisfies the following identity*:

$$S_q^{n,k}(\mathbf{a}) = \sum_{j=k}^{n} \binom{n}{j} q^{a_2(n-k)+k(a-a_1)} (-[-a_2]_q)^{n-j} S_q^{n,k}(\mathbf{a}^*), \tag{41}$$

where $\mathbf{a}^* = (a_1, a_1 + r, a_1 + 2r, a_1 + 3r, \ldots)$ *and* $a = a_1 + a_2$ *for some numbers* a_1 *and* a_2.

Proof. For $\Phi \in T^A(k, n-k)$, we substitute $j_i = |c|$ in (40). That is

$$\omega_A(\Phi) = \prod_{i=1}^{n-k} [a + rj_i]_q, \tag{42}$$

$j_i \in \{0, 1, 2, \ldots, k\}$. Suppose $a = a_1 + a_2$ for some numbers a_1 and a_2. Then with $\omega^*(j_i) = [a_1 + rj_i]_q$, we may write

$$\begin{aligned}
\omega_A(\Phi) &= \prod_{i=1}^{n-k} \left[a_2 + (a_1 + rj_i)\right]_q \\
&= \prod_{i=1}^{n-k} q^{a_2} \left(\omega^*(j_i) - [-a_2]_q\right) \\
&= q^{a_2(n-k)} \sum_{\ell=0}^{n-k} (-[-a_2]_q)^{n-k-\ell} \sum_{j_1 \le q_1 \le q_2 \le \cdots \le q_l \le j_{n-k}} \prod_{i=1}^{\ell} \omega^*(q_i).
\end{aligned}$$

Let B_Φ be the set of all A-tableaux corresponding to Φ such that for each $\psi \in B_\Phi$, one of the following is true:

ψ has no column whose weight is $-[-a_2]_q$;
ψ has one columns whose weight is $-[-a_2]_q$;
ψ has two columns whose weight is $-[-a_2]_q$;

$\vdots$

ψ has $n-k$ columns whose weight is $-[-a_2]_q$.

Thus, we have

$$\omega_A(\Phi) = \sum_{\psi \in B_\Phi} \omega_A(\psi).$$

If there are ℓ columns in ψ with weights other than $-[-a_2]_q$, then

$$\omega_A(\psi) = \prod_{c\in\psi} \omega^*(|c|) = q^{a_2(n-k)}\left(-[-a_2]_q\right)^{n-k-\ell} \prod_{i=1}^{\ell} \omega^*(q_i),$$

where $q_1, q_2, \ldots, q_r \in \{j_1, j_2, \ldots, j_{n-k}\}$. Hence, (39) may be rewritten into

$$q^{-r\binom{k}{2}-ka} S_q^{n,k}(\mathbf{a}) = \sum_{\Phi\in T^A(k,n-k)} \sum_{\psi\in B_\Phi} \omega_A(\psi). \tag{43}$$

It is known from [11] that for each ℓ, there correspond $\binom{n-k}{\ell}$ tableaux with ℓ columns having weights $\omega^*(q_i)$, $q_i \in \{j_1, j_2, j_3, \ldots, j_{n-k}\}$. Since $T^A(k, n-k)$ contains $\binom{n}{k}$ tableaux, then for each $\Phi \in T^A(k, n-k)$, there are $\binom{n}{k}\binom{n-k}{\ell}$ A-tableaux corresponding to Φ. But only $\binom{\ell+k}{\ell}$ of these tableaux are distinct. Hence, every tableau ψ with ℓ columns of weights other than $-[-a_2]_q$ appears

$$\frac{\binom{n}{k}\binom{n-k}{\ell}}{\binom{\ell+k}{\ell}} = \binom{n}{\ell+k}$$

times in the collection (cf. [11]). It then follows that

$$q^{-r\binom{k}{2}-ka} S_q^{n,k}(\mathbf{a}) = \sum_{\ell=0}^{n-k} \binom{n}{\ell+k} q^{a_2(n-k)}\left(-[-a_2]_q\right)^{n-k-\ell} \sum_{\psi\in\bar{B}_\ell} \prod_{c\in\psi} \omega^*(|c|),$$

where $\bar{B}_\ell$ denotes the set of all tableaux ψ having ℓ columns of weights $w^*(j_i)$. Reindexing the two sums give

$$q^{-r\binom{k}{2}-ka} S_q^{n,k}(\mathbf{a}) = \sum_{j=k}^{n} \binom{n}{j} q^{a_2(n-k)}\left(-[-a_2]_q\right)^{n-j} \sum_{\psi\in\bar{B}_{j-k}} \prod_{c\in\psi} \omega^*(|c|). \tag{44}$$

