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SPECTRA ORIGINATED FROM FREDHOLM THEORY AND BROWDER'S THEOREM

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Abstract. We give a new characterization of Browder's theorem through equality between the pseudo B-Weyl spectrum and the generalized Drazin spectrum. Also, we will give conditions under which pseudo B-Fredholm and pseudo B-Weyl spectrum introduced in [9] and [25] become stable under commuting Riesz perturbations.

1. Introduction and preliminaries

Throughout, X denotes a complex Banach space, $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X, let I be the identity operator, and for $T \in \mathcal{B}(X)$ we denote by T^*, $R(T)$, $R^\infty(T) = \bigcap_{n\geq 0} R(T^n)$, $\rho(T)$, $\sigma(T)$, $\sigma_p(T)$, $\sigma_{ap}(T)$ and $\sigma_{su}(T)$ respectively the adjoint, the range, the hyper-range, the resolvent set, the spectrum, the point spectrum, the approximate point spectrum and the surjectivety spectrum of T.

An operator $T \in \mathcal{B}(X)$ is said to be semi-regular, if $R(T)$ is closed and $N(T) \subseteq R^\infty(T)$. For subspaces M, N of X we write $M \subseteq^e N$ (M is essentially contained in N) if there exists a finite-dimensional subspace $F \subset X$ such that $M \subseteq N + F$. $T \in \mathcal{B}(X)$ is said to be essentially semi-regular, if $R(T)$ is closed and $N(T) \subseteq^e R^\infty(T)$. The corresponding spectra are the semi-regular spectrum $\sigma_{se}(T)$ and the essentially semi-regular spectrum $\sigma_{es}(T)$ defined by

$$\sigma_{se}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not semi-regular}\},$$

$$\sigma_{es}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not essentially semi-regular}\}, \text{ see [1]}.$$

Let E be a subset of X. E is said T-invariant if $T(E) \subseteq E$. We say that T is completely reduced by the pair (E, F) and we denote $(E, F) \in Red(T)$ if E and F are two closed T-invariant subspaces of X such that $X = E \oplus F$. In this case we write $T = T_{|E} \oplus T_{|F}$ and we say that T is the direct sum of $T_{|E}$ and $T_{|F}$. In the other hand, recall that an operator $T \in \mathcal{B}(X)$ admits a generalized Kato decomposition, (GKD for short), if there exists $(X_1, X_2) \in Red(T)$ such that

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$T_{|X_1}$ is semi-regular and $T_{|X_2}$ is quasi-nilpotent, in this case T is said a pseudo Fredholm operator. If we assume in the definition above that $T_{|X_2}$ is nilpotent, then T is said to be of Kato type. Clearly, every semi-regular operator is of Kato type and a quasi-nilpotent operator has a GKD, see [16, 18] for more information about generalized Kato decomposition.

Recall that $T \in \mathcal{B}(X)$ is said to be quasi-Fredholm if there exists $d \in \mathbb{N}$ such that

(1) $R(T^n) \cap N(T) = R(T^d) \cap N(T)$ for all $n \geq d$;
(2) $R(T^d) \cap N(T)$ and $R(T) + N(T^d)$ are closed in X.

An operator is quasi-Fredholm if it is quasi-Fredholm of some degree d. Note that semi-regular operators are quasi-Fredholm of degree 0 and by results of Labrousse [16], in the case of Hilbert spaces, the set of quasi-Fredholm operators coincides with the set of Kato type operators. For every bounded operator $T \in \mathcal{B}(X)$, let us define the essential quasi-Fredholm spectrum and generalized Kato spectrum respectively by:

$$\sigma_{eq}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not quasi-Fredholm}\};$$

$$\sigma_{gK}(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ does not admit a generalized Kato decomposition}\}.$$

It is know that $\sigma_{gK}(T)$ is always a compact subsets of the complex plane contained in the spectrum $\sigma(T)$ of T [12, Corollary 2.3]. Note that $\sigma_{gK}(T)$ is not necessarily non-empty. For example, all quasi-nilpotent operator has an empty generalized Kato spectrum, see [12, 13] for more information about $\sigma_{gK}(T)$.

A bounded linear operator is called an upper semi-Fredholm (resp, lower semi Fredholm) if $\dim N(T) < \infty$ and $R(T)$ is closed (resp, $\text{codim}R(T) < \infty$). T is semi-Fredholm if it is a lower or upper semi-Fredholm operator. The index of a semi-Fredholm operator T is defined by $ind(T) := \dim N(T) - \text{codim}R(T)$. Also, T is a Fredholm operator if it is a lower and upper semi-Fredholm operator, and T is called a Weyl operator if it is a Fredholm of index zero.

The essential and Weyl spectra of T are closed and defined by:

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a Fredholm operator}\};$$

$$\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not a Weyl operator}\}.$$

Recall that an operator $R \in \mathcal{B}(X)$ is said to be Riesz if $R - \mu I$ is Fredholm for every non-zero complex number μ. Of course compact and quasi-nilpotent operators are particular cases of Riesz operators.

Let $T \in \mathcal{B}(X)$, the ascent of T is defined by $a(T) = \min\{p \in \mathbb{N} : N(T^p) = N(T^{p+1})\}$ if such p does not exist we let $a(T) = \infty$. Analogously the descent of T is $d(T) = \min\{q \in \mathbb{N} : R(T^q) = R(T^{q+1})\}$ if such q does not exist we let $d(T) = \infty$ [23]. It is well known that if both $a(T)$ and $d(T)$ are finite, then $a(T) = d(T)$ and we have the decomposition $X = R(T^p) \oplus N(T^p)$ where $p = a(T) = d(T)$.

