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MOMENT APPROACH TO THE ADMISSIBLE CONTROL PROBLEM FOR LINEAR SYSTEM

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Abstract. In this paper, we consider the admissible control problem for the linear systems by using the solution of the Hausdorff moment problem. In addition, we consider the admissible control problem for SIR epidemic model.

1. Introduction

Consider the following linear continuous system

$$\dot{x}(t) = Ax(t) + bu(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector, $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$. Given an initial condition $x_0 \in \mathbb{R}^n$, and a time θ, find one of the controls $|u(t)| \leq 1$ such that the trajectory from x_0 of the system (1) arrives to the origin in time θ. This problem is called the admissible control (AC) problem ([3]).

The authors of [3] gave the following control with no restriction,

$$u(t) = -b^* e^{-A^* t} N^{-1}(\theta)\, x_0, \quad \text{where } N(\theta) = \int_0^\theta e^{-At} bb^* e^{-A^* t} \mathrm{d}t. \tag{2}$$

Moment problem is related to operator theory and has many applications (see [4], [5], [6], and [7], etc). For the bounded restriction, the solution of AC problem (1) is related to the following two moment problems.

1.1. The Markov moment problem and the Hausdorff moment problem

Let $\mathcal{C}_{0,L}$ be the set of all measurable functions on $[0,\theta]$ such that $0 \leq f(\tau) \leq L$ for all $\tau \in [0,\theta]$. Then the L-Markov moment problem (MMP) for an interval

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$[0,\theta]$ is stated as follows: Given a finite sequence of real numbers $c_0, c_1, \ldots, c_k$, find the set of functions $f \in \mathcal{C}_{0,L}$ such that

$$c_j = \int_0^\theta \tau^j f(\tau)\, \mathrm{d}\tau, \quad j = 0, 1, \ldots, k.$$

Let $\mathcal{M}[0,\theta]$ be the set of all nonnegative measures on $[0,\theta]$. Then the Hausdorff moment problem (HMP) for an interval $[0,\theta]$ is stated as follows: Given a finite sequence of real numbers $s_0, s_1, \ldots, s_k$, find the set of measure $\sigma \in \mathcal{M}[0,\theta]$ such that

$$s_j = \int_0^\theta \tau^j \mathrm{d}\sigma(\tau), \quad j = 0, 1, \ldots, k.$$

Recall from [7] that there is a bijection between the set $\mathcal{C}_{0,L}$ and measures $\sigma \in \mathcal{M}[0,\theta]$ satisfying $\int_0^\theta \mathrm{d}\sigma(\tau) = 1$ is given by

$$\int_0^\theta \frac{\mathrm{d}\sigma(\tau)}{\tau - z} = -\frac{1}{z} \exp\left(\frac{1}{L}\int_0^\theta \frac{f(\tau)\,\mathrm{d}\tau}{z - \tau}\right),$$

which determine the relation between $(c_j)_{j=0}^{k-1}$ and $(s_j)_{j=0}^{k}$: $s_0 = 1, s_1 = c_1$, and

$$s_k = \frac{1}{k!}\begin{vmatrix} c_1 & -1 & \cdots & 0 \\ 2c_2 & c_1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ (k-1)\,c_{k-1} & (k-2)\,c_{k-2} & \ddots & -(k-1) \\ kc_k & (k-1)\,c_{k-1} & \cdots & c_1 \end{vmatrix}, \quad k \geq 2. \tag{3}$$

By [7, Theorem 2.1], we have the following result.

Proposition 1. *The MMP is solvable with moments $(c_j)_{j=0}^{n-1}$ if and only if the HMP with $(s_j)_{j=0}^{n}$ is solvable.*

1.2. Algorithm for the control

In [3], the authors introduced the method for obtaining the admissible control of the system (1) with $A := (\delta_{i,j+1})_{i,j=1}^{n}$ and $b := (1, 0, \ldots, 0)^\intercal$, where $\delta_{i,j+1}$ is the Kronecker symbol, and $^\intercal$ denotes the transpose operation. We summarize that as following algorithm.