Since $\bar{B}_{j-k} = T^A(k, j-k)$, then

$$\sum_{\psi\in\bar{B}_{j-k}} \prod_{c\in\psi} \omega^*(|c|) = q^{-r\binom{k}{2}-ka_1} S_q^{n,k}(\mathbf{a}^*), \tag{45}$$

where $\mathbf{a}^* = (a_1, a_1+r, a_1+2r, a_1+3r, \ldots)$. Finally, combining this with (44) gives the desired result. □

For $A_1 = \{0, 1, 2, \ldots, p\}$ and $A_2 = \{p+1, p+2, \ldots, p+j+1\}$, let $\Phi_1 \in T^{A_1}(p, k-p)$ and $\Phi_2 \in T^{A_2}(j, n-k-j)$. We can generate an A-tableau Φ with $n-p-j$ columns whose lengths are in $A = \{0, 1, 2, \ldots, p+j+1\}$ by joining the columns of the tableaux Φ_1 and Φ_2. Hence, for $\Phi \in T^A(p+j+1, n-p-j)$, we can have

(46)
$$\sum_{\Phi \in T^A(p+j+1,n-p-j)} \omega_A(\Phi) = \sum_{k=0}^{n} \left\{ \sum_{\Phi_1 \in T^{A_1}(p,k-p)} \omega_{A_1}(\Phi_1) \cdot \sum_{\Phi_2 \in T^{A_2}(j,n-k-j)} \omega_{A_2}(\Phi_2) \right\}.$$

Clearly, by (39),

$$\sum_{\Phi \in T^A(p+j+1,n-p-j)} \omega_A(\Phi) = q^{-r\binom{p+j+1}{2}-(p+j+1)a} S_q^{n+1,p+j+1}(\mathbf{a}) \tag{47}$$

and

$$\sum_{\Phi_1 \in T^{A_1}(p,k-p)} \omega_{A_1}(\Phi_1) = q^{-r\binom{p}{2}-pa} S_q^{k,p}(\mathbf{a}). \tag{48}$$

Also,

$$\begin{aligned}\sum_{\Phi_2 \in T^{A_2}(j,n-k-j)} \omega_{A_2}(\Phi_2) &= \sum_{p+1 \le g_1 \le g_2 \le \cdots \le g_{n-k-j} \le p+j+1} \prod_{i=1}^{n-k-j} [a+rg_i]_q \\ &= \sum_{0 \le g_1 \le g_2 \le \cdots \le g_{n-k-j} \le j} \prod_{i=1}^{n-k-j} [a+r(p+1+g_i)]_q \\ &= \sum_{0 \le g_1 \le g_2 \le \cdots \le g_{n-k-j} \le j} \prod_{i=1}^{n-k-j} [(a+rp+r)+rg_i]_q \\ &= q^{-r\binom{j}{2}-j(a+rp+r)} S_q^{n-k,j}(\bar{\mathbf{a}}).\end{aligned}$$

Here, $\bar{\mathbf{a}} = (a+rp+r, a+rp+2r, a+rp+3r, \ldots)$. Thus,

$$\begin{aligned}&q^{-r\binom{p+j+1}{2}-(p+j+1)a} S_q^{n+1,p+j+1}(\mathbf{a}) \\ &= \sum_{k=0}^{n} q^{-r\binom{p}{2}-pa} S_q^{k,p}(\mathbf{a}) \cdot q^{-r\binom{j}{2}-j(a+rp+r)} S_q^{n-k,j}(\bar{\mathbf{a}}).\end{aligned} \tag{49}$$