An operator $T \in \mathcal{B}(X)$ is upper semi-Browder if T is upper semi-Fredholm and $a(T) < \infty$. If $T \in \mathcal{B}(X)$ is lower semi-Fredholm and $d(T) < \infty$, then T is lower semi-Browder. T is called Browder operator if it is a lower and upper Browder operator.

An operator $T \in \mathcal{B}(X)$ is said to be B-Fredholm if for some integer $n \geq 0$ the range $R(T^n)$ is closed and T_n, the restriction of T to $R(T^n)$ is a Fredholm operator. This class of operators, introduced and studied by Berkani et al. in a series of papers extends the class of semi-Fredholm operators. T is said to be a B-Weyl operator if T_n is a Fredholm operator of index zero. The B-Fredholm and B-Weyl spectra are defined by

$$\sigma_{BF}(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not B-Fredholm}\};$$
$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not B-Weyl}\}.$$

Note that T is a B-Fredholm operator if there exists $(X_1, X_2) \in Red(T)$ such that $T_{\upharpoonright X_1}$ is Fredholm and $T_{\upharpoonright X_2}$ is nilpotent, see [8, Theorem 2.7]. Also, T is a B-Weyl operator if and only if $T_{\upharpoonright X_1}$ is a Weyl operator and $T_{\upharpoonright X_2}$ is a nilpotent operator.

More recently, B-Fredholm and B-Weyl operators were generalized to pseudo B-Fredholm and pseudo B-Weyl, see [9,25], precisely, T is a pseudo B-Fredholm operator, if there exists $(X_1, X_2) \in Red(T)$ such that $T_{\upharpoonright X_1}$ is a Fredholm operator and $T_{\upharpoonright X_2}$ is a quasi-nilpotent operator. T is said to be pseudo B-Weyl operator if there exists $(X_1, X_2) \in Red(T)$ such that $T_{\upharpoonright X_1}$ is a Weyl operator and $T_{\upharpoonright X_2}$ is a quasi-nilpotent operator. The pseudo B-Fredholm and pseudo B-Weyl spectra are defined by:

$$\sigma_{pBF}(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not pseudo B-Fredholm}\};$$
$$\sigma_{gD\mathcal{W}}(T) = \{\lambda \in \mathbb{C} : \ T - \lambda I \text{ is not pseudo B-Weyl}\}.$$

Let $T \in \mathcal{B}(X)$, T is said to be Drazin invertible if there exist a positive integer k and an operator $S \in \mathcal{B}(X)$ such that

$$ST = TS, \ \ T^{k+1}S = T^k \quad \text{and} \quad S^2T = S.$$

Which is also equivalent to the fact that $T = T_1 \oplus T_2$; where T_1 is invertible and T_2 is nilpotent. The Drazin spectrum is defined by

$$\sigma_D(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not Drazin invertible}\}.$$

The concept of Drazin invertible operators has been generalized by Koliha [14]. In fact, $T \in \mathcal{B}(X)$ is generalized Drazin invertible if and only if $0 \notin acc(\sigma(T))$, where $acc(\sigma(T))$ is the set of accumulation points of $\sigma(T)$. This is also equivalent to the fact that there exists $(X_1, X_2) \in Red(T)$ such that $T_{\upharpoonright X_1}$ is invertible and $T_{\upharpoonright X_2}$ is quasi-nilpotent. The generalized Drazin spectrum is defined by

$$\sigma_{gD}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not generalized Drazin invertible}\}.$$

The concept of analytical core for an operator has been introduced by Vrbova in [24] and study by Mbekhta [18, 19], that is the following set:

$$K(T)=\{x\in X\ :\ \exists(x_n)_{n\geq 0}\subset X \text{ and } \delta>0 \text{ such that } x_0=x,\ Tx_n=x_{n-1}\ \forall n\geq 1 \text{ and } \|x_n\|\leq \delta^n\|x\|\}.$$

The quasi-nilpotent part of T, $H_0(T)$ is given by:

$$H_0(T):=\{x\in X; r_T(x)=0\} \text{ where } r_T(x)=\lim_{n\to+\infty}||T^nx||^{\frac{1}{n}}.$$

In [11], M. D. Cvetković and SČ. Živković-Zlatanović introduced and studied a new concept of generalized Drazin invertibility of bounded operators as a generalization of generalized Drazin invertible operators. In fact, an operator $T\in\mathcal{B}(X)$ is said to be generalized Drazin bounded below if $H_0(T)$ is closed and complemented with a subspace M in X such that $(M,H_0(T))\in Red(T)$ and $T(M)$ is closed which is equivalent to there exists $(M,N)\in Red(T)$ such that $T_{|M}$ is bounded below and $T_{|N}$ is quasi-nilpotent, see [11, Theorem 3.6]. An operator $T\in\mathcal{B}(X)$ is said to be generalized Drazin surjective if $K(T)$ is closed and complemented with a subspace N in X such that $N\subseteq H_0(T)$ and $(K(T),N)\in Red(T)$ which is equivalent to there exists $(M,N)\in Red(T)$ such that $T_{|M}$ is surjective and $T_{|N}$ is quasi-nilpotent, see [11, Theorem 3.7].

