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# A MODIFIED POLYNOMIAL SEQUENCE OF THE CHEBYSHEV POLYNOMIALS OF THE SECOND KIND

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Abstract. Dilcher and Stolarsky [1] recently studied a sequence resembling the Chebyshev polynomials of the first kind. In this paper, we follow their some research directions to the Chebyshev polynomials of the second kind. More specifically, we consider a sequence resembling the Chebyshev polynomials of the second kind in two different ways, and investigate its properties including relations between this sequence and the sequence studied in [1], zero distribution and the irreducibility.

## 1. Introduction

Chebyshev polynomials are of great importance in many areas of mathematics, particularly approximation theory. Many papers and books ([2], [3]) have been written about these polynomials. The well-known Chebyshev polynomials are $T_n(x)$ and $U_n(x)$ that are called the Chebyshev polynomials of first kind and of the second kind, respectively. These polynomials satisfy the recurrence relations

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x), \quad (n \geq 1)$$
$$U_0(x) = 1, \quad U_1(x) = 2x, \quad U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x) \quad (n \geq 1).$$

Dilcher and Stolarsky [1] recently studied a sequence resembling the Chebyshev polynomials of the first kind $\{V_n(x)\}$ defined by the nonlinear recurrence relation $V_0(x) = 1$, $V_1(x) = x$, and

$$V_{n+1}(x) = 2xV_n(x) - V_{n-1}(x) - x^{n+1} \quad (n \geq 1). \tag{1}$$

They gave an alternative definition of the polynomial sequence $\{V_n(x)\}$ and investigated several properties of $\{V_n(x)\}$ including relations between $\{V_n(x)\}$ and $\{T_n(x)\}$, their irreducibility and zero distribution. A next research in this vein seems to follow [1] to the study of the same kind of a modied polynomial sequence of the Chebyshev polynomials of the second kind. For this purpose, in this paper we define a sequence resembling the Chebyshev polynomials of the

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second kind $\{E_n(x)\}$ by the nonlinear recurrence relation $E_0(x) = 1$, $E_1(x) = 2x$, and

$$E_{n+1}(x) = 2xE_n(x) - E_{n-1}(x) - 2x^{n+1} \quad (n \geq 1) \tag{2}$$

and study its properties including relations between $\{E_n(x)\}$ and $\{U_n(x)\}$, their irreducibility and zero distribution.

## 2. The polynomial sequence $\{E_n(x)\}$

We start with providing an alternative definition of the polynomial sequence $\{E_n(x)\}$. If $p(x) \in \mathbb{Z}[x]$, let $B(p(x))$ be the polynomial obtained from $p(x)$ by dividing the leading coefcients by 2. For example, $B(2x^5 - 4x^3 + x^2 + 1) = x^5 - 4x^3 + x^2 + 1$. Using this, we define the sequence of polynomials $\{\tilde{E}_n(x)\}$ by $\tilde{E}_0(x) = 1 = E_0(x)$, $\tilde{E}_1(x) = 2x = E_1(x)$, and

$$\tilde{E}_{n+1}(x) = B(2x\tilde{E}_n(x) - \tilde{E}_{n-1}(x)) \quad (n \geq 1). \tag{3}$$

We first obtain some properties of the coefficients ${c_k}^{(n)}$ of

$$\tilde{E}_n(x) = \sum_{k=0}^{n} {c_k}^{(n)} x^{n-k}. \tag{4}$$

**Proposition 1.** *Let $n$ be a positive integer. Then*
 (a) $c_{2j+1}^{(n)} = 0$ *for all* $j \in \{0, 1, \ldots, \lfloor \frac{n-1}{2} \rfloor\}$,
 (b) $c_{2j}^{(n)}$ *is even for all* $j \in \{0, 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$ *and* $c_0^{(n)} = 2$.