Since

$$r\binom{p+j+1}{2} + (p+j+1)a - r\binom{p}{2} - pa - r\binom{j}{2} - j(a+rp+r) = a+rp, \tag{50}$$

then we get

$$S_q^{n+1,p+j+1}(\mathbf{a}) = \sum_{k=0}^{n} q^{a+rp} S_q^{k,p}(\mathbf{a}) S_q^{n-k,j}(\bar{\mathbf{a}}). \tag{51}$$

Similarly, for $B_1 = \{0, 1, 2, \ldots, k\}$ and $B_2 = \{k, k+1, k+2, \ldots, n\}$, let $\phi_1 \in T^{B_1}(k, p-k)$ and $\phi_2 \in T^{B_2}(n-k, j-n+k)$. Then we can generate an A-tableau ϕ with $p+j-n$ columns whose lengths are in $A = \{0, 1, 2, \ldots, n\}$ by joining the columns of ϕ_1 and ϕ_2. Hence, for $\phi \in T^A(n, p+j-n)$,

(52)
$$\sum_{\phi \in T^A(n,p+j-n)} \omega_A(\phi) = \sum_{k=0}^{n} \left\{ \sum_{\phi_1 \in T^{B_1}(k,p-k)} \omega_{B_1}(\phi_1) \cdot \sum_{\phi_2 \in T^{B_2}(n-k,j-n+k)} \omega_{B_2}(\phi_2) \right\}.$$

Again by (39),

(53)
$$\sum_{\phi \in T^A(n,p+j-n)} \omega_A(\phi) = q^{-r\binom{n}{2}-na} S_q^{p+j,n}(\mathbf{a})$$

and

(54)
$$\sum_{\phi_1 \in T^{B_1}(k,p-k)} \omega_{B_1}(\phi_1) = q^{-r\binom{k}{2}-ka} S_q^{p,k}(\mathbf{a}).$$

Furthermore,

$$\begin{aligned}
\sum_{\phi_2 \in T^{B_2}(n-k,j-n+k)} \omega_{B_2}(\phi_2) &= \sum_{k \le g_1 \le g_2 \le \cdots \le g_{j-n+k} \le n} \prod_{i=1}^{j-n+k} [a + rg_i]_q \\
&= \sum_{0 \le g_1 \le g_2 \le \cdots \le g_{j-n+k} \le n-k} \prod_{i=1}^{j-n+k} [a + r(k+g_i)]_q \\
&= \sum_{0 \le g_1 \le g_2 \le \cdots \le g_{j-n+k} \le n-k} \prod_{i=1}^{j-n+k} [(a+rk) + rg_i]_q \\
&= q^{-r\binom{n-k}{2}-(n-k)(a+rk)} S_q^{j,n-k}(\hat{\mathbf{a}}),
\end{aligned}$$

where $\hat{\mathbf{a}} = (a+rk, a+rk+r, a+rk+2r, \ldots)$. Thus,

(55)
$$q^{-r\binom{n}{2}-na} S_q^{p+j,n}(\mathbf{a}) = \sum_{k=0}^{n} q^{-r\binom{k}{2}-ka} S_q^{p,k}(\mathbf{a}) \cdot q^{-r\binom{n-k}{2}-(n-k)(a+rk)} S_q^{j,n-k}(\hat{\mathbf{a}}).$$

Finally, because

(56)
$$r\binom{n}{2} + na - r\binom{k}{2} - ka - r\binom{n-k}{2} - (n-k)(a+rk) = 0,$$

we get

(57)
$$S_q^{p+j,n}(\mathbf{a}) = \sum_{k=0}^{n} S_q^{p,k}(\mathbf{a}) S_q^{j,n-k}(\hat{\mathbf{a}}).$$

We formally state (51) and (57) in the next theorem.