The generalized Drazin bounded below and surjective spectra of $T\in\mathcal{B}(X)$ are defined respectively by:

$$\sigma_{gD\mathcal{M}}(T)=\{\lambda\in\mathbb{C},\ T-\lambda I \text{ is not generalized Drazin bounded below}\};$$
$$\sigma_{gD\mathcal{Q}}(T)=\{\lambda\in\mathbb{C},\ T-\lambda I \text{ is not generalized Drazin surjective}\}.$$

From [11], we have:

$$\sigma_{gD}(T)=\sigma_{gD\mathcal{M}}(T)\cup\sigma_{gD\mathcal{Q}}(T).$$

As a continuation of works [5–7, 9, 11, 25], we will study various spectra originated from Fredholm theory and related to Drazin spectrum. After given preliminaries results, in the second section of this work, we characterize the equality between the pseudo B-Weyl spectrum and generalized Drazin spectrum by means of the Browder's theorem. Also, we will give serval necessary and sufficient conditions for T to have equality between the spectra originated from Fredholm theory and Drazin invertibility. In the same direction as our work [22], we will give conditions under which pseudo B-Fredholm and pseudo B-Weyl spectrum are stable under commuting Riesz perturbations. In section four, we will prove that we can perturb a pseudo B-Fredholm (resp. pseudo Fredholm) operator $T\in\mathcal{B}(X)$ by a bounded operator S commuting with T to obtain a Fredholm (resp. semi-regular operator) $T+S$.

2. On pseudo semi B-Fredholm (Weyl) operators

In the following, we introduce the definition of pseudo upper B-Fredholm, pseudo lower B-Fredholm, generalized Drazin lower semi-Weyl, generalized Drazin upper semi-Weyl and pseudo semi B-Fredholm operators.

Definition 2.1 ([11])**.** An operator $T \in \mathcal{B}(X)$ is said to be pseudo upper B-Fredholm if there exist two T-invariant closed subspaces X_1 and X_2 of X such that $X = X_1 \oplus X_2$ and $T_{|X_1}$ is upper semi-Fredholm operator and $T_{|X_2}$ is quasi-nilpotent. If $ind(T_{|X_1}) \leq 0$, T is said to be generalized Drazin upper semi-Weyl.

Definition 2.2 ([11])**.** An operator $T \in \mathcal{B}(X)$ is said to be pseudo lower B-Fredholm if there exist two T-invariant closed subspaces X_1 and X_2 of X such that $X = X_1 \oplus X_2$ and $T_{|X_1}$ is lower semi-Fredholm operator and $T_{|X_2}$ is quasi-nilpotent. If $ind(T_{|X_1}) \leq 0$, T is said to be generalized Drazin lower semi-Weyl.

Definition 2.3. We say that $T \in \mathcal{B}(X)$ is pseudo semi B-Fredholm if T is pseudo lower B-Fredholm or pseudo upper B-Fredholm.

It is clear that T is a pseudo B-Fredholm operator if and only if T is a pseudo lower semi B-Fredholm operator and pseudo upper semi B-Fredholm operator. In the same way T is pseudo B-Weyl if and only if T is generalized Drazin lower semi-Weyl and generalized Drazin upper semi-Weyl. The generalized Drazin lower semi-Weyl and generalized Drazin upper semi-Weyl spectra of $T \in \mathcal{B}(X)$ are defined respectively by:

$$\sigma_{gD\mathcal{W}-}(T) = \{\lambda \in \mathbb{C},\ T - \lambda I \text{ is not generalized Drazin lower semi-Weyl}\};$$

$$\sigma_{gD\mathcal{W}+}(T) = \{\lambda \in \mathbb{C},\ T - \lambda I \text{ is not generalized Drazin upper semi-Weyl}\}.$$

From [11], we have:

$$\sigma_{gD\mathcal{W}}(T) = \sigma_{gD\mathcal{W}+}(T) \cup \sigma_{gD\mathcal{W}-}(T);$$

The pseudo upper and lower B-Fredholm spectra of $T \in \mathcal{B}(X)$ are defined respectively by:

$$\sigma_{puBF}(T) = \{\lambda \in \mathbb{C},\ T - \lambda I \text{ is not pseudo upper B-Fredholm}\};$$

$$\sigma_{plBF}(T) = \{\lambda \in \mathbb{C},\ T - \lambda I \text{ is not pseudo lower B-Fredholm}\}.$$

Also, from [11], we have:

$$\sigma_{pBF}(T) = \sigma_{puBF}(T) \cup \sigma_{plBF}(T).$$

The following results gives some relationship between pseudo upper/lower B-Fredholm operator in terms of generalized Drazin invertibility.

Proposition 2.1. *Let $T \in \mathcal{B}(X)$. If there exists $(N, F) \in Red(T)$ such that* $\operatorname{codim} F < \infty$, $\dim N < \infty$ *and $T_{|F}$ is generalized Drazin bounded below, then T is pseudo upper B-Fredholm.*

Proof. If there exists $(N, F) \in Red(T)$ such that $\operatorname{codim} F < \infty$, $\dim N < \infty$ and $T_{|F}$ is generalized Drazin bounded below, then $X = F \oplus N$. Since $T_{|F}$ is generalized Drazin bounded below, then there exist two closed T-invariant subspaces F_1 and F_2 of F such that $F = F_1 \oplus F_2$, $T_{|F_1}$ is bounded below and $T_{|F_2}$ is quasi-nilpotent, then $X = F_1 \oplus F_2 \oplus N$. Let $M = F_1 \oplus N$, $T(M) = T(F_1) +$

$T(N)$, since $T_{|F_1}$ is bounded below, then $T(F_1)$ is closed. Since $\dim N < \infty$, then $T(M)$ is closed. Now we have

$$N(T_{|M}) = N(T_{|F_1}) \oplus N(T_{|N}) = N(T_{|N}) \subseteq N,$$

because $T_{|F_1}$ is bounded below. Therefore, $T_{|M}$ is upper Fredholm and $T_{|F_2}$ is quasi-nilpotent. Thus T is pseudo upper B-Fredholm. □

Proposition 2.2. *Let $T \in \mathcal{B}(X)$. If T is pseudo lower B-Fredholm, then there exists $F \subseteq X$ such that* $\text{codim} F < \infty$ *and $T_{|F}$ is generalized Drazin surjective.*

Conversely, If there exists $(N, F) \in Red(T)$ such that $\text{codim} F < \infty$, $\dim N < \infty$ *and $T_{|F}$ is generalized Drazin surjective, then T is pseudo lower B-Fredholm*