*Proof.* Using (4), an easy summation change follows

$$2x\tilde{E}_n(x) - \tilde{E}_{n-1}(x) = \sum_{k=0}^{n+1} \left(2{c_k}^{(n)} - c_{k-2}^{(n-1)}\right) x^{n-k+1} \tag{5}$$

with the convention $c_{n+1}^{(n)} = c_{-2}^{(n-1)} = c_{-1}^{(n-1)} = 0$. From (3), (4) and (5), we get

$$\tilde{E}_{n+1}(x) = c_0^{(n)} x^{n+1} + \sum_{k=1}^{n+1} \left(2c_k^{(n)} - c_{k-2}^{(n-1)}\right) x^{n-k+1}$$

and

$$\tilde{E}_{n+1}(x) = \sum_{k=0}^{n+1} c_k^{(n+1)} x^{n-k+1},$$

respectively. So by comparing coefficients of $\tilde{E}_{n+1}(x)$ in above two equations, we have

$$c_0^{(n)} = {c_0}^{(n+1)}, \quad c_k^{(n+1)} = 2c_k^{(n)} - c_{k-2}^{(n-1)} \ (1 \leq k \leq n+1), \tag{6}$$

and

$$2x = \tilde{E}_1(x) = c_0^{(1)} x + c_1^{(1)}$$

implies that

$$c_0^{(1)} = 2 \quad \text{and} \quad c_1^{(1)} = 0. \tag{7}$$

We use induction on $n$ to prove all parts of the proposition. Part (a) is true for $n = 1$ since $c_1^{(1)} = 0$ from (7). Suppose that (a) is true up to a certain $n$. Then

$$2c_k^{(n)} - c_{k-2}^{(n-1)} = 0$$

for all odd integers $k$, and therefore, by (6), (a) also holds for n+1. Part (b) is once again true for $n = 1$ since $c_0^{(1)} = 2$ from (7). Suppose that the both assertions in (b) are true up to $n$. Then

$$2c_k^{(n)} - c_{k-2}^{(n-1)}$$

is even for all even integers $k$ with $2 \leq k \leq n$, while for $k = 0$, it is

$$2c_0^{(n)} - c_{-2}^{(n-1)} = 2c_0^{(n)} = 2c_0^{(1)} = 4.$$

Now (6) completes the proof. □

Using above proposition, we show that $\tilde{E}_n(x)$ is another expression of $E_n(x)$.

**Proposition 2.** *For any nonnegative integer $n$,*

$$E_n(x) = \tilde{E}_n(x).$$

*Proof.* By definitions, $E_0(x) = \tilde{E}_0(x) = 1$. We use induction on $n$. The case $n = 1$ is obvious. Suppose that the result holds up to a certain $n$. From (2), (3) and (4),

$$\begin{aligned} E_{n+1}(x) &= 2xE_n(x) - E_{n-1}(x) - 2x^{n+1} \\ &= 2x\tilde{E}_n - \tilde{E}_{n-1}(x) - 2x^{n+1} \\ &= \sum_{k=0}^{n+1} \left(2c_k{}^{(n)} - c_{k-2}^{(n-1)}\right) x^{n-k+1} - 2x^{n+1}. \end{aligned}$$

The leading coefficient of the last polynomial in above equations is $2c_0^{(n)} - 2 = 2$ by (b) of Proposition 1. This proves

$$E_{n+1}(x) = B(2x\tilde{E}_n - \tilde{E}_{n-1}(x)) = \tilde{E}_{n+1}(x),$$

which completes the proof. □

In the next proposition, we see that there is a close relation between the polynomial sequences $\{E_n(x)\}$ and $\{U_n(x)\}$.

**Proposition 3.** *For any integer $n \geq 2$, we have*

$$(x^2 - 1)E_n(x) = 2x^{n+2} - U_n(x) + x^2U_{n-2}(x). \tag{8}$$

*Proof.* We rewrite (8) as

$$U_n(x) - x^2U_{n-2}(x) = 2x^{n+2} - (x^2 - 1)E_n(x), \tag{9}$$

and denote the left-hand side and the right-hand side of (9) by $\Phi_n(x)$ and $\Psi_n(x)$, respectively, i.e., let $\Phi_n(x) = U_n(x) - x^2U_{n-2}(x)$ and $\Psi_n(x) = 2x^{n+2} - (x^2-1)E_n(x)$ for $n \geq 2$. Then it is easy to compute

$$\Phi_2(x) = 3x^2 - 1 = \Psi_2(x), \qquad \Phi_3(x) = 6x^3 - 4x = \Psi_3(x)$$

and

$$\Phi_{n+2}(x) = 2x\Phi_{n+1}(x) - \Phi_n(x) \qquad (n \geq 2) \tag{10}$$

using recurrence relations of $U_n(x)$. So it suffices to show that the polynomial sequence $\{\Psi_n(x)\}$ satisfies the same recurrence relation with (10) of $\{\Phi_n(x)\}$. But

$$\begin{aligned} &\Psi_{n+2}(x) - 2x\Psi_{n+1}(x) + \Psi_n(x) \\ &= 2x^{n+4} - 2x \cdot 2x^{n+3} + 2x^{n+2} - (x^2-1)\left(E_{n+2}(x) - 2xE_{n+1}(x) + E_n(x)\right) \\ &= -2x^{n+4} + 2x^{n+2} - (x^2-1)(-2x^{n+2}) = 0. \end{aligned}$$