I. Calculate all data $c_i = \frac{\theta^i + (-1)^i i! x_0^i}{2i}$, $i = 1, 2, \ldots$.
II. Calculate s_i by (3).

III. Calculate H_i, u_i, v_i, T and $R_T(z)$ by the following relations.

	if $n = 2k+1$	if $n = 2k$
H_1	$(s_{i+j+1})_{i,j=0}^{k}$	$(s_{i+j})_{i,j=0}^{k}$
H_2	$(\theta s_{i+j} - s_{i+j+1})_{i,j=0}^{k}$	$(\theta s_{i+j+1} - s_{i+j+2})_{i,j=0}^{k-1}$
u_1	$(-s_0, -s_1, \ldots, -s_k)^{\intercal}$	$(0, -s_0, \ldots, -s_{k-1})^{\intercal}$
T	$(\delta_{i,j+1})_{i,j=0}^{k}$	$(\delta_{i,j+1})_{i,j=0}^{k}$
u_2	$(\theta T - I)\, u_1$	$(s_1 - \theta s_0, s_2 - \theta s_1, \ldots, s_k - \theta s_{k-1})^{\intercal}$
v_1	$(1, 0, \ldots, 0)^{\intercal} \in \mathbb{R}^{k+1}$	$(1, 0, \ldots, 0)^{\intercal} \in \mathbb{R}^{k+1}$
v_2	$(1, 0, \ldots, 0)^{\intercal} \in \mathbb{R}^{k+1}$	$(1, 0, \ldots, 0)^{\intercal} \in \mathbb{R}^{k}$
$R_T(z)$	$(I - zT)^{-1}$	$(I - zT)^{-1}$

IV. Calculate U_{11}, U_{12}, U_{21}, and U_{22} by the following relations.

	if n is odd number	if n is even number
$U_{11}(z)$	$1 - zu_2^* R_{T_1^*}(z)\, H_2^{-1} v_1$	$1 - zu_1^* R_{T_1^*}(z)\, H_1^{-1} v_1$
$U_{12}(z)$	$u_1^* R_{T_1^*}(z)\, H_1^{-1} u_1$	$M - zu_1^* R_{T_1^*}(z)\, H_1^{-1} v_1 M + zu_1^* R_{T_1^*}(z)\, H_1^{-1} u_1$
$U_{21}(z)$	$-(\theta - z)\, zv_1^* R_{T_1^*}(z)\, H_2^{-1} v_1$	$-zv_1^* R_{T_1^*}(z)\, H_1^{-1} v_1$
$U_{22}(z)$	$1 + zv_1^* R_{T_1^*}(z)\, H_1^{-1} u_1$	$1 - zv_1^* R_{T_1^*}(z)\, H_1^{-1} v_1 M + zv_1^* R_{T_1^*}(z)\, H_1^{-1} u_1$
M		$\left(1 + \theta \left(u_1^* H_1^{-1} v_1 - u_2^* H_2^{-1} v_2\right)\right) \left(\theta v_1^* H_1^{-1} v_1\right)^{-1}$

V. Let $z = t + i\epsilon$, and calculate $-zs(z)$ by the following relations.

$$-zs(z) = \frac{U_{11}(\theta - (t + i\epsilon))(F + iG + i\pi) + U_{12}}{U_{21}(\theta - (t + i\epsilon))(F + iG + i\pi) + U_{22}},$$

where

$$F = \frac{1}{2} \ln \frac{(\theta - t)^2 + \epsilon^2}{t^2 + \epsilon^2}, \quad G = \arctan \frac{\theta \epsilon}{t^2 - Tt + \epsilon^2}.$$

VI. Let $\epsilon = 0$, and calculate the real part X and the imaginary part Y of $-zs(z)$.

VII. Finally, we can obtain

$$u(t) = -\frac{2}{\pi} \arg \frac{Y}{X} - 1.$$

Furthermore, in [2], the authors considered the extremal controls of the AC problem. In recent years, many researches studied SIR epidemic model (see [9], [10], etc). In this paper, we will obtain the clear solutions of the admissible control problem for the canonical system with $n = 1, 2$ and 3 by using the above algorithm. As in application, we consider the admissible control problem for SIR epidemic model.

2. The solutions of the AC problem

By using the algorithm step by step, we can obtain the results.