Theorem 3.3. *The q-Ruciński-Voigt numbers satisfy the following convolution-type identities*:

(58)
$$S_q^{n+1,p+j+1}(\mathbf{a}) = \sum_{k=0}^{n} q^{a+rp} S_q^{k,p}(\mathbf{a}) S_q^{n-k,j}(\bar{\mathbf{a}}),$$

$$S_q^{p+j,n}(\mathbf{a}) = \sum_{k=0}^{n} S_q^{p,k}(\mathbf{a}) S_q^{j,n-k}(\hat{\mathbf{a}}), \tag{59}$$

where $\bar{\mathbf{a}} = (a + rp + r, a + rp + 2r, a + rp + 3r, \ldots)$ *and* $\hat{\mathbf{a}} = (a + rk, a + rk + r, a + rk + 2r, \ldots)$.

Remark 3.4. When $r = m$, $a = r$ and $q \to 1$, we recover from this theorem the results recently obtained by Xu and Zhou [33, Theorem 2.4].

4. Matrix decompositions

In 2015, Pan [30] obtained a remarkable matrix decomposition that provides an explicit and nonrecursive manner of computing for the generalized Stirling numbers of Hsu and Shiue [18]. That is, if $\mathcal{S}_{\alpha,\beta,\gamma} = (S(n, k; \alpha, \beta, \gamma))$ is the matrix whose entries are the generalized Stirling numbers $S(n, k; \alpha, \beta, \gamma)$ defined by

$$(t|\alpha)_n = \sum_{k=0}^{n} S(n, k; \alpha, \beta, \gamma)(t - \gamma|\beta)_k, \tag{60}$$

where $(t|\alpha)_n = \prod_{i=1}^{n-1}(t - i\alpha)$, $(t|\alpha)_0 = 1$, then

$$\mathcal{S}_{\alpha,\beta,\gamma} = \mathcal{S}_{\alpha,0,0} \cdot \mathcal{S}_{0,0,\gamma} \cdot \mathcal{S}_{0,\beta,0} \tag{61}$$

(cf. [30, Theorem 7]). It is important to note that although the Ruciński-Voigt numbers are given by

$$S(n, k; 0, r, a) = S^{n,k}(\mathbf{a}), \tag{62}$$

it is not wise to assume that

$$\mathcal{S}_{0,r,a} = \mathcal{S}_{0,0,0} \cdot \mathcal{S}_{0,0,a} \cdot \mathcal{S}_{0,r,0}. \tag{63}$$

This is our justification in establishing the matrix decomposition of a matrix whose entries are the numbers $S^{n,k}(\mathbf{a})$.

First, we define $\widetilde{S}^{a,r}$ to be the matrix whose entries are the Ruciński-Voigt numbers. For clarity, we will refer to this matrix as the Ruciński-Voigt matrix. Also, we let

$$\mathcal{V}_r(x) = (1, x, (x|r)_2, (x|r)_3, \ldots, (x|r)_n, \ldots)^T \tag{64}$$

be an infinite column vector. Note that the defining relation in (1) can be rewritten into the form

$$(x + a)^n = \sum_{k=0}^{n} S^{n,k}(\mathbf{a})(x|r)_k. \tag{65}$$

Remark 4.1. The following identity is trivial:

$$\mathcal{V}_0(x + a) = \widetilde{S}^{a,r} \mathcal{V}_r(x). \tag{66}$$

Using $\mathbf{a}_6 = (a, a, a, \ldots)$ in place of $\mathbf{a}$ (the case when $r = 0$ in $\mathbf{a}$) in (65) gives

$$\sum_{k=0}^{n} S^{n,k}(\mathbf{a}_6)x^k = (x+a)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}x^k,$$

which implies that $S^{n,k}(\mathbf{a}_6) = a^{n-k}\binom{n}{k}$. On the other hand, replacing x with rx in (65) gives

$$r^n x^n = \sum_{k=0}^{n} S^{n,k}(\mathbf{a}) \prod_{i=0}^{k-1}(rx - a - ir), \tag{67}$$