This proves the proposition. □

We recall (1) and (2) for the definitions of $\{V_n(x)\}$ and $\{E_n(x)\}$. We now show a relation between $\{E_n(x)\}$ and $\{V_n(x)\}$. For any integer $n \geq 0$,

$$(x^2-1)V_n(x) = x^{n+2} - T_n(x) \tag{11}$$

and in [1], we see that

$$V_n(x) = x^n - \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n}{2k} (x^2-1)^{k-1} x^{n-2k}.$$

In particular,

$$V_n(1) = 1 - \binom{n}{2}, \qquad V_n(-1) = (-1)^n \left(1 - \binom{n}{2}\right). \tag{12}$$

With the well-known explicit expression

$$U_n(x) = (n+1)x^n + \sum_{k=1}^{\lfloor n/2 \rfloor} \binom{n+1}{2k+1} (x^2-1)^k x^{n-2k}$$

and (8), we may compute that

$$\begin{aligned} E_n(x) = 2x^n - &\sum_{k=1}^{\lfloor n/2 \rfloor - 1} \left( \binom{n+1}{2k+1} - \binom{n-1}{2k+1} \right) (x^2-1)^{k-1} x^{n-2k} - \\ &\binom{n+1}{2\lfloor n/2 \rfloor + 1} (x^2-1)^{\lfloor n/2 \rfloor - 1} x^{n - 2\lfloor n/2 \rfloor} \end{aligned} \tag{13}$$

and the special values

$$E_n(0) = \begin{cases} (-1)^{n/2} & \text{if } n \text{ is even}, \\ 0 & \text{if } n \text{ is odd}, \end{cases} \tag{14}$$

and

$$E_n(1) = 2 - (n-1)^2, \qquad E_n(-1) = (-1)^n \left(2 - (n-1)^2\right). \tag{15}$$

There is a relation between $\{E_n(x)\}$ and $\{V_n(x)\}$ as follows.

**Corollary 4.** *For any integer $n \geq 0$, we have*

$$E_n(x) = V_n(x) + xV_{n-1}(x).$$

*Proof.* Using (8), (11) and a well known identity $T_n(x) = U_n(x) - xU_{n-1}(x)$, we have

$$\begin{aligned}(x^2-1)(E_n(x) - V_n(x)) &= x^{n+2} - U_n(x) + x^2U_{n-2}(x) + T_n(x) \\ &= x^{n+2} - U_n(x) + x^2U_{n-2}(x) + U_n(x) - xU_{n-1}(x) \\ &= x^{n+2} + x(xU_{n-2}(x) - U_{n-1}(x)) \\ &= x^{n+2} - xT_n(x) = x(x^{n+1} - T_{n-1}(x)) \\ &= (x^2-1)xV_{n-1}(x),\end{aligned}$$

and so for $x \neq \pm 1$, $E_n(x) = V_n(x) + xV_{n-1}(x)$. The cases when $x = \pm 1$ can be checked from (12) and (15). □

## 3. Zero distribution and irreducibility

The polynomial

$$(x^2-1)V_n(x) = x^{n+2} - T_n(x)$$

has $n-2$ real zeros in the interval $(-1,1)$, zeros $\pm 1$ while the remaining two are real, large, and of opposite sign. For the proof, see Proposition 6 of [1]. In fact, we may show further that the consecutive polynomials $x^{n+2} - T_n(x)$ and $x^{n+1} - T_{n-1}(x)$ in the sequence $\{(x^2-1)V_n(x)\} = \{x^{n+2} - T_n(x)\}$ have no common zeros in $(-1,1)$. More generally, we can prove the following.