Theorem 2. *The AC problem* (*for* $n = 1$)

$$\dot{x} = u(t), \ |u| \le 1, \ x(0) = x_0, \ s.t. \ x(\theta) = 0, \tag{4}$$

is admissible if and only if $|x_0| \le \theta$. *In this case,* $u(t) = -\frac{2}{\pi}\arg\frac{Y}{X} - 1$, *where*

$$X = \frac{1}{8}\pi^2 c_{11} + \frac{1}{16}\left(c_{12}\left(\ln\frac{\theta - t}{t}\right) + c_{13}\right)\left(c_{14}\left(\ln\frac{\theta - t}{t}\right) + c_{15}\right),$$

$$Y = \frac{1}{16}\pi(t-\theta)c_{16} \le 0 \quad \textit{with}$$

$$c_{11} = t(t-\theta)^3(\theta - x_0)^2(2t - \theta - x_0), \quad c_{12} = 2t(t-\theta)^2(\theta - x_0),$$

$$c_{13} = (\theta + x_0)(2t - \theta + x_0), \quad c_{14} = (\theta - x_0)(t-\theta)(2t - \theta - x_0),$$

$$c_{15} = 2(\theta + x_0), \quad c_{16} = (\theta - x_0)^2(\theta + x_0)^2.$$

Corollary 3. *The AC problem (for* $n = 1$*)*

(5) $$\dot{x} = ax + bu(t), \ a < 0, \ b \ne 0, \ |u| \le 1, \ x(0) = x_0, \ \textit{s.t.} \ x(\theta) = 0,$$

is admissible if and only if $|x_0| \le \theta$. *In this case,* $u(t) = \frac{1}{b}\left(-\frac{2}{\pi}\arg\frac{Y}{X} - 1\right)e^{at}$, *where* X *and* Y *are as in Theorem 2.*

Proof. Let $z = xe^{-at}$, then by Theorem 2, we know that

$$\dot{z} = u(t)be^{-at} := \tilde{u}(t), |\tilde{u}| \le 1, \ z(0) = z_0, \ \text{s.t.} \ z(\theta) = 0,$$

is admissible if and only if $|z_0| \le \theta$, that is, $|x_0| \le \theta$. In this case, $\tilde{u}(t) = -\frac{2}{\pi}\arg\frac{Y}{X} - 1$, that is, $u(t) = \frac{1}{b}\left(-\frac{2}{\pi}\arg\frac{Y}{X} - 1\right)e^{at}$. □

Proposition 4. *The AC problem (for* $n = 2$*)*

$$\begin{aligned} \dot{x}_1 &= \tilde{u}, \qquad |\tilde{u}| \le 1, \\ \dot{x}_2 &= x_1, \\ \textit{such that } x_1(\theta) &= x_2(\theta) = 0 \end{aligned}$$

is admissible if and only if

$$|x_{01}| \le \theta \quad \textit{and} \quad |x_{02} + 2\theta x_{01}| \le \theta^2 - x_{01}^2.$$

In this case, $u(t) = -\frac{2}{\pi}\arg\frac{Y}{X} - 1$, *where*

$$X = \pi^2 c_{21} + t(t-\theta)\left(c_{22}\left(\ln\frac{\theta - t}{t}\right) + c_{23}\right)\left(c_{24}\left(\ln\frac{\theta - t}{t}\right) + c_{25}\right),$$

$$Y = \pi c_{26}(t - \theta) \quad \textit{with}$$

$$\begin{aligned} c_{21} = {} & t^2\theta^2\left(-4x_{02} + \theta^2 - x_{01}^2 - 2\theta x_{01}\right)^2(t-\theta)^2 \\ & \times\left(-4x_{02} + 4t\theta - 3\theta^2 - 4tx_{01} - x_{01}^2 + 2\theta x_{01}\right) \\ & \times\left(-4x_{02} + 4t\theta - \theta^2 - 4tx_{01} + x_{01}^2 - 2\theta x_{01}\right), \end{aligned}$$

$$\begin{aligned} c_{22} = {} & \theta\left(-4x_{02} + \theta^2 - x_{01}^2 - 2\theta x_{01}\right)(t-\theta) \\ & \times\left(-4x_{02} + 4t\theta - \theta^2 - 4tx_{01} + x_{01}^2 - 2\theta x_{01}\right), \end{aligned}$$

$$c_{23} = -\left(4x_{02} + \theta^2 - x_{01}^2 + 2\theta x_{01}\right)\left(4x_{02} + 4t\theta - 3\theta^2 + 4tx_{01} - x_{01}^2 - 2\theta x_{01}\right),$$

