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UNITARILY INVARIANT NORM INEQUALITIES INVOLVING G_1 OPERATORS

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Abstract. In this paper, we present some upper bounds for unitarily invariant norms inequalities. Among other inequalities, we show some upper bounds for the Hilbert-Schmidt norm. In particular, we prove

$$\|f(A)Xg(B) \pm g(B)Xf(A)\|_2 \leq \left\| \frac{(I+|A|)X(I+|B|)+(I+|B|)X(I+|A|)}{d_A d_B} \right\|_2,$$

where $A, B, X \in \mathbb{M}_n$ such that A, B are Hermitian with $\sigma(A) \cup \sigma(B) \subset \mathbb{D}$ and f, g are analytic on the complex unit disk $\mathbb{D}$, $g(0) = f(0) = 1$, $\mathrm{Re}(f) > 0$ and $\mathrm{Re}(g) > 0$.

1. Introduction

Let $\mathbb{B}(\mathbf{H})$ be the C^*-algebra of all bounded linear operators on a separable complex Hilbert space $\mathbf{H}$ with the identity I. In the case when $\dim\mathbf{H} = n$, we determine $\mathbb{B}(\mathbf{H})$ by the matrix algebra $\mathbb{M}_n$ of all $n \times n$ matrices having associated with entries in the complex field. If $z \in \mathbb{C}$, then we write z instead of zI. For any operator A in the algebra $\mathbb{K}(\mathbf{H})$ of all compact operators, we denote by $\{s_j(A)\}$ the sequence of singular values of A, i.e., the eigenvalues $\lambda_j(|A|)$, where $|A| = (A^*A)^{\frac{1}{2}}$, enumerated as $s_1(A) \geq s_2(A) \geq \cdots$ in decreasing order and repeated according to multiplicity. If the rank A is n, we put $s_k(A) = 0$ for any $k > n$. Note that $s_j(X) = s_j(X^*) = s_j(|X|)$ and $s_j(AXB) \leq \|A\|\|B\|s_j(X)\,(j = 1, 2, \ldots)$ for all $A, B \in \mathbb{B}(\mathbf{H})$ and all $X \in \mathbb{K}(\mathbf{H})$.

A unitarily invariant norm is a map $|||\cdot||| : \mathbb{K}(\mathbf{H}) \longrightarrow [0, \infty]$ given by $|||A||| = g(s_1(A), s_2(A), \ldots)$, where g is a symmetric norming function. The set $\mathcal{C}_{|||\cdot|||}$ including $\{A \in \mathbb{K}(\mathbf{H}) : |||A||| < \infty\}$ is a closed self-adjoint ideal $\mathcal{J}$ of $\mathbb{B}(\mathbf{H})$ containing finite rank operators. It enjoys the property [6]:

$$|||AXB||| \leq \|A\|\|B\||||X||| \tag{1}$$

for $A, B \in \mathbb{B}(\mathbf{H})$ and $X \in \mathcal{J}$. Inequality (1) implies that $|||UXV||| = |||X|||$, where U and V are arbitrary unitaries in $\mathbb{B}(\mathbf{H})$ and $X \in \mathcal{J}$. In addition,

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employing the polar decomposition of $X = W|X|$ with W a partial isometry and (1), we have $|||X||| = |||\,|X|\,|||$. An operator $A \in \mathbb{K}(\mathbf{H})$ is called Hilbert-Schmidt if $\|A\|_2 = \left(\sum_{j=1}^{\infty} s_j^2(A)\right)^{1/2} < \infty$. The Hilbert-Schmidt norm is a unitarily invariant norm. For $A = [a_{ij}] \in \mathbb{M}_n$, it holds that $\|A\|_2 = \left(\sum_{i,j=1}^{n} |a_{i,j}|^2\right)^{1/2}$. We use the notation $A \oplus B$ for the diagonal block matrix $\mathrm{diag}(A, B)$. Its singular values are $s_1(A), s_1(B), s_2(A), s_2(B), \ldots$. It is evident that

$$\left|\left|\left|\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right|\right|\right| = |||\,|A| \oplus |B|\,||| = |||A \oplus B|||\,,$$

$$||A \oplus B|| = \max\{\|A\|, \|B\|\} \quad \text{and} \quad ||A \oplus B||_2 = \left(\|A\|_2^2 + \|B\|_2^2\right)^{\frac{1}{2}}.$$

The inequalities involving unitarily invariant norms have been of special interest; see e.g., [4, 9] and references therein.