**Proposition 5.** *Let $n$ be an integer $\geq 2$. For $m > n$, let*

$$f_{m,n}(x) = x^m - T_n(x).$$

*Then $f_{m,n}(x) = x^m - T_n(x)$ and $f_{m-1,n-1}(x) = x^{m-1} - T_{n-1}(x)$ have no common zeros in the open interval $(-1,1)$.*

*Proof.* Consider the polynomial

$$F_{m,n}(x) := 2x^m - (T_n(x) + xT_{n-1}(x)) = f_{m,n}(x) + xf_{m-1,n-1}(x).$$

If $f_{m-1,n-1}(x) = 0$, then

$$\begin{aligned}F_{m,n}(x) &= x^m - T_n(x) = xT_{n-1}(x) - T_n(x) \\ &= (1-x^2)U_{n-2}(x),\end{aligned} \tag{16}$$

and if $f_{m,n}(x) = 0$, then

$$F_{m,n}(x) = x(x^{m-1} - T_{n-1}(x)) = x^m - xT_{n-1}(x) \tag{17}$$

$$= T_n(x) - xT_{n-1}(x) = -(1-x^2)U_{n-2}(x),$$

If $f_{m-1,n-1}(x)$ and $f_{m,n}(x)$ have a common zero in $(-1,1)$, it must be a zero of $U_{n-2}(x)$. But no zero of $U_{n-2}(x)$ satisfies

$$f_{m-1,n-1}(x) = 0, \quad \text{i.e.,} \quad x^{m-1} = T_{n-1}(x).$$

In fact, the $n-2$ critical points of $T_{n-1}(x)$ are the zeros of $U_{n-2}(x)$ in $(-1,1)$ and $T_{n-1}(x)$ has the values $\pm 1$ at those points. But the absolute values of $x^{m-1}$, where $-1 < x < 1$, is strictly less than 1. □

*Remark* 6. From many numerical computations, it seems that, like $V_n(x)$ in [1], all the zeros of $E_n(x)$ are real, and $n-2$ of them lie in the open interval $(-1,1)$, while the remaining two are large real numbers with opposite signs. From Corollary 4, we see that

$$\begin{aligned}(x^2-1)E_n(x) &= 2x^{n+2} - (T_n(x) + xT_{n-1}(x)) \\ &= f_{n+2,n}(x) + xf_{n+1,n-1}(x),\end{aligned} \tag{18}$$

where $f_{n+2,n}(x)$ and $f_{n+1,n-1}(x)$ are polynomials defined in Proposition 5. By Proposition 5, the polynomials $f_{n+2,n}(x)$ and $f_{n+1,n-1}(x)$ have no common zeros in $(-1,1)$. We now conjecture that the zeros of $f_{n+2,n}(x)$ and $f_{n+1,n-1}(x)$ interlace. A lot of computer algebra computations lead us to believe the truth of the conjecture, but the author does not know how to prove this. If this is true, we will have the proof that $n-2$ of the zeros of $E_n(x)$ lie in $(-1,1)$.

From Proposition 6 of [1], we see that the largest real zero of $f_{n+2,n}(x)$ and $f_{n+1,n-1}(x)$ are between

$$(\sqrt{2})^{n-1} - n(\sqrt{2})^{-n+1} \quad \text{and} \quad (\sqrt{2})^{n-1}$$

and

$$(\sqrt{2})^{n-2} - (n-1)(\sqrt{2})^{-n+2} \quad \text{and} \quad (\sqrt{2})^{n-2},$$

respectively, and we may check that for $n \geq 6$,

$$(\sqrt{2})^{n-2} < (\sqrt{2})^{n-1} - n(\sqrt{2})^{-n+1}.$$

Since both $f_{n+2,n}(x)$ and $f_{n+1,n-1}(x)$ are negative on the interval $(1, (\sqrt{2})^{n-2} - (n-1)(\sqrt{2})^{-n+2})$, and positive on the interval $((\sqrt{2})^{n-1}, \infty)$, the largest real zero of $E_n(x)$ must be between

$$(\sqrt{2})^{n-2} - (n-1)(\sqrt{2})^{-n+2} \quad \text{and} \quad (\sqrt{2})^{n-1}. \tag{19}$$