$$c_{24} = t\theta\left(-4x_{02} + \theta^2 - x_{01}^2 - 2\theta x_{01}\right)\left(-4x_{02} + 4t\theta - 3\theta^2 - 4tx_{01} - x_{01}^2 + 2\theta x_{01}\right),$$

$$c_{25} = -\left(4x_{02}+\theta^2-x_{01}^2+2\theta x_{01}\right)\left(4x_{02}+4t\theta-\theta^2+4tx_{01}+x_{01}^2+2\theta x_{01}\right),$$
$$c_{26} = t\theta^2\left(4x_{02}+\theta^2-x_{01}^2+2\theta x_{01}\right)^2\left(-4x_{02}+\theta^2-x_{01}^2-2\theta x_{01}\right)^2.$$

Let

$$S(\theta)=\{(x_{01},x_{02})\,|\,(x_1(0),x_2(0))=(x_{01},x_{02}) \text{ and } (x_1(\theta),x_2(\theta))=(0,0)\}.$$

Then $S(1), S(2)$ and $S(3)$ are as the following:

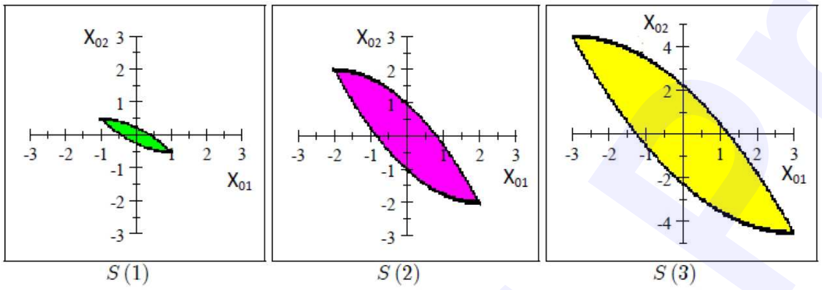

Fig. 1. The plots of $S(\theta)$

where

$$S(1)=\left\{(x_{01},x_{02})\,|\,|x_{01}|\le 1, \text{ and } |x_{02}+2x_{01}|\le 1-x_{01}^2\right\},$$
$$S(2)=\left\{(x_{01},x_{02})\,|\,|x_{01}|\le 2, \text{ and } |x_{02}+4x_{01}|\le 4-x_{01}^2\right\},$$
$$S(3)=\left\{(x_{01},x_{02})\,|\,|x_{01}|\le 3, \text{ and } |x_{02}+6x_{01}|\le 9-x_{01}^2\right\}.$$

Proposition 5. *The AC problem (for $n=3$)*

$$\dot{x}_1=\tilde{u}, \qquad |\tilde{u}|\le 1,$$
$$\dot{x}_2=x_1,$$
$$\dot{x}_3=x_2,$$
$$\text{such that } x_1(\theta)=x_2(\theta)=x_3(\theta)=0$$

is admissible if and only if $|x_{01}|\le\theta$ and

$$\left(3\theta^4-48x_{02}^2-x_{01}^4-6\theta^2x_{01}^2+96x_{01}x_{03}\right)\ge\left|96\theta x_{03}-4\theta x_{01}^3+48\theta^2x_{02}+12\theta^3x_{01}\right|.$$

In this case, $u(t)=-\frac{2}{\pi}\arg\frac{Y}{X}-1$, where

$$X = 4\pi^2t^2(t-\theta)^3c_{31}^2c_{32}c_{33}+t\left(4t(t-\theta)^2c_{31}c_{33}\left(\ln\frac{\theta-t}{t}\right)+c_{34}c_{35}\right)$$
$$\times\left((t-\theta)c_{31}c_{32}\left(\ln\frac{\theta-t}{t}\right)+4c_{34}c_{36}\right),$$
$$Y=\pi t c_{31}^2c_{34}^2(t-\theta) \qquad \textit{with}$$
$$c_{31}=3\theta^4-48x_{02}^2-x_{01}^4-6\theta^2x_{01}^2-96\theta x_{03}+96x_{01}x_{03}+4\theta x_{01}^3-48\theta^2x_{02}$$
$$-12\theta^3x_{01},$$

