

SOME RESULTS ON STABLE f -HARMONIC MAPS

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ABSTRACT. In this paper, we prove that any stable f -harmonic map from sphere \mathbb{S}^n to Riemannian manifold (N, h) is constant, where f is a smooth positive function on $\mathbb{S}^n \times N$ satisfying one condition with $n > 2$. We also prove that any stable f -harmonic map φ from a compact Riemannian manifold (M, g) to \mathbb{S}^n ($n > 2$) is constant where, in this case, f is a smooth positive function on $M \times \mathbb{S}^n$ satisfying $\Delta^{\mathbb{S}^n}(f) \circ \varphi \leq 0$.

1. Preliminaries and notations

We give some definitions. (1) Let (M, g) be a Riemannian manifold. By R^M and Ric^M we denote respectively the Riemannian curvature tensor and the Ricci tensor of (M, g) . Then R^M and Ric^M are defined by

$$(1) \quad R^M(X, Y)Z = \nabla_X^M \nabla_Y^M Z - \nabla_Y^M \nabla_X^M Z - \nabla_{[X, Y]}^M Z,$$

$$(2) \quad \text{Ric}^M(X, Y) = g(R^M(X, e_i)e_i, Y),$$

where ∇^M is the Levi-Civita connection with respect to g , $\{e_i\}$ is an orthonormal frame, and $X, Y, Z \in \Gamma(TM)$. The divergence of $(0, p)$ -tensor α on M is defined by

$$(3) \quad (\text{div}^M \alpha)(X_1, \dots, X_{p-1}) = (\nabla_{e_i}^M \alpha)(e_i, X_1, \dots, X_{p-1}),$$

where $X_1, \dots, X_{p-1} \in \Gamma(TM)$, and $\{e_i\}$ is an orthonormal frame. Given a smooth function λ on M , the gradient of λ is defined by

$$(4) \quad g(\text{grad}^M \lambda, X) = X(\lambda),$$

the Hessian of λ is defined by

$$(5) \quad (\text{Hess}^M \lambda)(X, Y) = g(\nabla_X^M \text{grad} \lambda, Y),$$

where $X, Y \in \Gamma(TM)$, the Laplacian of λ is defined by

$$(6) \quad \Delta^M(\lambda) = \text{trace}_g \text{Hess}^M \lambda,$$

(for more details, see for example [9]).

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(2) Let $\varphi : (M, g) \rightarrow (N, h)$ be a smooth map between two Riemannian manifolds, $\tau(\varphi)$ the tension field of φ (see [1, 2, 6]), and let f be a smooth positive function on $M \times N$, the f -tension field of φ is given by

$$(7) \quad \tau_f(\varphi) = f_\varphi \tau(\varphi) + d\varphi(\text{grad}^M f_\varphi) - e(\varphi)(\text{grad}^N f) \circ \varphi,$$

where f_φ is a smooth positive function on M defined by

$$(8) \quad f_\varphi(x) = f(x, \varphi(x)), \quad \forall x \in M,$$

$e(\varphi)$ is the energy density of φ (see [1]), and grad^M (resp. grad^N) denotes the gradient operator with respect to g (resp. h). Then φ is called f -harmonic if the f -tension field vanishes, i.e., $\tau_f(\varphi) = 0$ (for more details on the concept of f -harmonic maps see [4, 5, 10]). We define the index form for f -harmonic maps by

$$(9) \quad I_f^\varphi(v, w) = \int_M h(J_f^\varphi(v), w) v^M$$

for all $v, w \in \Gamma(\varphi^{-1}TN)$, where

$$(10) \quad \begin{aligned} J_f^\varphi(v) = & -f_\varphi \text{trace}_g R^N(v, d\varphi)d\varphi - \text{trace}_g \nabla^\varphi f_\varphi \nabla^\varphi v \\ & + e(\varphi)(\nabla_v^N \text{grad}^N f) \circ \varphi - d\varphi(\text{grad}^M v(f)) \\ & - v(f)\tau(\varphi) + \langle \nabla^\varphi v, d\varphi \rangle (\text{grad}^N f) \circ \varphi, \end{aligned}$$

R^N is the curvature tensor of (N, h) , ∇^N is the Levi-Civita connection of (N, h) , ∇^φ denote the pull-back connection on $\varphi^{-1}TN$, and v^M is the volume form of (M, g) (see [1], [9]). If φ is a f -harmonic map and for any vector field v along φ , the index form satisfies $I_f^\varphi(v, v) \geq 0$, then φ is called a stable f -harmonic map. Note that, the definition of stable f -harmonic maps is a generalization of stable harmonic maps if $f = 1$ on M (see [11]).