An operator $A \in \mathbb{B}(\mathbf{H})$ is called G_1 operator if the growth condition

$$\|(z - A)^{-1}\| = \frac{1}{\mathrm{dist}(z, \sigma(A))}$$

holds for all z not in the spectrum $\sigma(A)$ of A, where $\mathrm{dist}(z, \sigma(A))$ denotes the distance between z and $\sigma(A)$. It is known that normal (more generally, hyponormal) operators are G_1 operators (see e.g., [15]). Let $A \in \mathbb{B}(\mathbf{H})$ and f be a function which is analytic on an open neighborhood Ω of $\sigma(A)$ in the complex plane. Then $f(A)$ denotes the operator defined on $\mathbf{H}$ by the Riesz-Dunford integral as

$$f(A) = \frac{1}{2\pi i} \int_C f(z)(z - A)^{-1} dz,$$

where C is a positively oriented simple closed rectifiable contour surrounding $\sigma(A)$ in Ω (see e.g., [8, p. 568]). The spectral mapping theorem asserts that $\sigma(f(A)) = f(\sigma(A))$. Throughout this note, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denotes the unit disk, $\partial\mathbb{D}$ stands for the boundary of $\mathbb{D}$ and $d_A = \mathrm{dist}(\partial\mathbb{D}, \sigma(A))$. In addition, we denote

$$\mathfrak{A} = \{f : \mathbb{D} \to \mathbb{C} : f \text{ is analytic, } \mathrm{Re}(f) > 0 \text{ and } f(0) = 1\}.$$

The Sylvester type equations $AXB \pm X = C$ have been investigated in matrix theory; see [5]. Several perturbation bounds for the norm of sum or difference of operators have been presented in the literature by employing some integral representations of certain functions; see [3, 11, 12, 16] and references therein.

In the recent paper [12], Kittaneh showed that the following inequality involving $f \in \mathfrak{A}$

$$|||f(A)X - Xf(B)||| \le \frac{2}{d_A d_B} |||AX - XB|||,$$

where $A, B, X \in \mathbb{B}(\mathbf{H})$ such that A and B are G_1 operators with $\sigma(A)\cup\sigma(B) \subset \mathbb{D}$. In [13], the authors extended this inequality for two functions $f, g \in \mathfrak{A}$ as follows

$$|||f(A)X - Xg(B)||| \leq \frac{2\sqrt{2}}{d_A d_B}|||\,|AX| + |XB|\,||| \tag{2}$$

and

$$|||f(A)X + Xg(B)||| \leq \frac{2\sqrt{2}}{d_A d_B}|||\,|AXB| + |X|\,|||, \tag{3}$$

in which $A, B, X \in \mathbb{B}(\mathbf{H})$ such that A and B are G_1 operators with $\sigma(A) \cup \sigma(B) \subset \mathbb{D}$. They also showed that

$$|||f(A)Xg(B) - X||| \leq \frac{2\sqrt{2}}{d_A d_B}|||\,|AX| + |XB|\,||| \tag{4}$$

and

$$|||f(A)Xg(B) + X||| \leq \frac{2\sqrt{2}}{d_A d_B}|||\,|AXB| + |X|\,|||, \tag{5}$$

where $A, B, X \in \mathbb{B}(\mathbf{H})$ such that A and B are G_1 operators with $\sigma(A)\cup\sigma(B) \subset \mathbb{D}$.

In this paper, by using some ideas from [12, 13] we present some upper bounds for unitarily invariant norms of the forms $|||f(A)X + X\bar{f}(A)|||$ and $|||f(A)X - X\bar{f}(A)|||$ involving G_1 operator and $f \in \mathfrak{A}$. We also present the Hilbert-Schmidt norm inequality of the form

$$\begin{aligned} &\|f(A)Xg(B) \pm g(B)Xf(A)\|_2 \\ &\leq \left\| \frac{(I + |A|)X(I + |B|) + (I + |B|)X(I + |A|)}{d_A d_B} \right\|_2, \end{aligned}$$

where $A, B, X \in \mathbb{M}_n$ such that A and B are Hermitian matrices with $\sigma(A) \cup \sigma(B) \subset \mathbb{D}$ and $f, g \in \mathfrak{A}$.

2. Main results

Our first result is some upper bounds for the Hilbert-Schmidt norm inequalities.