$$
\begin{aligned}
c_{32} = {} & 24t^2\theta^2 + 3\theta^4 - 24t^2x_{01}^2 - 192tx_{03} - 48x_{02}^2 - x_{01}^4 - 6\theta^2x_{01}^2 \\
& + 96\theta x_{03} + 96x_{01}x_{03} - 24t\theta^3 + 8tx_{01}^3 - 96t^2x_{02} - 4\theta x_{01}^3 \\
& + 48\theta^2x_{02} + 12\theta^3x_{01} + 24t\theta x_{01}^2 + 24t\theta^2x_{01} - 48t^2\theta x_{01}, \\
c_{33} = {} & -48x_{03} - 3\theta^3 - 24tx_{02} - x_{01}^3 - 12\theta x_{02} - 12x_{01}x_{02} \\
& + 6t\theta^2 - 6tx_{01}^2 - 3\theta x_{01}^2 + 3\theta^2x_{01} - 12t\theta x_{01}, \\
c_{34} = {} & 3\theta^4 - 48x_{02}^2 - x_{01}^4 - 6\theta^2x_{01}^2 + 96\theta x_{03} + 96x_{01}x_{03} - 4\theta x_{01}^3 \\
& + 48\theta^2x_{02} + 12\theta^3x_{01}, \\
c_{35} = {} & 24t^2\theta^2 + 3\theta^4 - 24t^2x_{01}^2 + 192tx_{03} - 48x_{02}^2 - x_{01}^4 - 6\theta^2x_{01}^2 \\
& - 96\theta x_{03} + 96x_{01}x_{03} - 24t\theta^3 - 8tx_{01}^3 + 96t^2x_{02} \\
& + 4\theta x_{01}^3 - 48\theta^2x_{02} - 12\theta^3x_{01} + 24t\theta x_{01}^2 - 24t\theta^2x_{01} + 48t^2\theta x_{01}, \\
c_{36} = {} & 48x_{03} - 3\theta^3 + 24tx_{02} + x_{01}^3 + 12\theta x_{02} - 12x_{01}x_{02} \\
& + 6t\theta^2 - 6tx_{01}^2 - 3\theta x_{01}^2 - 3\theta^2x_{01} + 12t\theta x_{01}.
\end{aligned}
$$

3. Some examples

Example 1. The case of $n = 1$. Let $\theta = 3, x_0 = 1$, then

$$
\begin{aligned}
u(t) = {} & -\frac{2}{\pi}\arg\frac{g_1(t)}{h_1(t)} - 1, \quad g_1(t) = 4\pi(t-3), \\
h_1(t) = {} & \pi^2 t(t-2)(t-3)^3 \\
& + \left((t-2)(t-3)\ln\frac{(3-t)}{t} + 2\right)\left(t(t-3)^2\ln\frac{(3-t)}{t} + 2(t-1)\right).
\end{aligned}
$$

Since the roots of $h_1(t)$ on $[0,3]$ are

$$
t_1 \approx 0.033658, \quad t_2 \approx 2.11099, \quad t_3 \approx 2.23092,
$$

we obtain

$$
u(t) = \begin{cases}
-\frac{2}{\pi}\left(\arctan\frac{g_1(t)}{h_1(t)} - \pi\right) - 1, & \text{if } \ 0 \le t \le 0.033658, \\
-\frac{2}{\pi}\left(\arctan\frac{g_1(t)}{h_1(t)}\right) - 1, & \text{if } \ 0.033658 < t \le 2.11099, \\
-\frac{2}{\pi}\left(\arctan\frac{g_1(t)}{h_1(t)} - \pi\right) - 1, & \text{if } \ 2.11099 < t \le 2.23092, \\
-\frac{2}{\pi}\left(\arctan\frac{g_1(t)}{h_1(t)}\right) - 1, & \text{if } \ 2.23092 < t \le 3.
\end{cases}
$$

Example 2. The case of $n = 2$. Let $x_{01} = 0, x_{02} = 1, \theta = 3$. Then

$$
\begin{aligned}
u(t) = {} & -\frac{2}{\pi}\arg\frac{g_2(t)}{h_2(t)} - 1, \quad g_2(t) = 38\,025\pi t(t-3), \\
h_2(t) = {} & 225\pi^2t^2(t-3)^2(12t-31)(12t-13) \\
& + t(t-3)\left(-156t + 299 + \left(180t^2 - 735t + 585\right)\ln\frac{(3-t)}{t}\right)
\end{aligned}
$$

$$\times \left(-156t + 65 + \left(180t^2 - 465t\right) \ln \frac{(3-t)}{t} \right).$$