which, in return, gives

$$x^n = \sum_{k=0}^{n} r^{k-n} S^{n,k}(\mathbf{a}_7)(x)_k \tag{68}$$

when $\mathbf{a}$ is replaced with $\mathbf{a}_7 = (0, r, 2r, 3r, \ldots)$ (the case when $a = 0$ in $\mathbf{a}$). Comparing the coefficients of $(x)_k$ with the horizontal generating functions of $S(n,k)$ (cf. [9]) gives $S^{n,k}(\mathbf{a}_7) = r^{n-k}S(n,k)$. It is, therefore, clear that

$$\widetilde{S}^{0,r} = \left(r^{n-k}S(n,k)\right) \text{ and } \widetilde{S}^{a,0} = \left(a^{n-k}\binom{n}{k}\right), \tag{69}$$

and because

$$\mathcal{V}_0(x) = \widetilde{S}^{0,r}\mathcal{V}_r(x) \text{ and } \mathcal{V}_0(x+a) = \widetilde{S}^{a,0}\mathcal{V}_0(x), \tag{70}$$

then

$$\mathcal{V}_0(x+a) = \widetilde{S}^{a,0}\widetilde{S}^{0,r}\mathcal{V}_r(x). \tag{71}$$

Comparing this with (66) yields

$$\left(\widetilde{S}^{a,r} - \widetilde{S}^{a,0}\widetilde{S}^{0,r}\right)\mathcal{V}_r(x) = \mathbf{0}, \tag{72}$$

where $\mathbf{0}$ is the infinite-dimensional zero matrix. Since x is an arbitrary real or complex number and $\mathcal{V}_r(x)$ is a nonzero vector, then we obtain the following theorem:

Theorem 4.2. *The Ruciński-Voigt matrix $\widetilde{S}^{a,r}$ has the following decomposition:*

$$\widetilde{S}^{a,r} = \widetilde{S}^{a,0} \cdot \widetilde{S}^{0,r}. \tag{73}$$

We might as well extend this result to the q-Ruciński-Voigt numbers. We start by expressing (19) as

$$[x+a]_q^n = \sum_{k=0}^{n} S_q^{n,k}(\mathbf{a})[x|r]_k, \tag{74}$$

where $[x|r]_k = \prod_{i=0}^{k-1}[x-ir]_q$, $[x|r]_0 = 1$. Next, we define $\widetilde{S}_q^{a,r} = \left(S_q^{n,k}(\mathbf{a})\right)$ to be the q-Ruciński-Voigt matrix and let

$$\mathcal{V}_{q,r}[x] = \left(1, [x]_q, [x|r]_2, [x|r]_3, \ldots, [x|r]_q, \ldots\right)^T. \tag{75}$$

Remark 4.3. Clearly,

$$\mathcal{V}_{q,0}[x+a] = \widetilde{S}_q^{a,r}\mathcal{V}_{q,r}[x]. \tag{76}$$

Combining (74) with the defining relation of the q-analogue of the translated Whitney numbers of the second kind [23, Equation 4] yields

$$\sum_{k=0}^{n} S_q^{n,k}(\mathbf{a}_7)[x|r]_k = [x]_q^n = \sum_{k=0}^{n} w_{(r)}^2[n,k]_q[x|r]_k.$$

Obviously, $S_q^{n,k}(\mathbf{a}_7) = w_{(r)}^2[n,k]_q$. On the other hand, replace $\mathbf{a}$ with $\mathbf{a}_6$ in (19) and we obtain

$$\begin{aligned}\sum_{k=0}^{n} S_q^{n,k}(\mathbf{a}_6)[x-a]^k &= [x]_q^n \\ &= ([x]_q - [a]_q + [a]_q)^n \\ &= \sum_{k=0}^{n}\binom{n}{k}[a]_q^{n-k}q^{ak}[x-a]^k.\end{aligned}$$

Hence, $S_q^{n,k}(\mathbf{a}_6) = q^{ak}\binom{n}{k}[a]_q^{n-k}$. Moreover, we have

$$\widetilde{S}_q^{0,r} = \left(w_{(r)}^2[n,k]_q\right) \text{ and } \widetilde{S}_q^{a,0} = \left(q^{ak}\binom{n}{k}[a]_q^{n-k}\right). \tag{77}$$

We are now ready for the next theorem.