Proof. If T is pseudo lower B-Fredholm, then there exist two closed T-invariant subspaces X_1 and X_2 of X such that $X = X_1 \oplus X_2$ and $T_1 = T_{|X_1}$ is lower semi-Fredholm and $T_2 = T_{|X_2}$ is quasi-nilpotent. Since T_1 is lower semi-Fredholm, then $\text{codim} R(T_1) < \infty$, hence there exists $N \subseteq X_1$ such that, $\dim N < \infty$ and $X_1 = R(T_1) \oplus N$. Thus, $X = N \oplus R(T_1) \oplus X_2$. Let $F = R(T_1) \oplus X_2$, then $\text{codim} F < \infty$ and $T_{|R(T_1)}$ is surjective and T_2 is quasi-nilpotent, so T is generalized Drazin surjective.

Conversely, if there exists $(N, F) \in Red(T)$ such that $\text{codim} F < \infty$, $\dim N < \infty$ and $T_{|F}$ is generalized Drazin surjective. Since $T_{|F}$ is generalized Drazin surjective, then there exist two closed T-invariant subspaces F_1 and F_2 of F such that $F = F_1 \oplus F_2$ and $T_{|F_1}$ is surjective and $T_{|F_2}$ is quasi-nilpotent, then $X = F_1 \oplus F_2 \oplus N$. Let $M = F_1 \oplus N$, since $T_{|F_1}$ is surjective, then $T_{|F_1}$ is lower Fredholm. Since $T_{|N}$ is finite rank operator, so $T_{|M} = T_{|F_1} \oplus T_{|N}$ is lower Fredholm. Therefore, $T_{|F_1} \oplus T_{|N}$ is lower Fredholm and $T_{|F_2}$ is quasi-nilpotent. So, T is pseudo lower B-Fredholm. □

Recall that $T \in \mathcal{B}(X)$ is said to have the single valued extension property at $\lambda_0 \in \mathbb{C}$ (SVEP for short) if for every open neighbourhood $U \subseteq \mathbb{C}$ of λ_0, the only analytic function $f : U \longrightarrow X$ which satisfies the equation $(T - zI)f(z) = 0$ for all $z \in U$ is the function $f \equiv 0$. An operator T is said to have the SVEP if T has the SVEP for every $\lambda \in \mathbb{C}$. Obviously, every operator $T \in \mathcal{B}(X)$ has the SVEP at every $\lambda \in \rho(T) = \mathbb{C} \setminus \sigma(T)$, hence T and T^* have the SVEP at every point of the boundary $\partial(\sigma(T))$ of the spectrum. Also, we have the implication

$$a(T) < \infty \Longrightarrow T \text{ has SVEP at } 0,$$

$$d(T) < \infty \Longrightarrow T^* \text{ has SVEP at } 0.$$

In [11], the authors gave some examples showing that $\sigma_{gD\mathcal{M}}(T) \subset \sigma_{gD\mathcal{W}+}(T)$, $\sigma_{gD\mathcal{Q}}(T) \subset \sigma_{gD\mathcal{W}-}(T)$ and $\sigma_{gD}(T) \subset \sigma_{gD\mathcal{W}}(T)$ can be proper. In the following results we give serval necessary and sufficient conditions for T to have equality.

Proposition 2.3. *Let $T \in \mathcal{B}(X)$. Then $\sigma_{gD\mathcal{M}}(T) = \sigma_{gD\mathcal{W}+}(T)$ if and only if T has SVEP at every $\lambda \notin \sigma_{gD\mathcal{W}+}(T)$.*

Proof. Assume that T has SVEP at every $\lambda \notin \sigma_{gD\mathcal{W}+}(T)$. If $\lambda \notin \sigma_{gD\mathcal{W}+}(T)$, then $T-\lambda I$ is generalized Drazin upper semi-Weyl, then there exists $(M,N) \in Red(T)$ such that $(T-\lambda I)_{|M}$ is semi-regular and $(T-\lambda I)_{|N}$ is quasi-nilpotent. T has SVEP at every $\lambda \notin \sigma_{gD\mathcal{W}+}(T)$, it follows that $(T-\lambda I)_{|M}$ has the SVEP at 0, then $(T-\lambda I)_{|M}$ is bounded below. Hence $T-\lambda I$ is generalized Drazin bounded below, $\lambda \notin \sigma_{gD\mathcal{M}}(T)$, and since the reverse implication holds for every operator we conclude that $\sigma_{gD\mathcal{M}}(T) = \sigma_{gD\mathcal{W}+}(T)$. Conversely, suppose that $\sigma_{gD\mathcal{M}}(T) = \sigma_{gD\mathcal{W}+}(T)$. If $\lambda \notin \sigma_{gD\mathcal{W}+}(T)$, then $T-\lambda I$ is generalized Drazin bounded below so $H_0(T-\lambda I)$ is closed. By [3, Theorem 1.7], T has SVEP at λ. □

We denote by $\sigma_{lB}(T)$ and $\sigma_{lW}(T)$ respectively the lower Browder and lower Weyl spectra. In the same way we have the following result.