Theorem 2.1. *Let $A, B \in \mathbb{M}_n$ be Hermitian matrices with $\sigma(A) \cup \sigma(B) \subset \mathbb{D}$ and $f, g \in \mathfrak{A}$. Then*

$$\begin{aligned} &\|f(A)X + Xg(B) \pm f(A)Xg(B)\|_2 \\ &\leq \left\| \frac{X + |A|X}{d_A} + \frac{X + X|B|}{d_B} + \frac{(I + |A|)\,X\,(I + |B|)}{d_A d_B} \right\|_2 \end{aligned}$$

and

$$\|f(A)Xg(B) \pm g(B)Xf(A)\|_2 \leq \left\| \frac{(I+|A|)X(I+|B|)+(I+|B|)X(I+|A|)}{d_A d_B} \right\|_2,$$

where $X \in \mathbb{M}_n$.

Proof. Let $A=U\Lambda U^*$ and $B=V\Gamma V^*$ be the spectral decomposition of A and B such that $\Lambda=\operatorname{diag}(\lambda_1,\dots,\lambda_n)$, $\Gamma=\operatorname{diag}(\gamma_1,\dots,\gamma_n)$ and let $U^*XV:=[y_{jk}]$. It follows from $|e^{i\alpha}-\lambda_j|\geq d_A$ and $|e^{i\beta}-\gamma_k|\geq d_B$ that

$$\begin{aligned}
&\|f(A)X+Xg(B)\pm f(A)Xg(B)\|_2^2\\
=&\sum_{j,k}|f(\lambda_j)y_{j,k}+y_{j,k}g(\gamma_k)\pm f(\lambda_j)y_{j,k}g(\gamma_k)|^2\\
=&\sum_{j,k}|f(\lambda_j)\pm f(\lambda_j)g(\gamma_k)+g(\gamma_k)|^2|y_{j,k}|^2\\
=&\sum_{j,k}\left|\int_0^{2\pi}\int_0^{2\pi}\tfrac{e^{i\alpha}+\lambda_j}{e^{i\alpha}-\lambda_j}+\tfrac{e^{i\beta}+\gamma_k}{e^{i\beta}-\gamma_k}\pm\tfrac{(e^{i\alpha}+\lambda_j)(e^{i\beta}+\gamma_k)}{(e^{i\alpha}-\lambda_j)(e^{i\beta}-\gamma_k)}d\mu(\alpha)d\mu(\beta)\right|^2|y_{j,k}|^2\\
\leq&\sum_{j,k}\left(\int_0^{2\pi}\int_0^{2\pi}\tfrac{|e^{i\alpha}+\lambda_j|}{|e^{i\alpha}-\lambda_j|}+\tfrac{|e^{i\beta}+\gamma_k|}{|e^{i\beta}-\gamma_k|}+\tfrac{|e^{i\alpha}+\lambda_j||e^{i\beta}+\gamma_k|}{|e^{i\alpha}-\lambda_j||e^{i\beta}-\gamma_k|}d\mu(\alpha)d\mu(\beta)\right)^2|y_{j,k}|^2\\
\leq&\sum_{j,k}\left(\int_0^{2\pi}\int_0^{2\pi}\tfrac{1+|\lambda_j|}{d_A}+\tfrac{(1+|\lambda_j|)(1+|\gamma_k|)}{d_Ad_B}+\tfrac{1+|\gamma_k|}{d_B}d\mu(\alpha)d\mu(\beta)\right)^2|y_{j,k}|^2\\
\leq&\sum_{j,k}\left(\tfrac{1+|\lambda_j|}{d_A}+\tfrac{1+|\gamma_k|}{d_B}+\tfrac{(1+|\lambda_j|)(1+|\gamma_k|)}{d_Ad_B}\right)^2|y_{j,k}|^2\\
=&\left\|\tfrac{X+|A|X}{d_A}+\tfrac{X+X|B|}{d_B}+\tfrac{(I+|A|)X(I+|B|)}{d_Ad_B}\right\|_2^2.
\end{aligned}$$