2. Main results

Theorem 2.1. *Any stable f -harmonic map φ from sphere (\mathbb{S}^n, g) ($n > 2$) to Riemannian manifold (N, h) is constant, where f is a smooth positive function on $\mathbb{S}^n \times N$ satisfying $\text{trace}_g h((\nabla d\varphi)(\cdot, \text{grad}^{\mathbb{S}^n} f), d\varphi(\cdot)) \geq 0$.*

Proof. Choose a normal orthonormal frame $\{e_i\}$ at point x_0 in \mathbb{S}^n . Set

$$\lambda(x) = \langle \alpha, x \rangle_{\mathbb{R}^{n+1}}$$

for all $x \in \mathbb{S}^n$, where $\alpha \in \mathbb{R}^{n+1}$ and let $v = \text{grad}^{\mathbb{S}^n} \lambda$. Note that

$$v = \langle \alpha, e_i \rangle e_i, \quad \nabla_X^{\mathbb{S}^n} v = -\lambda X \text{ for all } X \in \Gamma(T\mathbb{S}^n),$$

$$\text{trace}_g (\nabla^{\mathbb{S}^n})^2 v = \nabla_{e_i}^{\mathbb{S}^n} \nabla_{e_i}^{\mathbb{S}^n} v - \nabla_{\nabla_{e_i}^{\mathbb{S}^n} e_i}^{\mathbb{S}^n} v = -v,$$

where $\nabla^{\mathbb{S}^n}$ is the Levi-Civita connection on \mathbb{S}^n with respect to the standard metric g of the sphere (see [11]). At point x_0 , we have

$$(11) \quad \nabla_{e_i}^\varphi f_\varphi \nabla_{e_i}^\varphi d\varphi(v) = \nabla_{\text{grad}^{\mathbb{S}^n} f_\varphi}^\varphi d\varphi(v) + f_\varphi \nabla_{e_i}^\varphi \nabla_{e_i}^\varphi d\varphi(v),$$

the first term of (11) is given by

$$\begin{aligned}
\nabla_{\text{grad } f_\varphi}^\varphi d\varphi(v) &= \nabla_v^\varphi d\varphi(\text{grad}^{\mathbb{S}^n} f_\varphi) + d\varphi([\text{grad}^{\mathbb{S}^n} f_\varphi, v]) \\
&= \nabla_v^\varphi d\varphi(\text{grad}^{\mathbb{S}^n} f_\varphi) + d\varphi(\nabla_{\text{grad}^{\mathbb{S}^n} f_\varphi}^{\mathbb{S}^n} v) \\
(12) \quad &\quad - d\varphi(\nabla_v^{\mathbb{S}^n} \text{grad}^{\mathbb{S}^n} f_\varphi),
\end{aligned}$$

the seconde term of (11) is given by

$$\begin{aligned}
f_\varphi \nabla_{e_i}^\varphi \nabla_{e_i}^\varphi d\varphi(v) &= f_\varphi \nabla_{e_i}^\varphi \nabla_v^\varphi d\varphi(e_i) + f_\varphi \nabla_{e_i}^\varphi d\varphi([e_i, v]) \\
&= f_\varphi R^N(d\varphi(e_i), d\varphi(v))d\varphi(e_i) + f_\varphi \nabla_v^\varphi \nabla_{e_i}^\varphi d\varphi(e_i) \\
(13) \quad &\quad + f_\varphi d\varphi([e_i, [e_i, v]]) + 2f_\varphi \nabla_{[e_i, v]}^\varphi d\varphi(e_i),
\end{aligned}$$