In the next proposition, we obtain a sharper bound for the size of a large real zero of $E_n(x)$. In fact, the real zero $r_n$ in the proposition below seems to be the largest real zero of $E_n(x)$, and for example, with $n = 20$, compared with (19) and the inequality (20) below,

$$\begin{aligned}&(\sqrt{2})^{n-2} - (n-1)(\sqrt{2})^{-n+2} = 511.96\cdots \\ &< \sqrt{3}\cdot(\sqrt{2})^{n-3} - \sqrt{3}\cdot n\cdot(\sqrt{2})^{-n+3} = 626.97\cdots\end{aligned}$$

$$< \sqrt{3}\cdot(\sqrt{2})^{n-3} = 627.06\cdots < (\sqrt{2})^{n-1} = 724.07\ldots,$$

and the larger $n$ give sharper bounds in (20).

**Proposition 7.** *For any integer $n \geq 8$, there is a real zero $r_n$ of $E_n(x)$ such that*

$$\sqrt{3}\cdot(\sqrt{2})^{n-3} - \sqrt{3}\cdot n\cdot(\sqrt{2})^{-n+3} < r_n < \sqrt{3}\cdot(\sqrt{2})^{n-3}. \tag{20}$$

*Proof.* For $n \geq 2$ and $x \geq 1$, it can be easily checked that $T_n(x) < 2^{n-1}x^n$ and so, with (18) and $x = \sqrt{3}\cdot 2^{\frac{n-3}{2}}$

$$\begin{aligned}
(x^2-1)E_n(x) &> (\sqrt{3})^{n+2}2^{1+\frac{(n-3)(n+2)}{2}} - (\sqrt{3})^n 2^{(n-1)+\frac{n(n-3)}{2}} \\
&\quad - (\sqrt{3})^n 2^{(n+2)+\frac{n(n-3)}{2}} \\
&= (\sqrt{3})^{n+2}2^{\frac{n^2-n-4}{2}} - (\sqrt{3})^n 2^{\frac{n^2-n-2}{2}} - (\sqrt{3})^n 2^{\frac{n^2-n-4}{2}} \\
&= (\sqrt{3})^n 2^{\frac{n^2-n-4}{2}}(3-2-1) = 0.
\end{aligned}$$

We now find a near-by point $x$ to $\sqrt{3}\cdot 2^{\frac{n-3}{2}}$ satisfying $(x^2-1)E_n(x) < 0$. Using well-known inequalities $\sqrt{x^2-1} > x - \frac{1}{x}$ for $x > 1$ and $(1-z)^n \geq (1-zn)$ for $n \geq 1$, $0 \leq z \leq 1$, we have that for $x > 1$,

$$\begin{aligned}
T_n(x) &> \frac{1}{2}(x+\sqrt{x^2-1})^n > \frac{1}{2}\left(2x - \frac{1}{x}\right)^n \\
&= \frac{1}{2}(2x)^n\left(1-\frac{1}{2x^2}\right)^n \geq \frac{1}{2}(2x)^n\left(1-\frac{n}{2x^2}\right),
\end{aligned}$$

and so

$$\begin{aligned}
(x^2-1)E_n(x) &< 2x^{n+2} - \left(\frac{1}{2}(2x)^n\left(1-\frac{n}{2x^2}\right) + \frac{1}{2}(2x)^{n-1}\left(1-\frac{n-1}{2x^2}\right)x\right) \\
&= 2x^{n-2}\left[x^4 - 2^{n-2}\left(x^2-\frac{n}{2}\right) - 2^{n-3}\left(x^2-\frac{n-1}{2}\right)\right] \\
&= 2x^{n-2}\left[x^4 - 3\cdot 2^{n-3}x^2 + 2^{n-4}(3n-1)\right]
\end{aligned}$$

We now consider the polynomial in the last

$$e_n(x) := x^4 - 3\cdot 2^{n-3}x^2 + 2^{n-4}(3n-1).$$

Write

$$x = x_n := \sqrt{3}\cdot 2^{\frac{n-3}{2}} - \sqrt{3}\cdot n\cdot 2^{-\frac{n-3}{2}}.$$