Proposition 2.4. *Let $T \in \mathcal{B}(X)$. Then $\sigma_{gD\mathcal{Q}}(T) = \sigma_{gD\mathcal{W}-}(T)$ if and only if T^* has SVEP at every $\lambda \notin \sigma_{gD\mathcal{W}-}(T)$.*

Proof. Suppose that T has SVEP at every $\lambda \notin \sigma_{gD\mathcal{W}-}(T)$. If $\lambda \notin \sigma_{gD\mathcal{W}-}(T)$, then by [11, Theorem 3.7], $T-\lambda I$ admits GKD and $\lambda \notin acc\sigma_{lW}(T)$. T^* has SVEP at every $\lambda \notin \sigma_{gD\mathcal{W}-}(T)$, then T^* has SVEP at every $\lambda \notin \sigma_{lW}(T)$, and so $\sigma_{lB}(T) = \sigma_{lW}(T)$. Then $\lambda \notin acc\sigma_{lB}(T)$. Therefore, $T-\lambda I$ is generalized Drazin surjective according to [11, Theorem 3.7], $\lambda \notin \sigma_{gD\mathcal{Q}}(T)$ and since the reverse implication holds for every operator we conclude that $\sigma_{gD\mathcal{Q}}(T) = \sigma_{gD\mathcal{W}-}(T)$. Conversely, suppose that $\sigma_{gD\mathcal{Q}}(T) = \sigma_{gD\mathcal{W}-}(T)$. If $\lambda \notin \sigma_{gD\mathcal{W}-}(T)$, then $T-\lambda I$ is generalized Drazin surjective then $K(T-\lambda I)$ is closed and complemented with a subspace N in X such that $N \subseteq H_0(T-\lambda I)$ and $(K(T-\lambda I), N) \in Red(T-\lambda I)$, so $K(T-\lambda) + H_0(T-\lambda) = X$. From [3, Theorem 1.7], T^* has the SVEP at λ. □

As a consequence of the two previous results we have the following proposition.

Proposition 2.5. *Let $T \in \mathcal{B}(X)$. Then $\sigma_{gD}(T) = \sigma_{gD\mathcal{W}}(T)$ if and only if T and T^* have the SVEP at every $\lambda \notin \sigma_{gD\mathcal{W}}(T)$*

A bounded linear operator T is said to satisfy Browder's theorem if $\sigma_W(T) = \sigma_B(T)$, or equivalently $acc\sigma(T) \subseteq \sigma_W(T)$, where $\sigma_B(T)$ is the Browder spectrum of T.

It is known from [2] that a-Browder's theorem holds for T if $\sigma_{uW}(T) = \sigma_{uB}(T)$, or equivalently $acc\sigma_{ap}(T) \subseteq \sigma_{uW}(T)$, where $\sigma_{uB}(T)$ and $\sigma_{uW}(T)$ are the upper semi-Browder and upper semi-Weyl spectra of T.

The following result shows that Browder's (a-Browder's) theorem holds for T precisely when $\sigma_{gD}(T) = \sigma_{gD\mathcal{W}}(T)$ $(\sigma_{gD\mathcal{M}}(T) = \sigma_{gD\mathcal{W}+}(T))$, which give new characterizations for Browder's and a-Browder's theorems.

Theorem 2.1. *Let $T \in \mathcal{B}(X)$. Then*

1) *a-Browder's theorem holds for T if and only if $\sigma_{gD\mathcal{M}}(T) = \sigma_{gD\mathcal{W}+}(T)$.*

2) *a-Browder's theorem holds for* T^* *if and only if* $\sigma_{gD\mathcal{Q}}(T) = \sigma_{gD\mathcal{W}-}(T)$.
3) *Browder's theorem holds for* T *if and only if* $\sigma_{gD}(T) = \sigma_{gD\mathcal{W}}(T)$.

Proof. 1) Suppose that a-Browder's theorem holds for T implies $\sigma_{uB}(T) = \sigma_{uW}(T)$.
Using [11, Theorems 3.4 and 3.6], we conclude that

$$\begin{aligned}\lambda \notin \sigma_{gD\mathcal{M}}(T) &\iff T-\lambda I \text{ is generalized Drazin bounded below}\\ &\iff T-\lambda I \text{ admits a GKD and } \lambda \notin acc\sigma_{uB}(T)\\ &\iff T-\lambda I \text{ admits a GKD and } \lambda \notin acc\sigma_{uW}(T)\\ &\iff T-\lambda I \text{ is generalized Drazin upper semi-Weyl}\\ &\iff \lambda \notin \sigma_{gD\mathcal{W}+}(T).\end{aligned}$$

Hence $\sigma_{gD\mathcal{M}}(T) = \sigma_{gD\mathcal{W}+}(T)$. Conversely, if $\sigma_{gD\mathcal{M}}(T) = \sigma_{gD\mathcal{W}+}(T)$, from Proposition 2.3, T has SVEP at every $\lambda \notin \sigma_{gD\mathcal{W}+}(T)$. Since $\sigma_{gD\mathcal{W}+}(T) \subseteq \sigma_{uW}(T)$, T has SVEP at every $\lambda \notin \sigma_{uW}(T)$, so a-Browder's theorem holds for T, see [2, Theorem 4.34].
2) Suppose that a-Browder's theorem holds for T^* then $\sigma_{lB}(T) = \sigma_{lW}(T)$.
Using [11, Theorems 3.4 and 3.7] we have

$$\begin{aligned}\lambda \notin \sigma_{gD\mathcal{Q}}(T) &\iff T-\lambda I \text{ is generalized Drazin surjective}\\ &\iff T-\lambda I \text{ admits a GKD and } \lambda \notin acc\sigma_{lB}(T)\\ &\iff T-\lambda I \text{ admits a GKD and } \lambda \notin acc\sigma_{lW}(T)\\ &\iff T-\lambda I \text{ is generalized Drazin lower semi-Weyl}\\ &\iff \lambda \notin \sigma_{gD\mathcal{W}-}(T).\end{aligned}$$