Theorem 4.4. *The q-Ruciński-Voigt matrix $\widetilde{S}_q^{a,r}$ has the following decomposition:*

$$\widetilde{S}_q^{a,r} = \widetilde{S}_q^{a,0}\cdot\widetilde{S}_q^{0,r}. \tag{78}$$

Proof. When $a=0$, we have $\mathcal{V}_{q,0}[x] = \widetilde{S}_q^{0,r}\mathcal{V}_{q,r}[x]$, while when $r=0$, $\mathcal{V}_{q,0}[x+a] = \widetilde{S}_q^{a,0}\mathcal{V}_{q,0}[x]$. Hence,

$$\mathcal{V}_{q,0}[x+a] = \widetilde{S}_q^{a,0}\widetilde{S}_q^{0,r}\mathcal{V}_{q,r}[x]. \tag{79}$$

Compare this with (76) and we have

$$\left(\widetilde{S}_q^{a,r} - \widetilde{S}_q^{a,0}\widetilde{S}_q^{0,r}\mathcal{V}_{q,r}[x]\right) = \mathbf{0}. \tag{80}$$

Since x is arbitrary and $\mathcal{V}_{q,r}[x]$ is nonzero, then we obtain the desired result. □

The results in Theorems 4.2 and 4.4 can be used to compute for the values of the Ruciński-Voigt and the q-Ruciński-Voigt numbers, respectively, for nonnegative integers n and k ($k \le n$), and complex numbers a and r in an explicit but nonrecursive manner.

References

[1] M. Arik and D. D. Coon, *Hilbert spaces of analytic functions and generalized coherent states*, J. Mathematical Phys. **17** (1976), no. 4, 524–527.
[2] H. Belbachir and I. E. Bousbaa, *Translated Whitney and r-Whitney numbers: a combinatorial approach*, J. Integer Seq. **16** (2013), no. 8, Article 13.8.6, 7 pp.
[3] M. Benoumhani, *On Whitney numbers of Dowling lattices*, Discrete Math. **159** (1996), no. 1-3, 13–33.
[4] ———, *On some numbers related to Whitney numbers of Dowling lattices*, Adv. in Appl. Math. **19** (1997), no. 1, 106–116.
[5] A. Z. Broder, *The r-Stirling numbers*, Discrete Math. **49** (1984), no. 3, 241–259.
[6] L. Carlitz, *q-Bernoulli numbers and polynomials*, Duke Math. J. **15** (1948), 987–1000.
[7] ———, *Weighted Stirling numbers of the first and second kind. I*, Fibonacci Quart. **18** (1980), no. 2, 147–162.
[8] J. Cigler, *A new q-analog of Stirling numbers*, Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II **201** (1992), no. 1-10, 97–109.
[9] L. Comtet, *Advanced Combinatorics*, revised and enlarged edition, D. Reidel Publishing Co., Dordrecht, 1974.
[10] R. B. Corcino, *The (r,β)-Stirling numbers*, The Mindanao Forum **14** (1999), 91–99.
[11] R. B. Corcino, R. B. Corcino, J. M. Ontolan, C. M. Perez-Fernandez, and E. R. Cantallopez, *The Hankel transform of q-noncentral Bell numbers*, Int. J. Math. Math. Sci. **2015** (2015), Art. ID 417327, 10 pp.
[12] R. B. Corcino and J. C. Fernandez, *A combinatorial approach for q-analogue of r-Stirling numbers*, British J. Math. Comput. Sci. **4** (2014), 1268–1279.
[13] R. B. Corcino and M. M. Mangontarum, *On multiparameter q-noncentral Stirling and Bell numbers*, Ars Combin. **118** (2015), 201–220.
[14] R. B. Corcino and C. B. Montero, *A q-analogue of Rucinski-Voigt numbers*, ISRN Discrete Math. **2012** (2012), Article ID 592818, 18 pages.
[15] R. Ehrenborg, *Determinants involving q-Stirling numbers*, Adv. in Appl. Math. **31** (2003), no. 4, 630–642.
[16] B. S. El-Desouky, R. S. Gomaa, and N. P. Cakić, *q-analogues of multiparameter noncentral Stirling and generalized harmonic numbers*, Appl. Math. Comput. **232** (2014), 132–143.
[17] H. W. Gould, *The q-Stirling numbers of first and second kinds*, Duke Math. J. **28** (1961), 281–289.
[18] L. C. Hsu and P. J.-S. Shiue, *A unified approach to generalized Stirling numbers*, Adv. in Appl. Math. **20** (1998), no. 3, 366–384.
[19] J. Katriel, *Stirling number identities: interconsistency of q-analogues*, J. Phys. A **31** (1998), no. 15, 3559–3572.
[20] M.-S. Kim and J.-W. Son, *A note on q-difference operators*, Commun. Korean Math. Soc. **17** (2002), no. 3, 423–430.
[21] M. Koutras, *Noncentral Stirling numbers and some applications*, Discrete Math. **42** (1982), no. 1, 73–89.
[22] M. M. Mangontarum, *Some theorems and applications of the (q,r)-Whitney numbers*, J. Integer Seq. **20** (2017), no. 2, Art. 17.2.5, 26 pp.
[23] M. M. Mangontarum, O. I. Cauntongan, and A. M. Dibagulun, *A note on the translated Whitney numbers and their q-analogues*, Turkish J. Anal. Number Theory **4** (2016), 74–81.
[24] M. M. Mangontarum, O. I. Cauntongan, and A. P. Macodi-Ringia, *The noncentral version of the Whitney numbers: a comprehensive study*, Int. J. Math. Math. Sci. **2016** (2016), Art. ID 6206207, 16 pp.