Then we get the first inequality. Similarly,

$$\begin{aligned}
&\|f(A)Xg(B)\pm g(B)Xf(A)\|_2^2\\
=&\sum_{j,k}|f(\lambda_j)y_{j,k}g(\gamma_k)\pm g(\gamma_j)y_{j,k}f(\lambda_k)|^2\\
=&\sum_{j,k}|f(\lambda_j)g(\gamma_k)\pm g(\gamma_j)f(\lambda_k)|^2|y_{j,k}|^2\\
=&\sum_{j,k}\left|\int_0^{2\pi}\int_0^{2\pi}\tfrac{(e^{i\alpha}+\lambda_j)(e^{i\beta}+\gamma_k)}{(e^{i\alpha}-\lambda_j)(e^{i\beta}-\gamma_k)}\pm\tfrac{(e^{i\beta}+\gamma_j)(e^{i\alpha}+\lambda_k)}{(e^{i\beta}-\gamma_j)(e^{i\alpha}-\lambda_k)}d\mu(\alpha)d\mu(\beta)\right|^2|y_{j,k}|^2\\
\leq&\sum_{j,k}\left(\int_0^{2\pi}\int_0^{2\pi}\tfrac{|e^{i\alpha}+\lambda_j||e^{i\beta}+\gamma_k|}{|e^{i\alpha}-\lambda_j||e^{i\beta}-\gamma_k|}+\tfrac{|e^{i\beta}+\gamma_j||e^{i\alpha}+\lambda_k|}{|e^{i\beta}-\gamma_j||e^{i\alpha}-\lambda_k|}d\mu(\alpha)d\mu(\beta)\right)^2|y_{j,k}|^2\\
\leq&\sum_{j,k}\left(\int_0^{2\pi}\int_0^{2\pi}\tfrac{(1+|\lambda_j|)(1+|\gamma_k|)}{d_Ad_B}+\tfrac{(1+|\gamma_j|)(1+|\lambda_k|)}{d_Ad_B}d\mu(\alpha)d\mu(\beta)\right)^2|y_{j,k}|^2\\
\leq&\sum_{j,k}\left(\tfrac{(1+|\lambda_j|)(1+|\gamma_k|)}{d_Ad_B}+\tfrac{(1+|\gamma_j|)(1+|\lambda_k|)}{d_Ad_B}\right)^2|y_{j,k}|^2
\end{aligned}$$

$$\leq \sum_{j,k}\left(\tfrac{(1+|\lambda_j|)y_{j,k}(1+|\gamma_k|)}{d_Ad_B}+\tfrac{(1+|\gamma_j|)y_{j,k}(1+|\lambda_k|)}{d_Ad_B}\right)^2$$
$$=\left\|\tfrac{(I+|A|)X(I+|B|)+(I+|B|)X(I+|A|)}{d_Ad_B}\right\|_2. \qquad \square$$

Now, if we put $X=I$ in Theorem 2.1, then we get the next result.

Corollary 2.2. *Let $A, B \in \mathbb{M}_n$ be Hermitian matrices with $\sigma(A)\cup\sigma(B)\subset\mathbb{D}$ and $f, g\in\mathfrak{A}$. Then*

$$\|f(A)+g(B)\pm f(A)g(B)\|_2 \leq \left\|\frac{I+|A|}{d_A}+\frac{I+|B|}{d_B}+\frac{(I+|A|)\,(I+|B|)}{d_Ad_B}\right\|_2$$

and

$$\|f(A)g(B)\pm g(B)f(A)\|_2 \leq \left\|\frac{(I+|A|)(I+|B|)+(I+|B|)(I+|A|)}{d_Ad_B}\right\|_2.$$

To prove the next results, the following lemma is required.

Lemma 2.3. *Let $A, B, X, Y \in \mathbb{B}(\mathbf{H})$ such that X and Y are compact. Then*
(a) $s_j(AX\pm YB)\leq 2\sqrt{\|A\|\|B\|}s_j(X\oplus Y)$ $(j=1,2,\ldots)$;
(b) $|||(AX\pm YB)\oplus 0|||\leq 2\sqrt{\|A\|\|B\|}\,|||X\oplus Y|||$.

Proof. Using [11, Theorem 2.2] we have

$$s_j(AX\pm YB)\leq(\|A\|+\|B\|)s_j(X\oplus Y)\ (j=1,2,\ldots).$$

If we replace A, B, X and Y by tA, $\frac{B}{t}$, $\frac{X}{t}$ and tY, respectively, then we get

$$s_j(AX\pm YB)\leq(t\|A\|+\frac{\|B\|}{t})s_j(X\oplus Y)\ (j=1,2,\ldots).$$

It follows from $\min_{t>0}(t\|A\|+\frac{\|B\|}{t}) = 2\sqrt{\|A\|\|B\|}$ that we reach the first inequality. The second inequality can be proven by the first inequality and the Ky Fan dominance theorem [6, Theorme IV.2.2]; see also [1]. $\square$

Now, by applying Lemma 2.3 we obtain the following result.