from the definition of tension field, we get

$$\begin{aligned}
f_\varphi \nabla_{e_i}^\varphi \nabla_{e_i}^\varphi d\varphi(v) &= -f_\varphi R^N(d\varphi(v), d\varphi(e_i))d\varphi(e_i) + f_\varphi \nabla_v^\varphi \tau(\varphi) \\
&\quad + f_\varphi \nabla_v^\varphi d\varphi(\nabla_{e_i}^{\mathbb{S}^n} e_i) + f_\varphi d\varphi(\nabla_{e_i}^{\mathbb{S}^n} \nabla_{e_i}^{\mathbb{S}^n} v) \\
&\quad - f_\varphi d\varphi(\nabla_{e_i}^{\mathbb{S}^n} \nabla_v^{\mathbb{S}^n} e_i) + 2f_\varphi \nabla_{[e_i, v]}^\varphi d\varphi(e_i) \\
&= -f_\varphi R^N(d\varphi(v), d\varphi(e_i))d\varphi(e_i) + \nabla_v^\varphi f_\varphi \tau(\varphi) - v(f_\varphi)\tau(\varphi) \\
&\quad + f_\varphi \nabla_v^\varphi d\varphi(\nabla_{e_i}^{\mathbb{S}^n} e_i) + f_\varphi d\varphi(\nabla_{e_i}^{\mathbb{S}^n} \nabla_{e_i}^{\mathbb{S}^n} v) \\
(14) \quad &\quad - f_\varphi d\varphi(\nabla_{e_i}^{\mathbb{S}^n} \nabla_v^{\mathbb{S}^n} e_i) + 2f_\varphi \nabla_{[e_i, v]}^\varphi d\varphi(e_i),
\end{aligned}$$

by equations (11), (12), (14), and the f -harmonicity condition of φ , we have

$$\begin{aligned}
\nabla_{e_i}^\varphi f_\varphi \nabla_{e_i}^\varphi d\varphi(v) &= d\varphi(\nabla_{\text{grad}^{\mathbb{S}^n} f_\varphi}^{\mathbb{S}^n} v) - d\varphi(\nabla_v^{\mathbb{S}^n} \text{grad}^{\mathbb{S}^n} f_\varphi) \\
&\quad - f_\varphi R^N(d\varphi(v), d\varphi(e_i))d\varphi(e_i) \\
&\quad + \nabla_v^\varphi e(\varphi)(\text{grad}^N f) \circ \varphi - v(f_\varphi)\tau(\varphi) \\
&\quad + f_\varphi d\varphi(\nabla_v^{\mathbb{S}^n} \nabla_{e_i}^{\mathbb{S}^n} e_i) + f_\varphi d\varphi(\nabla_{e_i}^{\mathbb{S}^n} \nabla_{e_i}^{\mathbb{S}^n} v) \\
(15) \quad &\quad - f_\varphi d\varphi(\nabla_{e_i}^{\mathbb{S}^n} \nabla_v^{\mathbb{S}^n} e_i) + 2f_\varphi \nabla_{\nabla_{e_i}^{\mathbb{S}^n} v}^\varphi d\varphi(e_i),
\end{aligned}$$

by the definition of Ricci tensor, we get

$$\begin{aligned}
\nabla_{e_i}^\varphi f_\varphi \nabla_{e_i}^\varphi d\varphi(v) &= d\varphi(\nabla_{\text{grad}^{\mathbb{S}^n} f_\varphi}^{\mathbb{S}^n} v) - d\varphi(\nabla_v^{\mathbb{S}^n} \text{grad}^{\mathbb{S}^n} f_\varphi) \\
&\quad - f_\varphi R^N(d\varphi(v), d\varphi(e_i))d\varphi(e_i) + \nabla_v^\varphi e(\varphi)(\text{grad}^N f) \circ \varphi \\
&\quad - v(f_\varphi)\tau(\varphi) + f_\varphi d\varphi(\text{Ricci}^{\mathbb{S}^n} v) + f_\varphi d\varphi(\text{trace}(\nabla^{\mathbb{S}^n})^2 v) \\
(16) \quad &\quad + 2f_\varphi \nabla_{\nabla_{e_i}^{\mathbb{S}^n} v}^\varphi d\varphi(e_i),
\end{aligned}$$