Hence $\sigma_{gD\mathcal{Q}}(T) = \sigma_{gD\mathcal{W}-}(T)$. Conversely, if $\sigma_{gD\mathcal{Q}}(T) = \sigma_{gD\mathcal{W}-}(T)$, from Proposition 2.4, T^* has SVEP at every $\lambda \notin \sigma_{gD\mathcal{W}-}(T)$. Since $\sigma_{gD\mathcal{W}-}(T) \subseteq \sigma_{lW}(T)$, T^* has SVEP at every $\lambda \notin \sigma_{lW}(T)$, so a-Browder's theorem holds for T^*, see [2, Theorem 4.34].
3) Suppose that Browder's theorem holds for T then $\sigma_B(T) = \sigma_W(T)$.
Using [11, Theorems 3.4 and 3.9] we have

$$\begin{aligned}\lambda \notin \sigma_{gD}(T) &\iff T-\lambda I \text{ is generalized Drazin invertible}\\ &\iff T-\lambda I \text{ admits a GKD and } \lambda \notin acc\sigma_{B}(T)\\ &\iff T-\lambda I \text{ admits a GKD and } \lambda \notin acc\sigma_{W}(T)\\ &\iff T-\lambda I \text{ is generalized Drazin Weyl}\\ &\iff \lambda \notin \sigma_{gD\mathcal{W}}(T).\end{aligned}$$

Hence $\sigma_{gD}(T) = \sigma_{gD\mathcal{W}}(T)$. Conversely, if $\sigma_{gD}(T) = \sigma_{gD\mathcal{W}}(T)$, from Proposition 2.5, T and T^* has SVEP at every $\lambda \notin \sigma_{gD\mathcal{W}}(T)$. Since $\sigma_{gD\mathcal{W}}(T) \subseteq \sigma_W(T)$, T has SVEP at every $\lambda \notin \sigma_W(T)$, so Browder's theorem holds for T, see [2, Theorem 4.23]. □

It will be said that generalized Browder's theorem holds for $T \in \mathcal{B}(X)$ if $\sigma_{BW}(T) = \sigma(T)\setminus\Pi(T)$, equivalently, $\sigma_{BW}(T) = \sigma_D(T)$, where $\Pi(T)$ is the set of all poles of the resolvent of T ([4]). A classical result of the second author and H. Zguitti [6, Theorem 2.1] shows that Browder's theorem and generalized Browder's theorem are equivalent. According to the previous results and the equivalent between Browder's theorem and generalized Browder's theorem [6, Theorem 2.1] we have the following theorem.

Theorem 2.2. *Let $T \in \mathcal{B}(X)$. The statements are equivalent:*
1) *Browder's theorem holds for T;*
2) *generalized Browder's theorem holds for T;*
3) *T and T^* have SVEP at every $\lambda \notin \sigma_{gD\mathcal{W}}(T)$;*
4) *$\sigma_{gD}(T) = \sigma_{gD\mathcal{W}}(T)$.*

In the same way we have the following result.

Theorem 2.3. *Let $T \in \mathcal{B}(X)$. The statements are equivalent:*
1) *a-Browder's theorem holds for T;*
2) *generalized a-Browder's theorem holds for T;*
3) *T has SVEP at every $\lambda \notin \sigma_{gD\mathcal{W}+}(T)$;*
4) *$\sigma_{gD\mathcal{M}}(T) = \sigma_{gD\mathcal{W}+}(T)$.*

We denote by $\sigma_{lf}(T)$ and $\sigma_{uf}(T)$, $T \in \mathcal{B}(X)$, respectively the lower and upper semi-Fredholm spectra. Concerning the pseudo upper/lower B-Fredholm spectrum and the generalized Drazin bounded below/surjective spectrum, we have the following characterization. Note that $\sigma_{puBF}(T) \subset \sigma_{gD\mathcal{M}}(T)$, $\sigma_{plBF}(T) \subset \sigma_{gD\mathcal{Q}}(T)$ and $\sigma_{pBF}(T) \subset \sigma_{gD}(T)$ are strict [11].

Theorem 2.4. *Let $T \in \mathcal{B}(X)$. The statements are equivalent:*
1) *$\sigma_{uf}(T) = \sigma_{uB}(T)$;*
2) *T has SVEP at every $\lambda \notin \sigma_{uf}(T)$;*
3) *T has SVEP at every $\lambda \notin \sigma_{puBF}(T)$;*
4) *$\sigma_{gD\mathcal{M}}(T) = \sigma_{puBF}(T)$.*

Proof. 1) $\Longleftrightarrow$ 2): Suppose that T has SVEP at every $\lambda \notin \sigma_{uf}(T)$. If $\lambda \notin \sigma_{uf}(T)$, $T-\lambda I$ is upper semi-Fredholm. T has SVEP at λ, then $a(T-\lambda I) < \infty$, see [1, Theorem 3.16]. So $\lambda \notin \sigma_{uB}(T)$. Now, Suppose that $\sigma_{uf}(T) = \sigma_{uB}(T)$. Let $\lambda \notin \sigma_{uf}(T)$, $\lambda \notin \sigma_{uB}(T)$ then $a(T-\lambda I) < \infty$, hence T has SVEP at λ by [1].

3) $\Longleftrightarrow$ 4): Suppose that T has SVEP at every $\lambda \notin \sigma_{puBF}(T)$. If $\lambda \notin \sigma_{puBF}(T)$, $T-\lambda I$ is pseudo upper B-Fredholm, then there exists $(M,N) \in Red(T)$ such that $(T-\lambda I)_{|M}$ is semi-regular and $(T-\lambda I)_{|N}$ is quasinilpotent. T has SVEP at every $\lambda \notin \sigma_{puBF}(T)$ implies $(T-\lambda I)_{|M}$ has the SVEP at 0, it follows that $(T-\lambda I)_{|M}$ is bounded below. Hence $T-\lambda I$ is generalized Drazin bounded below, $\lambda \notin \sigma_{gD\mathcal{M}}(T)$, and since the reverse implication holds for every operator we conclude that $\sigma_{gD\mathcal{M}}(T) = \sigma_{puBF}(T)$. Conversely, assume that $\sigma_{gD\mathcal{M}}(T) = \sigma_{puBF}(T)$. If $\lambda \notin \sigma_{puBF}(T)$, then $T-\lambda I$ is generalized Drazin bounded below so $H_0(T-\lambda I)$ is closed. By [3, Theorem 1.7], T has the SVEP at λ.