Theorem 2.4. *Let $A, B, X, Y \in \mathbb{B}(\mathbf{H})$ and $f, g\in\mathfrak{A}$. Then*

$$\left|\left|\left|\big((f(A)-g(B))X\pm Y(f(B)-g(A))\big)\oplus 0\right|\right|\right| \leq \frac{4\sqrt{2}}{d_Ad_B}\,|||A|+|B|||\,|||X\oplus Y|||$$

and

$$\left|\left|\left|\big((f(A)+g(B))X\pm Y(f(B)+g(A))\big)\oplus 0\right|\right|\right| \leq \frac{4\sqrt{2}}{d_Ad_B}\,\|I+|AB|\|\,|||X\oplus Y|||,$$

where X, Y are compact and A, B are G_1 operators with $\sigma(A)\cup\sigma(B)\subset\mathbb{D}$.

Proof. Using Lemma 2.3 and inequalities (2) and (3) we have

$$\left|\left|\left|\big((f(A)-g(B))X\pm Y(f(B)-g(A))\big)\oplus 0\right|\right|\right|$$
$$\leq 2\|f(A)-g(B)\|^{\frac{1}{2}}\|f(B)-g(A)\|^{\frac{1}{2}}|||X\oplus Y|||\ \text{(by Lemma 2.3)}$$

$$\leq 2\sqrt{\frac{2\sqrt{2}}{d_Ad_B}|||A|+|B|||}\sqrt{\frac{2\sqrt{2}}{d_Ad_B}|||B|+|A|||}\,|||X\oplus Y|||$$
(by inequality (2))
$$=\frac{4\sqrt{2}}{d_Ad_B}\,|||A|+|B|||\,|||X\oplus Y|||\,.$$

Similarly,

$$\left|\left|\left|\left((f(A)+g(B))X\pm Y(f(B)+g(A))\right)\oplus 0\right|\right|\right|$$
$$\leq 2\|f(A)+g(B)\|^{\frac{1}{2}}\|f(B)+g(A)\|^{\frac{1}{2}}|||X\oplus Y|||\ \text{(by Lemma 2.3)}$$
$$\leq 2\sqrt{\frac{2\sqrt{2}}{d_Ad_B}\|I+|AB|\|}\sqrt{\frac{2\sqrt{2}}{d_Ad_B}\|I+|AB|\|}\,|||X\oplus Y|||$$
(by inequality (3))
$$=\frac{4\sqrt{2}}{d_Ad_B}\,\|I+|AB|\|\,|||X\oplus Y|||\,.$$
□

Theorem 2.5. *Let* $A,B\in\mathbb{B}(\mathbf{H})$ *be* G_1 *operators with* $\sigma(A)\cup\sigma(B)\subset\mathbb{D}$ *and* $f\in\mathfrak{A}$. *Then for every* $X\in\mathbb{B}(\mathbf{H})$

$$\left|\left|\left|f(A)X+X\bar{f}(B)\right|\right|\right|\leq\frac{2}{d_Ad_B}\,|||X-AXB^*||| \tag{6}$$

and

$$\left|\left|\left|f(A)X-X\bar{f}(B)\right|\right|\right|\leq\frac{2\sqrt{2}}{d_Ad_B}\,|||\,|AX|+|XB^*|\,|||\,. \tag{7}$$

Proof. Using the Herglotz representation theorem (see e.g., [7, p. 21]) we have

$$f(z)=\int_0^{2\pi}\frac{e^{i\alpha}+z}{e^{i\alpha}-z}d\mu(\alpha)+i\mathrm{Im}\,f(0)=\int_0^{2\pi}\frac{e^{i\alpha}+z}{e^{i\alpha}-z}d\mu(\alpha),$$

where μ is a positive Borel measure on the interval $[0,2\pi]$ with finite total mass $\int_0^{2\pi}d\mu(\alpha)=f(0)=1$. Hence

$$\bar{f}(z)=\overline{\int_0^{2\pi}\frac{e^{i\alpha}+z}{e^{i\alpha}-z}d\mu(\alpha)}=\int_0^{2\pi}\frac{e^{-i\alpha}+\bar{z}}{e^{-i\alpha}-\bar{z}}d\mu(\alpha),$$

where $\bar{f}$ is the conjugate function of f (i.e., $\bar{f}f=|f|^2$). So

$$\begin{aligned}&f(A)X+X\bar{f}(B)\\&=\int_0^{2\pi}\left(e^{i\alpha}+A\right)\left(e^{i\alpha}-A\right)^{-1}X+X\left(e^{-i\alpha}+B^*\right)\left(e^{-i\alpha}-B^*\right)^{-1}d\mu(\alpha)\\&=\int_0^{2\pi}\left(e^{i\alpha}-A\right)^{-1}\Big[\left(e^{i\alpha}+A\right)X\left(e^{-i\alpha}-B^*\right)\\&\qquad+\left(e^{i\alpha}-A\right)X\left(e^{-i\alpha}+B^*\right)\Big]\left(e^{-i\alpha}-B^*\right)^{-1}d\mu(\alpha)\end{aligned}$$

$$= 2\int_0^{2\pi} \left(e^{i\alpha} - A\right)^{-1} \left(X - AXB^*\right) \left(e^{-i\alpha} - B^*\right)^{-1} d\mu(\alpha).$$