from the property $\nabla_X^{\mathbb{S}^n} v = -\lambda X$, we obtain

$$\begin{aligned}
\nabla_{e_i}^\varphi f_\varphi \nabla_{e_i}^\varphi d\varphi(v) &= -\lambda d\varphi(\text{grad}^{\mathbb{S}^n} f_\varphi) - d\varphi(\nabla_v^{\mathbb{S}^n} \text{grad}^{\mathbb{S}^n} f_\varphi) \\
&\quad - f_\varphi R^N(d\varphi(v), d\varphi(e_i))d\varphi(e_i) \\
&\quad + \langle \nabla_{e_i}^\varphi d\varphi(v), d\varphi(e_i) \rangle (\text{grad}^N f) \circ \varphi
\end{aligned}$$

$$\begin{aligned}
& -h(d\varphi(\nabla_{e_i}^{\mathbb{S}^n} v), d\varphi(e_i))(\text{grad}^N f) \circ \varphi \\
& + e(\varphi)\nabla_v^\varphi(\text{grad}^N f) \circ \varphi \\
& - v(f)\tau(\varphi) - d\varphi(v)(f)\tau(\varphi) + f_\varphi d\varphi(\text{Ricci}^{\mathbb{S}^n} v) \\
(17) \quad & + f_\varphi d\varphi(\text{trace}(\nabla^{\mathbb{S}^n})^2 v) - 2\lambda f_\varphi \tau(\varphi).
\end{aligned}$$

From the definition of Jacobi operator (10) and equation (17) we have

$$\begin{aligned}
J_f^\varphi(d\varphi(v)) &= \lambda d\varphi(\text{grad}^{\mathbb{S}^n} f_\varphi) + d\varphi(\nabla_v^{\mathbb{S}^n} \text{grad}^{\mathbb{S}^n} f_\varphi) \\
& + \lambda h(d\varphi(e_i), d\varphi(e_i))(\text{grad}^N f) \circ \varphi + v(f)\tau(\varphi) \\
& - f_\varphi d\varphi(\text{Ricci}^{\mathbb{S}^n} v) - f_\varphi d\varphi(\text{trace}(\nabla^{\mathbb{S}^n})^2 v) \\
(18) \quad & - d\varphi(\text{grad}^{\mathbb{S}^n} d\varphi(v)(f)) + 2\lambda f_\varphi \tau(\varphi),
\end{aligned}$$

since $\text{trace}_g(\nabla^{\mathbb{S}^n})^2 v = -v$ and $\text{Ricci}^{\mathbb{S}^n} v = (n-1)v$ (see [1, 11]), we conclude

$$\begin{aligned}
h(J_f^\varphi(d\varphi(v)), d\varphi(v)) &= \lambda h(d\varphi(\text{grad}^{\mathbb{S}^n} f_\varphi), d\varphi(v)) \\
& + h(d\varphi(\nabla_v^{\mathbb{S}^n} \text{grad}^{\mathbb{S}^n} f_\varphi), d\varphi(v)) \\
& + 2\lambda h(e(\varphi)(\text{grad}^N f) \circ \varphi, d\varphi(v)) \\
& + v(f)h(\tau(\varphi), d\varphi(v)) \\
& - (n-2)f_\varphi h(d\varphi(v), d\varphi(v)) \\
& - h(d\varphi(\text{grad}^{\mathbb{S}^n} d\varphi(v)(f)), d\varphi(v)) \\
(19) \quad & + 2\lambda f_\varphi h(\tau(\varphi), d\varphi(v)),
\end{aligned}$$

by (19) and the f -harmonicity condition of φ , it follows that

$$\begin{aligned}
\text{trace}_\alpha h(J_f^\varphi(d\varphi(v)), d\varphi(v)) &= h(d\varphi(\nabla_{e_j}^{\mathbb{S}^n} \text{grad}^{\mathbb{S}^n} f_\varphi), d\varphi(e_j)) \\
& + h(\tau(\varphi), d\varphi(\text{grad}^{\mathbb{S}^n} f)) - (n-2)f_\varphi |d\varphi|^2 \\
(20) \quad & - h(d\varphi(\text{grad}^{\mathbb{S}^n} d\varphi(e_j)(f)), d\varphi(e_j)),
\end{aligned}$$