Since the roots of $h_2(t)$ on $[0,3]$ are

$$t_1 \approx 0.0197331, \quad t_2 \approx 1.11522, \quad t_3 \approx 2.52602, \quad t_4 \approx 2.90912,$$

we obtain

$$u(t) = \begin{cases} -\frac{2}{\pi}\left(\arctan \frac{g_2(t)}{h_2(t)} - \pi\right) - 1, & \text{if} \quad 0 \le t \le 0.0197331, \\ -\frac{2}{\pi}\left(\arctan \frac{g_2(t)}{h_2(t)}\right) - 1, & \text{if} \quad 0.0197331 < t \le 1.11522, \\ -\frac{2}{\pi}\left(\arctan \frac{g_2(t)}{h_2(t)} - \pi\right) - 1, & \text{if} \quad 1.11522 < t \le 2.52602, \\ -\frac{2}{\pi}\left(\arctan \frac{g_2(t)}{h_2(t)}\right) - 1, & \text{if} \quad 2.52602 < t \le 2.90912, \\ -\frac{2}{\pi}\left(\arctan \frac{g_2(t)}{h_2(t)} - \pi\right) - 1, & \text{if} \quad 2.90912 < t \le 3. \end{cases}$$

Example 3. The case of $n = 3$. Let $\theta = 3, x_{01} = 0, x_{02} = \frac{1}{2}, x_{03} = 1$, then

$$u(t) = -\frac{2}{\pi} \arg \frac{g_3(t)}{h_3(t)} - 1, \quad g_3(t) = 40262429025\pi t\,(t-3),$$

$$\begin{aligned} h_3(t) = {} & 131469156t^2\pi^2 (2t-7) \left(-40t + 8t^2 + 35\right)(t-3)^3 \\ & + 194481t \left(13(t-3)\left(-40t + 8t^2 + 35\right) \ln \frac{(3-t)}{t} - 20(22t-5) \right) \\ & \times \left(52t(2t-7)(t-3)^2 \ln \frac{(3-t)}{t} - 5\left(-152t + 88t^2 - 91\right) \right). \end{aligned}$$

Since the roots of $h_3(t)$ on $[0,3]$ are

$$t_1 \approx 0.0210242, \quad t_2 \approx 1.11293, \quad t_3 \approx 2.39258,$$

we obtain

$$u(t) = \begin{cases} -\frac{2}{\pi}\left(\arctan \frac{g_3(t)}{h_3(t)} - \pi\right) - 1, & \text{if} \quad 0 \le t \le 0.0210242, \\ -\frac{2}{\pi}\left(\arctan \frac{g_3(t)}{h_3(t)}\right) - 1, & \text{if} \quad 0.0210242 < t \le 1.11293, \\ -\frac{2}{\pi}\left(\arctan \frac{g_3(t)}{h_3(t)} - \pi\right) - 1, & \text{if} \quad 1.11293 < t \le 2.39258, \\ -\frac{2}{\pi}\left(\arctan \frac{g_3(t)}{h_3(t)}\right) - 1, & \text{if} \quad 2.39258 < t \le 3. \end{cases}$$

4. The AC problem for SIR epidemic model

Epidemic models divide the population into three classes: the susceptible S, the infective I, and the removed R. Mena-Lorca and Hethcote founded the following SIR epidemic model:

$$\begin{cases} \dot{S} = \Lambda - \beta SI - dS + cI + \delta R, \\ \dot{I} = \beta SI - (r + d + \alpha + c)I, \\ \dot{R} = rI - (d + \delta)R, \end{cases} \tag{6}$$

where Λ is the recruitment rate, d the natural rate, β the infection rate, α the death rate to disease, r the recovery rate, c is the sensible rate without immunity, and δ the rate that the removed return to the susceptible. Let $R_0 = \frac{\beta\Lambda}{d(r+d+\alpha+c)}$. If $R_0 < 1$, there only exists the disease free equilibrium point $P(\frac{\Lambda}{d}, 0, 0)$, which is globally asymptotically stable (see [9]). In [1] and [8], the authors considered the domain of attraction for (6).