Then

$$(x_n^2 - 3\cdot 2^{n-4})^2 = (3\cdot 2^{n-4} - 6n + 3n^2 2^{-n+3})^2 < (3\cdot 2^{n-4} - 3n)^2, \tag{21}$$

where the last inequality can be easily shown for $n \geq 8$. So

$$\begin{aligned}
e_n(x_n) &= (x_n^2 - 3\cdot 2^{n-4})^2 - 9\cdot 2^{2n-8} + 2^{n-4}(3n-1) \\
&< 9\cdot 2^{2n-8} - 18\cdot 2^{n-4}n + 9n^2 - 9\cdot 2^{2n-8} + 2^{n-4}(3n-1)
\end{aligned}$$

$$= \frac{1}{16}(144n^2 - 2^n(15n+1)) < 0.$$

This completes the proof of $(x^2-1)E_n(x) < 0$ at $x = x_n$ for $n \geq 8$. □

The Chebyshev polynomial of the second kind $U_n(x)$ have a well-known factorization over $\mathbb{Q}$ (see p. 229 of [3]) and $U_n(x)$ is reducible over $\mathbb{Q}$ for every $n \geq 2$. However $E_n(x)$ seems to behave differently. If $n$ is an even integer, then the leading coefficient of $E_n(x)$ is 2, and so it follows from Proposition 1 and (14) that $E_n(x)$ is in Eisenstein form with respect to the prime 2. That is, $E_n(x)$ is irreducible over $\mathbb{Q}$. The case when $n$ is odd is not obvious to decide the irreducibility of $E_n(x)$. It seems that $E_n(x)/(2x)$ is irreducible over $\mathbb{Q}$ by many numerical computations. This remains an open problem, but the coefficients of $E_n(x)/(2x)$ does not give enough clues to the irreducibility. In fact, we may compute the coefficients as follows which is also of independent interest.

**Proposition 8.** *Let $n$ be an odd integers $\geq 1$. Then with $n = 2m+1$,*

$$\frac{E_n(x)}{2x} = x^{2m} - \sum_{r=0}^{m-2}(-1)^r\left(\sum_{k=r+1}^{m-1}\binom{2m}{2k+1}\frac{(2k+1)(2m+1-k)}{(2m-2k)(2m-2k+1)}\binom{k-1}{r}\right)$$
$$x^{2m-2r-2} - (m+1)\sum_{r=0}^{m-1}\binom{m-1}{r}(-1)^{m-1-r}x^{2r}.$$

*Proof.* Let $n = 2m+1$, Then by (13),

$$\frac{E_n(x)}{2x} = x^{2m} - \sum_{k=1}^{m-1}\binom{2m}{2k+1}\frac{(2k+1)(2m+1-k)}{(2m-2k)(2m-2k+1)}(x^2-1)^{k-1}x^{2m-2k}$$
$$- (m+1)(x^2-1)^{m-1}.$$

We use the binomial expansion

$$(x^2-1)^{k-1} = \sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^{k-1-j}x^{2j}.$$

Then with $r := k - j$,

$$\sum_{k=1}^{m-1}\binom{2m}{2k+1}\frac{(2k+1)(2m+1-k)}{(2m-2k)(2m-2k+1)}(x^2-1)^{k-1}x^{2m-2k}$$
$$= \sum_{k=1}^{m-1}\binom{2m}{2k+1}\frac{(2k+1)(2m+1-k)}{(2m-2k)(2m-2k+1)}\sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^{k-1-j}x^{2m-2k+2j}$$
$$= \sum_{k=1}^{m-1}\binom{2m}{2k+1}\frac{(2k+1)(2m+1-k)}{(2m-2k)(2m-2k+1)}\sum_{r=1}^{k}\binom{k-1}{r-1}(-1)^{r-1}x^{2m-2r}$$
$$= \sum_{k=1}^{m-1}\binom{2m}{2k+1}\frac{(2k+1)(2m+1-k)}{(2m-2k)(2m-2k+1)}\sum_{r=0}^{k-1}\binom{k-1}{r}(-1)^{r}x^{2m-2r-2}$$

$$= \sum_{r=0}^{m-2} (-1)^r \left( \sum_{k=r+1}^{m-1} \binom{2m}{2k+1} \frac{(2k+1)(2m+1-k)}{(2m-2k)(2m-2k+1)} \binom{k-1}{r} \right) x^{2m-2r-2}$$

and

$$(m+1)(x^2-1)^{m-1} = (m+1) \sum_{r=0}^{m-1} \binom{m-1}{r} (-1)^{m-1-r} x^{2r}.$$

These completes the proof. □

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