Hence

$$\begin{aligned}
&\left|\left|\left| f(A)X + X\bar{f}(B) \right|\right|\right| \\
&= \left|\left|\left| \int_0^{2\pi} \left(e^{i\alpha} + A\right)\left(e^{i\alpha} - A\right)^{-1} X + X\left(e^{-i\alpha} + B^*\right)\left(e^{-i\alpha} - B^*\right)^{-1} d\mu(\alpha) \right|\right|\right| \\
&= 2\left|\left|\left| \int_0^{2\pi} \left(e^{i\alpha} - A\right)^{-1} \left(X - AXB^*\right) \left(e^{-i\alpha} - B^*\right)^{-1} d\mu(\alpha) \right|\right|\right| \\
&\le 2\int_0^{2\pi} \left|\left|\left| \left(e^{i\alpha} - A\right)^{-1} \left(X - AXB^*\right) \left(e^{-i\alpha} - B^*\right)^{-1} \right|\right|\right| d\mu(\alpha) \\
&\le 2\int_0^{2\pi} \left\| \left(e^{i\alpha} - A\right)^{-1} \right\| \left\|(e^{i\alpha} - B)^{-1}\right\| \, |||X - AXB^*||| \, d\mu(\alpha) \\
&\qquad \text{(by inequality (1))}.
\end{aligned}$$

Since A and B are G_1 operators, it follows from $\left\|\left(e^{i\alpha} - A\right)^{-1}\right\| = \frac{1}{\operatorname{dist}(e^{i\alpha},\sigma(A))}$ $\le \frac{1}{\operatorname{dist}(\partial\mathbb{D},\sigma(A))} = \frac{1}{d_A}$ and $\left\|\left(e^{i\alpha} - B\right)^{-1}\right\| \le \frac{1}{d_B}$ that

$$\begin{aligned}
\left|\left|\left| f(A)X + X\bar{f}(B) \right|\right|\right| &\le \left(\frac{2}{d_A d_B}\int_0^{2\pi} d\mu(\alpha)\right) |||X - AXB^*||| \\
&= \left(\frac{2}{d_A d_B} f(0)\right) |||X - AXB^*||| \\
&= \frac{2}{d_A d_B} \, |||X - AXB^*||| \,.
\end{aligned}$$

Then we have the first inequality. Using the inequality

$$\begin{aligned}
|||e^{-i\alpha}AX + e^{i\alpha}XB^*||| &= \left|\left|\left| \begin{bmatrix} e^{-i\alpha}AX + e^{i\alpha}XB^* & 0 \\ 0 & 0 \end{bmatrix} \right|\right|\right| \\
&= \left|\left|\left| \begin{bmatrix} e^{-i\alpha} & e^{i\alpha} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} AX & 0 \\ XB^* & 0 \end{bmatrix} \right|\right|\right| \\
&\le \left\| \begin{bmatrix} e^{-i\alpha} & e^{i\alpha} \\ 0 & 0 \end{bmatrix} \right\| \left|\left|\left| \begin{bmatrix} AX & 0 \\ XB^* & 0 \end{bmatrix} \right|\right|\right| \\
&\qquad \text{(by inequality (1))} \\
&= \sqrt{2} \left|\left|\left| \begin{bmatrix} AX & 0 \\ XB^* & 0 \end{bmatrix} \right|\right|\right| \\
&= \sqrt{2} \left|\left|\left| (|AX|^2 + |XB^*|^2)^{\frac{1}{2}} \oplus 0 \right|\right|\right| \\
&\le \sqrt{2} \, |||(|AX| + |XB^*|) \oplus 0||| \\
&\qquad \text{(applying [2, p. 775] to the function } h(t) = t^{\frac{1}{2}})
\end{aligned}$$

the Ky Fan dominance theorem we have

$$|||e^{-i\beta}AX + e^{i\alpha}XB^*||| \leq \sqrt{2}\,|||\,|AX| + |XB^*|\,|||\,. \tag{8}$$

It follows from (8) and the same argument of the proof of the first inequality that we have the second inequality and this completes the proof. □