In this paper, we consider the case of $\beta = 0$, then $R_0 < 1$, so the disease free equilibrium point $P(\frac{\Lambda}{d}, 0, 0)$ is globally asymptotically stable. Let $x = S - \frac{\Lambda}{d}, y = I, z = R$, then P changes to origin that is the unique equilibrium point of the following linear system:

$$\begin{cases} \dot{x} = -dx + cy + \delta z, \\ \dot{y} = -(r + d + \alpha + c)y, \\ \dot{z} = ry - (d + \delta)z. \end{cases} \tag{7}$$

4.1. With no restriction

Let $\alpha = r = d = \delta = c = \frac{1}{2}$, then

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & -2 & 0 \\ 0 & \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u. \tag{8}$$

Since the controllability matrix

$$Q_c = \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 1 & -2 & 4 \\ 0 & \frac{1}{2} & -\frac{5}{2} \end{bmatrix}$$

is invertible, (8) is completely controllable. Take $x(0) = y(0) = z(0) = 1$, by (2), we can obtain

$$u(t) = 1.0627e^{2t} - 11.973e^{t} + 13.801e^{\frac{1}{2}t}.$$

4.2. With bounded restriction

We take the following linear transform

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & \frac{1}{3} & 1 \\ 0 & -2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix},$$

then system (8) changes to the following:

$$\begin{cases} \dot{\xi} = -\xi + \frac{1}{2}u(t), \\ \dot{\eta} = -2\eta - \frac{1}{2}u(t), \\ \dot{\zeta} = -\frac{1}{2}\zeta + \frac{5}{3}u(t). \end{cases} \tag{9}$$

The initial state point is $(1,1,1)^{\intercal}$ and $\theta = 2$. In this case, by using Corollary 3, we can obtain

$$\begin{aligned} X =& \left(\left(\ln\frac{2-t}{t}\right)(2t-3)(t-2)+6\right)\left(6t+2\left(\ln\frac{2-t}{t}\right)t(t-2)^2-3\right) \\ &+2\pi^2 t(2t-3)(t-2)^3, \\ Y =& \ 9\pi(t-2). \end{aligned}$$

The numerical root of X in $[0,2]$ is $t_1 \approx 6.218\,5\times10^{-2}$. Hence the control is given by the following:

$$u(t) = \begin{cases} -\frac{2}{\pi}\left(\arctan\left(\frac{Y}{X}\right)-\pi\right)-1, & \text{if } 0 \le t \le 0.062185, \\ -\frac{2}{\pi}\left(\arctan\left(\frac{Y}{X}\right)\right)-1, & \text{if } 0.062185 \le t \le 2. \end{cases}$$

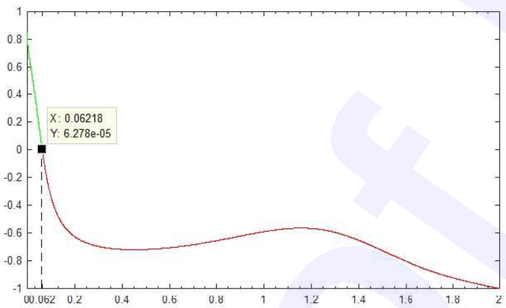

Fig. 2. The plot of $u(t)$

The plots of state vector of system (8) are as following. The left one is the trajectory from $(1,1,1)^{\intercal}$ to the origin in time $\theta = 2$ with no restriction; the right one is the trajectory from $\left(\frac{1}{3},-2,2\right)^{\intercal}$ to the origin in time $\theta = 2$ with bounded restriction.

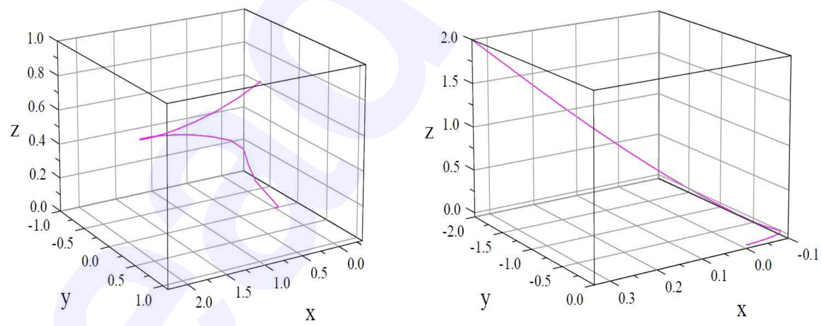

Fig. 3. The plots of state vector of system (5)

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