1) $\Longleftrightarrow$ 4): Suppose that $\sigma_{uf}(T) = \sigma_{uB}(T)$.
According to [11, Theorems 3.4 and 3.6] we have

$$\begin{aligned}
\lambda \notin \sigma_{gD\mathcal{M}}(T) &\Longleftrightarrow T - \lambda I \text{ is generalized Drazin bounded below}\\
&\Longleftrightarrow T - \lambda I \text{ admits a GKD and } \lambda \notin acc\sigma_{uB}(T)\\
&\Longleftrightarrow T - \lambda I \text{ admits a GKD and } \lambda \notin acc\sigma_{uf}(T)\\
&\Longleftrightarrow T - \lambda I \text{ is pseudo upper B-Fredholm}\\
&\Longleftrightarrow \lambda \notin \sigma_{puBF}(T).
\end{aligned}$$

Hence $\sigma_{gD\mathcal{M}}(T) = \sigma_{puBF}(T)$. Conversely, if $\sigma_{gD\mathcal{M}}(T) = \sigma_{puBF}(T)$, then by 3) $\Longleftrightarrow$ 4), T has SVEP at every $\lambda \notin \sigma_{puBF}(T)$. Since $\sigma_{puBF}(T) \subseteq \sigma_{uf}(T)$, T has SVEP at every $\lambda \notin \sigma_{uf}(T)$, 1) $\Longleftrightarrow$ 2) gives the result. □

Theorem 2.5. *Let $T \in \mathcal{B}(X)$. The statements are equivalent*:
1) $\sigma_{lf}(T) = \sigma_{lB}(T)$;
2) T^* *has SVEP at every* $\lambda \notin \sigma_{lf}(T)$;
3) T^* *has SVEP at every* $\lambda \notin \sigma_{plBF}(T)$;
4) $\sigma_{gD\mathcal{Q}}(T) = \sigma_{plBF}(T)$.

Proof. 1) $\Longleftrightarrow$ 2): Suppose that T^* has SVEP at every $\lambda \notin \sigma_{lf}(T)$. $\lambda \notin \sigma_{lf}(T)$ implies that $T - \lambda I$ is lower semi-Fredholm. T^* has SVEP at λ, then $d(T - \lambda I) < \infty$, see [1, Theorem 3.17]. So $\lambda \notin \sigma_{lB}(T)$. Now, Suppose that $\sigma_{lf}(T) = \sigma_{lB}(T)$. Let $\lambda \notin \sigma_{lf}(T)$, $\lambda \notin \sigma_{lB}(T)$ then $d(T - \lambda I) < \infty$, hence T^* has SVEP at λ by [1].

3) $\Longleftrightarrow$ 4): Suppose that T^* has SVEP at every $\lambda \notin \sigma_{plBF}(T)$. If $\lambda \notin \sigma_{plBF}(T)$, $T - \lambda I$ admits GKD and $\lambda \notin acc\sigma_{lf}(T)$ by [11, Theorem 3.4]. T^* has SVEP at every $\lambda \notin \sigma_{plBF}(T)$, it follows that T^* has SVEP at every $\lambda \notin \sigma_{lf}(T)$, then $\sigma_{lB}(T) = \sigma_{lf}(T)$ so $\lambda \notin acc\sigma_{lB}(T)$. Therefore, $T - \lambda I$ is generalized Drazin surjective [11, Theorem 3.7], $\lambda \notin \sigma_{gD\mathcal{Q}}(T)$ and since the reverse implication holds for every operator we conclude that $\sigma_{gD\mathcal{Q}}(T) = \sigma_{plBF}(T)$. Conversely, suppose that $\sigma_{gD\mathcal{Q}}(T) = \sigma_{plBF}(T)$, if $\lambda \notin \sigma_{plBF}(T)$, then $T - \lambda I$ is generalized Drazin surjective then $K(T - \lambda I)$ is closed and complemented with a subspace N in X such that $N \subseteq H_0(T - \lambda I)$ and $(K(T - \lambda I), N) \in Red(T - \lambda I)$, so $K(T - \lambda I) + H_0(T - \lambda I) = X$. From [3, Theorem 1.7], T^* has SVEP at λ.

1) $\Longleftrightarrow$ 4): Suppose that $\sigma_{lf}(T) = \sigma_{lB}(T)$.
According to [11, Theorems 3.4 and 3.7] we have

$$\begin{aligned}
\lambda \notin \sigma_{gD\mathcal{Q}}(T) &\Longleftrightarrow T - \lambda I \text{ is generalized Drazin surjective}\\
&\Longleftrightarrow T - \lambda I \text{ admits a GKD and } \lambda \notin acc\sigma_{lB}(T)\\
&\Longleftrightarrow T - \lambda I \text{ admits a GKD and } \lambda \notin acc\sigma_{lf}(T)\\
&\Longleftrightarrow T - \lambda I \text{ is pseudo lower B-Fredholm}\\
&\Longleftrightarrow \lambda \notin \sigma_{plBF}(T).
\end{aligned}$$

Hence $\sigma_{gD\mathcal{Q}}(T) = \sigma_{plBF}(T)$. Conversely, if $\sigma_{gD\mathcal{Q}}(T) = \sigma_{plBF}(T)$, by 3) $\Longleftrightarrow$ 4), T^* has SVEP at every $\lambda \notin \sigma_{plBF}(T)$. Since $\sigma_{plBF}(T) \subseteq \sigma_{lf}(T)$, T has SVEP at every $\lambda \notin \sigma_{lf}(T)$, according to 1) $\Longleftrightarrow$ 2) we obtain the result. □

As a direct consequence of the Theorem 2.4 and Theorem 2.5 we have the following corollary.