by the following formulas

$$\begin{aligned}
h(d\varphi(\nabla_{e_j}^{\mathbb{S}^n} \text{grad}^{\mathbb{S}^n} f_\varphi), d\varphi(e_j)) &= h(d\varphi(\nabla_{e_j}^{\mathbb{S}^n} \text{grad}^{\mathbb{S}^n} f), d\varphi(e_j)) \\
& + h(\nabla_{e_j}^\varphi d\varphi(e_i), \text{grad}^N f)h(d\varphi(e_i), d\varphi(e_j)) \\
& + h(d\varphi(e_i), \nabla_{e_j}^\varphi \text{grad}^N f)h(d\varphi(e_i), d\varphi(e_j)), \\
d\varphi(\text{grad}^{\mathbb{S}^n} d\varphi(e_j)(f)) &= e_i[h(d\varphi(e_j), \text{grad}^N f)]d\varphi(e_i) \\
& = h(\nabla_{e_i}^\varphi d\varphi(e_j), \text{grad}^N f)d\varphi(e_i) \\
& + h(d\varphi(e_j), \nabla_{e_i}^\varphi \text{grad}^N f)d\varphi(e_i), \\
-h(d\varphi(\text{grad}^{\mathbb{S}^n} d\varphi(e_j)(f)), d\varphi(e_j)) &= -h(\nabla_{e_i}^\varphi d\varphi(e_j), \text{grad}^N f)h(d\varphi(e_i), d\varphi(e_j)) \\
& - h(d\varphi(e_j), \nabla_{e_i}^\varphi \text{grad}^N f)h(d\varphi(e_i), d\varphi(e_j)),
\end{aligned}$$

and equation (20), it follows that

$$\begin{aligned} \text{trace}_\alpha h(J_f^\varphi(d\varphi(v)), d\varphi(v)) &= h(d\varphi(\nabla_{e_j}^{\mathbb{S}^n} \text{grad}^{\mathbb{S}^n} f), d\varphi(e_j)) \\ &\quad + h(\tau(\varphi), d\varphi(\text{grad}^{\mathbb{S}^n} f)) - (n-2)f_\varphi|d\varphi|^2, \end{aligned}$$

note that

$$\begin{aligned} h(\tau(\varphi), d\varphi(\text{grad}^{\mathbb{S}^n} f)) &= h(\nabla_{e_i}^\varphi d\varphi(e_i), d\varphi(\text{grad}^{\mathbb{S}^n} f)) \\ &= \text{div}^{\mathbb{S}^n} \eta - h(d\varphi(e_i), \nabla_{e_i}^\varphi d\varphi(\text{grad}^{\mathbb{S}^n} f)), \end{aligned}$$

with $\eta(X) = h(d\varphi(X), d\varphi(\text{grad}^{\mathbb{S}^n} f))$, $\forall X \in \Gamma(T\mathbb{S}^n)$. We obtain

$$(21) \quad \begin{aligned} \text{trace}_\alpha h(J_f^\varphi(d\varphi(v)), d\varphi(v)) &= -h((\nabla d\varphi)(e_j, \text{grad}^{\mathbb{S}^n} f), d\varphi(e_j)) \\ &\quad + \text{div}^{\mathbb{S}^n} \eta - (n-2)f_\varphi|d\varphi|^2, \end{aligned}$$

since $h((\nabla d\varphi)(e_j, \text{grad}^{\mathbb{S}^n} f), d\varphi(e_j)) \geq 0$, from the stable f -harmonic condition, and equation (21), we get

$$\begin{aligned} 0 &\leq \text{trace}_\alpha I_f^\varphi(d\varphi(v), d\varphi(v)) + \int_{\mathbb{S}^n} h((\nabla d\varphi)(e_j, \text{grad}^{\mathbb{S}^n} f), d\varphi(e_j))v^{\mathbb{S}^n} \\ &= -(n-2) \int_{\mathbb{S}^n} f_\varphi|d\varphi|^2v^{\mathbb{S}^n} \leq 0. \end{aligned}$$

Consequently, $|d\varphi| = 0$, that is φ is constant, because $n > 2$. \square

If $f = 1$ on $\mathbb{S}^n \times N$, we get the following result:

Corollary 2.2 ([12]). *Any stable harmonic map φ from sphere \mathbb{S}^n ($n > 2$) to Riemannian manifold (N, h) is constant.*

Using the similar technique we have:

Theorem 2.3. *Let (M, g) be a compact Riemannian manifold. When $n > 2$, any stable f -harmonic map $\varphi : M \rightarrow \mathbb{S}^n$ must be constant, where f is a smooth positive function on $M \times \mathbb{S}^n$, with $\Delta^{\mathbb{S}^n}(f) \circ \varphi \leq 0$.*

Proof. Choose a normal orthonormal frame $\{e_i\}$ at point x_0 in M . When the data is the same as those in the previous proof, we have

$$(22) \quad \nabla_{e_i}^\varphi f_\varphi \nabla_{e_i}^\varphi (v \circ \varphi) = \nabla_{\text{grad}^M f_\varphi}^\varphi (v \circ \varphi) + f_\varphi \nabla_{e_i}^\varphi \nabla_{e_i}^\varphi (v \circ \varphi),$$

the first term of (22) is given by

$$(23) \quad \nabla_{\text{grad}^M f_\varphi}^\varphi (v \circ \varphi) = -(\lambda \circ \varphi)d\varphi(\text{grad}^M f_\varphi),$$

the seconde term of (22) is given by

$$(24) \quad \begin{aligned} f_\varphi \nabla_{e_i}^\varphi \nabla_{e_i}^\varphi (v \circ \varphi) &= -f_\varphi \nabla_{e_i}^\varphi (\lambda \circ \varphi)d\varphi(e_i) \\ &= -f_\varphi d\varphi(\text{grad}^M(\lambda \circ \varphi)) - (\lambda \circ \varphi)f_\varphi \tau(\varphi), \end{aligned}$$

by the definition of gradient operator, we get

$$(25) \quad -f_\varphi d\varphi(\text{grad}^M(\lambda \circ \varphi)) = -f_\varphi \langle d\varphi(e_i), v \circ \varphi \rangle d\varphi(e_i),$$

substituting the formulas (23), (24), (25) into (22) gives

$$\begin{aligned} \nabla_{e_i}^\varphi f_\varphi \nabla_{e_i}^\varphi (v \circ \varphi) &= -(\lambda \circ \varphi) d\varphi(\text{grad}^M f_\varphi) - f_\varphi \langle d\varphi(e_i), v \circ \varphi \rangle d\varphi(e_i) \\ (26) \qquad \qquad \qquad &\quad - (\lambda \circ \varphi) f_\varphi \tau(\varphi), \end{aligned}$$

from the f -harmonicity condition of φ , and equation (26), we have

$$\begin{aligned} \langle \nabla_{e_i}^\varphi f_\varphi \nabla_{e_i}^\varphi (v \circ \varphi), v \circ \varphi \rangle &= -f_\varphi \langle d\varphi(e_i), v \circ \varphi \rangle \langle d\varphi(e_i), v \circ \varphi \rangle \\ (27) \qquad \qquad \qquad &\quad - (\lambda \circ \varphi) e(\varphi) \langle (\text{grad}^{\mathbb{S}^n} f) \circ \varphi, v \circ \varphi \rangle, \end{aligned}$$

since the sphere \mathbb{S}^n has constant curvature, we obtain

$$\begin{aligned} \langle f_\varphi R^{\mathbb{S}^n}(v \circ \varphi, d\varphi(e_i)) d\varphi(e_i), v \circ \varphi \rangle \\ (28) \qquad \qquad \qquad &= f_\varphi |d\varphi|^2 \langle v \circ \varphi, v \circ \varphi \rangle - f_\varphi \langle d\varphi(e_i), v \circ \varphi \rangle \langle d\varphi(e_i), v \circ \varphi \rangle, \end{aligned}$$

by the definition of Jacobi operator and equations (27), (28), we get

$$\begin{aligned} \langle J_f^\varphi(v \circ \varphi), v \circ \varphi \rangle &= 2f_\varphi \langle d\varphi(e_i), v \circ \varphi \rangle \langle d\varphi(e_i), v \circ \varphi \rangle \\ &\quad - f_\varphi |d\varphi|^2 \langle v \circ \varphi, v \circ \varphi \rangle \\ &\quad + (\lambda \circ \varphi) e(\varphi) \langle (\text{grad}^{\mathbb{S}^n} f) \circ \varphi, v \circ \varphi \rangle \\ &\quad + e(\varphi) \langle (\nabla_{v \circ \varphi}^{\mathbb{S}^n} \text{grad}^{\mathbb{S}^n} f) \circ \varphi, v \circ \varphi \rangle \\ &\quad - \langle d\varphi(\text{grad}^M(v \circ \varphi)(f)), v \circ \varphi \rangle \\ &\quad - \langle (v \circ \varphi)(f) \tau(\varphi), v \circ \varphi \rangle \\ (29) \qquad \qquad \qquad &\quad + \langle \nabla^\varphi v \circ \varphi, d\varphi \rangle \langle (\text{grad}^{\mathbb{S}^n} f) \circ \varphi, v \circ \varphi \rangle, \end{aligned}$$