Remark 2.6. Let $f(x+yi) = u(x,y) + v(x,y)i$, where u, v are the real and imaginary parts of f, respectively. If $f, \bar{f} \in \mathfrak{A}$, then the Cauchy-Riemann equations for complex analytic functions (i.e., $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$) implies that $v(x,y) = k$ for some $k \in \mathcal{C}$. The condition $f(0) = 1$ conclude that $v(x,y) = 0$. Hence, f is a real valued function. So, for arbitrary functions $f, g \in \mathfrak{A}$, we can not replace g by $\bar{f}$ in inequalities (2) and (3). Thus, in Theorem 2.5 we have been established some upper bounds for $|||f(A)X + X\bar{f}(B)|||$ and $|||f(A)X - X\bar{f}(B)|||$ in terms of $|||X - AXB^*|||$ and $|||\,|AX| + |XB^*|\,|||$, respectively, that can not be derived from inequality (2) and (3) for an arbitrary function $f \in \mathfrak{A}$.

Remark 2.7. If $A, B \in \mathbb{B}(\mathbf{H})$ are G_1 operators with $\sigma(A) \cup \sigma(B) \subset \mathbb{D}$ and $f \in \mathfrak{A}$, then with a similar argument in the proof of Theorem 2.5 we get the following inequalities

$$\left|\left|\left|\bar{f}(A)X + Xf(B)\right|\right|\right| \leq \frac{2}{d_A d_B}\,|||X - A^*XB||| \tag{9}$$

and

$$\left|\left|\left|\bar{f}(A)X - Xf(B)\right|\right|\right| \leq \frac{2\sqrt{2}}{d_A d_B}\,|||\,|A^*X| + |XB|\,|||\,,$$

where $X \in \mathbb{B}(\mathbf{H})$.

Remark 2.8. For an arbitrary operator $A \in \mathbb{B}(\mathbf{H})$, the numerical range is definition by $W(A) = \{\langle Ax, x\rangle : x \in \mathbf{H}, \|x\| = 1\}$. It is well-known that $W(A)$ is a bounded convex subset of the complex plane $\mathbb{C}$. Its closure $\overline{W(A)}$ contains $\sigma(A)$ and is contained in $\{z \in \mathbb{C} : |z| \leq \|A\|\}$. In [10], it is shown

$$\frac{1}{\operatorname{dist}(z, \sigma(A))} \leq \left\|(z - A)^{-1}\right\| \qquad (z \notin \sigma(A))$$

and

$$\left\|(z - A)^{-1}\right\| \leq \frac{1}{\operatorname{dist}(z, \overline{W(A)})} \qquad (z \notin \overline{W(A)}).$$

Now, if we replace the hypophysis G_1 operators by the conditions $\overline{W(A)} \cup \overline{W(B)} \subseteq \mathbb{D}$ in Theorem 2.5, then in inequalities (2)-(5), the constants d_A and d_B interchange to D_A and D_B, respectively, where $D_A = \operatorname{dist}(\partial\mathbb{D}, \overline{W(A)})$, $D_B = \operatorname{dist}(\partial\mathbb{D}, \overline{W(A)})$. Also inequalities (6) and (7) appear of the forms

$$\left|\left|\left|f(A)X + X\bar{f}(B)\right|\right|\right| \leq \frac{2}{D_A D_B}\,|||X - AXB^*|||$$

and

$$\left|\left|\left|f(A)X - X\bar{f}(B)\right|\right|\right| \leq \frac{2\sqrt{2}}{D_A D_B} \left|\left|\left| \, |AX| + |XB^*| \, \right|\right|\right|.$$

where $f \in \mathfrak{A}$. For example, for every contraction operator A (i.e., $A^*A \leq I$) and $0 < \epsilon < 1$, the operator ϵA has the property $\overline{W(\epsilon A)} \subseteq \mathbb{D}$.

If we take $X = I$ in Theorem 2.5, then we get the following result.

Corollary 2.9. *Let $A, B \in \mathbb{B}(\mathbf{H})$ be normal operators with $\sigma(A) \cup \sigma(B) \subset \mathbb{D}$ and $f \in \mathfrak{A}$. Then for every $X \in \mathbb{B}(\mathbf{H})$*

$$\left|\left|\left|f(A) + \bar{f}(B)\right|\right|\right| \leq \frac{2}{d_A d_B} \left|\left|\left|I - AB^*\right|\right|\right|.$$

In particular, for $B = A$ we have

$$\left|\left|\left|Re(f(A))\right|\right|\right| \leq \frac{1}{d_A^2} \left|\left|\left|I - AA^*\right|\right|\right|.$$

For the next result we need the following lemma (see also [14]).