[25] M. M. Mangontarum and A. M. Dibagulun, *On the translated Whitney numbers and their combinatorial properties*, British J. Appl. Sci. Technology **11** (2015), 1–15.
[26] M. M. Mangontarum and J. Katriel, *On q-boson operators and q-analogues of the r-Whitney and r-Dowling numbers*, J. Integer Seq. **18** (2015), no. 9, Article 15.9.8, 23 pp.
[27] M. M. Mangontarum, A. P. Macodi-Ringia, and N. S. Abdulcarim, *The translated Dowling polynomials and numbers*, International Scholarly Research Notices **2014** (2014), Article ID 678408, 8 pages.
[28] A. de Médicis and P. Leroux, *Generalized Stirling numbers, convolution formulae and p,q-analogues*, Canad. J. Math. **47** (1995), no. 3, 474–499.
[29] I. Mező, *A new formula for the Bernoulli polynomials*, Results Math. **58** (2010), no. 3-4, 329–335.
[30] J. Pan, *Matrix decomposition of the unified generalized Stirling numbers and inversion of the generalized factorial matrices*, J. Integer Seq. **15** (2012), no. 6, Article 12.6.6, 9 pp.
[31] A. Ruciński and B. Voigt, *A local limit theorem for generalized Stirling numbers*, Rev. Roumaine Math. Pures Appl. **35** (1990), no. 2, 161–172.
[32] J. Stirling, *Methodus Differentialissme Tractus de Summatione et Interpolatione Serierum Infinitarum*, London, 1730.
[33] A. Xu and T. Zhou, *Some identities related to the r-Whitney numbers*, Integral Transforms Spec. Funct. **27** (2016), no. 11, 920–929.

Wardah M. Bent-Usman
Department of Mathematics
Mindanao State University-Main Campus
Marawi City 9700, Philippines
Email address: wardah_bentusman@yahoo.com

Amerah M. Dibagulun
Department of Mathematics
Mindanao State University-Main Campus
Marawi City 9700, Philippines
Email address: a.dibagulun@yahoo.com

Mahid M. Mangontarum
Department of Mathematics
Mindanao State University-Main Campus
Marawi City 9700, Philippines
Email address: mmangontarum@yahoo.com, mangontarum.mahid@msumain.edu.ph

Charles B. Montero
Department of Mathematics
Mindanao State University-Main Campus
Marawi City 9700, Philippines
Email address: charlesbmontero@yahoo.com