so that

$$\begin{aligned} \text{trace}_\alpha \langle J_f^\varphi(v \circ \varphi), v \circ \varphi \rangle &= (2-n) f_\varphi |d\varphi|^2 \\ &\quad + e(\varphi) \text{trace}_\alpha(\text{Hess}^{\mathbb{S}^n} f)(v \circ \varphi, v \circ \varphi) \\ &\quad - \text{trace}_\alpha \langle d\varphi(\text{grad}^M(v \circ \varphi)(f)), v \circ \varphi \rangle \\ &\quad - \text{trace}_\alpha \langle \tau(\varphi), v \circ \varphi \rangle \langle (v \circ \varphi)(f) \rangle \\ (30) \qquad \qquad \qquad &\quad + \text{trace}_\alpha \langle \nabla^\varphi v \circ \varphi, d\varphi \rangle \langle (\text{grad}^{\mathbb{S}^n} f) \circ \varphi, v \circ \varphi \rangle, \end{aligned}$$

where $\text{Hess}^{\mathbb{S}^n} f$ is the hessian of the function f on \mathbb{S}^n , by the following formulas

$$\begin{aligned} d\varphi(\text{grad}^M(v \circ \varphi)(f)) &= e_i \langle v \circ \varphi, \text{grad}^{\mathbb{S}^n} f \rangle d\varphi(e_i) \\ &= \langle \nabla_{e_i}^\varphi (v \circ \varphi), \text{grad}^{\mathbb{S}^n} f \rangle d\varphi(e_i) \\ &\quad + \langle v \circ \varphi, \nabla_{e_i}^\varphi \text{grad}^{\mathbb{S}^n} f \rangle d\varphi(e_i) \\ &= -(\lambda \circ \varphi) \langle d\varphi(e_i), \text{grad}^{\mathbb{S}^n} f \rangle d\varphi(e_i) \\ &\quad + \langle v \circ \varphi, \nabla_{d\varphi(e_i)}^{\mathbb{S}^n} \text{grad}^{\mathbb{S}^n} f \rangle d\varphi(e_i), \\ -\text{trace}_\alpha \langle d\varphi(\text{grad}^M(v \circ \varphi)(f)), v \circ \varphi \rangle &= -(\text{Hess}^{\mathbb{S}^n} f)(d\varphi(e_i), d\varphi(e_i)), \end{aligned}$$

$$\begin{aligned}\langle \nabla^\varphi v \circ \varphi, d\varphi \rangle &= \langle \nabla_{e_i}^\varphi v \circ \varphi, d\varphi(e_i) \rangle \\ &= -(\lambda \circ \varphi) \langle d\varphi(e_i), d\varphi(e_i) \rangle \\ &= -(\lambda \circ \varphi) |d\varphi|^2,\end{aligned}$$

$$\text{trace}_\alpha \langle \nabla^\varphi v \circ \varphi, d\varphi \rangle \langle (\text{grad}^{\mathbb{S}^n} f) \circ \varphi, v \circ \varphi \rangle = 0,$$

$$\begin{aligned}-\text{trace}_\alpha \langle \tau(\varphi), v \circ \varphi \rangle (v \circ \varphi)(f) &= -\langle \tau(\varphi), \text{grad}^{\mathbb{S}^n} f \rangle \\ &= -\langle \nabla_{e_i}^\varphi d\varphi(e_i), \text{grad}^{\mathbb{S}^n} f \rangle \\ &= -\text{div } w + \langle d\varphi(e_i), \nabla_{d\varphi(e_i)}^{\mathbb{S}^n} \text{grad}^{\mathbb{S}^n} f \rangle,\end{aligned}$$