Lemma 2.10. *If $A, B, X \in \mathbb{B}(\mathbf{H})$ such that A and B are self-adjoint and $0 < mI \leq X$ for some positive real number m, then*

$$m \left|\left|\left|A - B\right|\right|\right| \leq \left|\left|\left|AX + XB\right|\right|\right|.$$

Proof.

$$\begin{aligned} m \left|\left|\left|A - B\right|\right|\right| &\leq \frac{1}{2} \left|\left|\left|(A-B)X + X(A-B)\right|\right|\right| \qquad \text{(by [17, Lemma 3.1])} \\ &= \frac{1}{2} \left|\left|\left|AX - XB + (XA - BX)\right|\right|\right| \\ &\leq \frac{1}{2} \left(\left|\left|\left|AX - XB\right|\right|\right| + \left|\left|\left|XA - BX\right|\right|\right|\right) \\ &= \left|\left|\left|AX - XB\right|\right|\right| \qquad (\text{since } \|A\| = \|A^*\|). \end{aligned}$$

□

Proposition 2.11. *Let $A, B \in \mathbb{B}(\mathbf{H})$ be G_1 operators with $\sigma(A) \cup \sigma(B) \subset \mathbb{D}$, let $X \in \mathbb{B}(\mathbf{H})$ such that $0 < mI \leq X$ for some positive real number m and $f \in \mathfrak{A}$. Then*

(10)

$$m \left|\left|\left|Re(f(A)) - Re(f(B))\right|\right|\right| \leq \frac{1}{d_A d_B} \left(\left|\left|\left|X - AXB^*\right|\right|\right| + \left|\left|\left|X - A^*XB\right|\right|\right|\right).$$

In particular, if A and B are unitary operators, then

$$m \left|\left|\left|Re(f(A)) - Re(f(B))\right|\right|\right| \leq \frac{2}{d_A d_B} \left|\left|\left|X - AXB^*\right|\right|\right|.$$

Proof.

$$m \left|\left|\left|\mathrm{Re}(f(A)) - \mathrm{Re}(f(B))\right|\right|\right| \leq \left|\left|\left|\mathrm{Re}(f(A))X + X\mathrm{Re}(f(B))\right|\right|\right| \qquad \text{(by Lemma 2.10)}$$

$$
\begin{aligned}
&= \frac{1}{2}\left|\left|\left|f(A)X + X\bar{f}(B) + \bar{f}(A)X + Xf(B)\right|\right|\right| \\
&\leq \frac{1}{2}\left(\left|\left|\left|f(A)X + X\bar{f}(B)\right|\right|\right| + \left|\left|\left|\bar{f}(A)X + Xf(B)\right|\right|\right|\right) \\
&\leq \frac{1}{d_A d_B}\left(|||X - AXB^*||| + |||X - A^*XB|||\right) \\
&\qquad\qquad \text{(by inequalities (6) and (9)).}
\end{aligned}
$$

Hence we get the first inequality. Especially, it follows from inequality (10) and equation

$$|||X - AXB^*||| = |||A(A^*XB - X)B^*||| = |||A^*XB - X||| = |||X - A^*XB|||\,.$$

□

Remark 2.12. Using Lemma 2.3 we have

$$
\begin{aligned}
&\left|\left|\left|((f(A) + \bar{f}(B))X - Y(f(B) + \bar{f}(A))) \oplus 0\right|\right|\right| \\
&\qquad \leq 2\|f(A) + \bar{f}(B)\|^{\frac{1}{2}}\|f(B) + \bar{f}(A)\|^{\frac{1}{2}}|||X \oplus Y||| \\
&\qquad = 2\|f(A) + \bar{f}(B)\||||X \oplus Y|||.
\end{aligned}
$$

Now, if we apply inequality (6), then we reach

$$\|f(A) + \bar{f}(B)\||||X \oplus Y||| \leq \frac{2}{d_A d_B}\|I - AB^*\|\,|||X \oplus Y|||\,,$$

whence

$$\left|\left|\left|\big((f(A) + \bar{f}(B))X - Y(f(B) + \bar{f}(A))\big) \oplus 0\right|\right|\right| \leq \frac{4}{d_A d_B}\|I - AB^*\|\,|||X \oplus Y|||\,.$$

Hence, if we put $B = A$, then we get

$$|||\mathrm{Re}(f(A))X - Y\mathrm{Re}(f(A)) \oplus 0||| \leq \frac{2}{d_A^2}\,\|I - AA^*\|||\,|||X \oplus Y|||\,.$$

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