where $w(X) = \langle d\varphi(X), \text{grad}^{\mathbb{S}^n} f \rangle$, $\forall X \in \Gamma(TM)$, and (30), we have

$$\begin{aligned}\text{trace}_\alpha \langle J_f^\varphi(v \circ \varphi), v \circ \varphi \rangle &= (2-n) f_\varphi |d\varphi|^2 \\ &\quad + e(\varphi) \Delta^{\mathbb{S}^n}(f) \circ \varphi \\ &\quad - (\text{Hess}^{\mathbb{S}^n} f)(d\varphi(e_i), d\varphi(e_i)) \\ &\quad - \text{div } w + (\text{Hess}^{\mathbb{S}^n} f)(d\varphi(e_i), d\varphi(e_i)),\end{aligned}$$

where $\Delta^{\mathbb{S}^n}(f) \circ \varphi = \text{trace}_\alpha(\text{Hess}^{\mathbb{S}^n} f)(v \circ \varphi, v \circ \varphi)$, so that

$$\begin{aligned}\text{trace}_\alpha I_f^\varphi(v \circ \varphi, v \circ \varphi) &= (2-n) \int_M f_\varphi |d\varphi|^2 v^M \\ &\quad + \int_M e(\varphi) [\Delta^{\mathbb{S}^n}(f) \circ \varphi] v^M.\end{aligned}\tag{31}$$

Hence Theorem 2 follows from (31) and the stable f -harmonicity condition of φ with $n > 2$ and $\Delta^{\mathbb{S}^n}(f) \circ \varphi \leq 0$. \square

From Theorem 2, we deduce:

Corollary 2.4 ([7, 8]). *Let (M, g) be a compact Riemannian manifold. When $n > 2$, any stable harmonic map $\varphi : M \rightarrow \mathbb{S}^n$ must be constant.*

Corollary 2.5 ([3]). *Let (M, g) be a compact Riemannian manifold. When $n > 2$, any stable f -harmonic map $\varphi : M \rightarrow \mathbb{S}^n$ must be constant, where f is a smooth positive function on M .*

References

- [1] P. Baird and J. C. Wood, *Harmonic morphisms between Riemannian manifolds*, London Mathematical Society Monographs. New Series, 29, The Clarendon Press, Oxford University Press, Oxford, 2003.
- [2] R. Caddeo, S. Montaldo, and C. Oniciuc, *Biharmonic submanifolds of S^3* , Internat. J. Math. **12** (2001), no. 8, 867–876.
- [3] A. M. Cherif, M. Djaa, and K. Zegga, *Stable f -harmonic maps on sphere*, Commun. Korean Math. Soc. **30** (2015), no. 4, 471–479.
- [4] N. Course, *f -harmonic maps which map the boundary of the domain to one point in the target*, New York J. Math. **13** (2007), 423–435.

- [5] M. Djaa, A. M. Cherif, K. Zagga, and S. Ouakkas, *On the generalized of harmonic and bi-harmonic maps*, Int. Electron. J. Geom. **5** (2012), no. 1, 90–100.
- [6] J. Eells, Jr. and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math. **86** (1964), 109–160.
- [7] R. Howard and S. W. Wei, *Nonexistence of stable harmonic maps to and from certain homogeneous spaces and submanifolds of Euclidean space*, Trans. Amer. Math. Soc. **294** (1986), no. 1, 319–331.
- [8] Y. Ohnita, *Stability of harmonic maps and standard minimal immersions*, Tohoku Math. J. (2) **38** (1986), no. 2, 259–267.
- [9] B. O'Neill, *Semi-Riemannian Geometry*, Pure and Applied Mathematics, **103**, Academic Press, Inc., New York, 1983.
- [10] S. Ouakkas, R. Nasri, and M. Djaa, *On the f -harmonic and f -biharmonic maps*, J. P. J. Geom. Topol. **10** (2010), no. 1, 11–27.
- [11] Y. Xin, *Geometry of Harmonic Maps*, Progress in Nonlinear Differential Equations and their Applications, **23**, Birkhäuser Boston, Inc., Boston, MA, 1996.
- [12] Y. L. Xin, *Some results on stable harmonic maps*, Duke Math. J. **47** (1980), no. 3, 609–613.

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