Corollary 2.1. *Let $T \in \mathcal{B}(X)$. The statements are equivalent*:
1) $\sigma_e(T) = \sigma_B(T)$;
2) *T and T^* have SVEP at every $\lambda \notin \sigma_e(T)$;*
3) *T and T^* have SVEP at every $\lambda \notin \sigma_{BF}(T)$;*
4) $\sigma_{BF}(T) = \sigma_D(T)$;
5) *T and T^* have SVEP at every $\lambda \notin \sigma_{pBF}(T)$;*
6) $\sigma_{gD}(T) = \sigma_{pBF}(T)$.

3. Perturbations

Now, we consider the classes of operators introduced in [11]:

$$gDR := \{T \in \mathcal{B}(X);\text{ there exists } (M,N) \in Red(T) \text{ such that } T_{|M} \in R \text{ and } T_{|N} \text{ is quasinilpotent}\}.$$

$$DR := \{T \in \mathcal{B}(X);\text{ there exists } (M,N) \in Red(T) \text{ such that } T_{|M} \in R \text{ and } T_{|N} \text{ is nilpotent}\}.$$

Where R denote any of the following classes: bounded below/surjective operators, upper(lower) semi-Fredholm operators, Fredholm operator, upper(lower) semi-Weyl operators.

Proposition 3.1. *Let $T \in \mathcal{B}(X)$. If $T \in gDR$, then there exists $\alpha > 0$ such that for every $S \in \mathcal{B}(X)$ invertible operator satisfying $ST = TS$ and $||S|| < \alpha$, we have $T - S \in DR$.*

Proof. If $T \in gDR$, then T admits a GKD and $0 \in acc\sigma_R(T)$, see ([11]). From [10, Theorem 2.1] $T - S$ is semi-regular, and since $acc\sigma_R(T-S) = acc\sigma_R(T)$, $\sigma_R(T)$ the spectrum associated to the class R, then T is of Kato type and $0 \in acc\sigma_R(T-S)$. According to [11, Theorem 4.1], $T - S \in DR$. □

Let $\mathcal{F}(X)$ denote the ideal of finite rank operators on X. A bounded linear operator $F \in \mathcal{B}(X)$ is power finite rank if $F^n \in \mathcal{F}(X)$ for some $n \in \mathbb{N}$. In what follow, we will prove that pseudo B-Weyl operators satisfying Browder's theorem is stable by power finite rank perturbations.

Proposition 3.2. *Let $T \in \mathcal{B}(X)$, $F^n \in \mathcal{F}(X)$ for some $n \in \mathbb{N}$ commutes with T. Then*:
(1) *If T satisfy Browder theorem, then $\sigma_{gD\mathcal{W}}(T+F) = \sigma_{gD\mathcal{W}}(T)$;*
(2) *If T and T^* have SVEP at every $\lambda \notin \sigma_e(T)$, then $\sigma_{pBF}(T+F) = \sigma_{pBF}(T)$.*

Proof. (1) According to [26, Theorem 2.2], we have $acc(\sigma(T)) = acc(\sigma(T+F))$. Then $\lambda \in \sigma_{gD}(T)$ if and only if $\lambda \in acc(\sigma(T))$ if and only if $\lambda \in acc(\sigma(T+F))$ if and only if $\lambda \in \sigma_{gD}(T+F)$. So $\sigma_{gD}(T+F) = \sigma_{gD}(T)$. Theorem 2.1 and Corollary 2.1 give the result. □

By the same argument we have the following proposition.

Proposition 3.3. *Let $T \in \mathcal{B}(X)$ satisfy Browder theorem, Q a quasi-nilpotent operator commutes with T. Then:*

1. *If T satisfy Browder theorem, then $\sigma_{gD\mathcal{W}}(T+Q) = \sigma_{gD\mathcal{W}}(T)$;*
2. *If T and T^* have SVEP at every $\lambda \notin \sigma_e(T)$, then $\sigma_{pBF}(T+Q) = \sigma_{pBF}(T)$.*

Proof. Since $\sigma_{gD}(T+Q) = \sigma_{gD}(T)$, from Theorem 2.1 and Corollary 2.1 we have the result. □

Remark 1. Let $T \in \mathcal{B}(X)$, we have $\sigma_{pBF}(T) \subset \sigma_e(T)$, $\sigma_{gD\mathcal{W}}(T) \subset \sigma_W(T)$ and $\sigma_{gD}(T) \subset \sigma_D(T)$ but generally these inclusions are proper. Indeed, let T and S defined on $l^2(\mathbb{N})$ by

$$T(x_1, x_2, x_3, \ldots) = (\frac{1}{2}x_2, \frac{1}{3}x_3, \ldots); \quad S(x_1, x_2, x_3, \ldots) = (0, \frac{1}{2}x_1, 0, 0, \ldots).$$

Then T is quasi-nilpotent with infinite ascent and hence

$$\sigma_{gD}(T) = \emptyset \text{ but } \sigma_D(T) = \{0\}.$$

Furthermore,

$$\sigma_{pBF}(S) = \sigma_{gD\mathcal{W}}(S) = \emptyset \text{ but } \sigma_e(S) = \sigma_W(S) = \{0\}.$$

The following lemma, will be needed in the sequel to study Riesz perturbations.

Lemma 3.1. *Let $T \in \mathcal{B}(X)$.*

1. $\sigma_D(T) = \sigma_{gD}(T) \cup iso(\sigma_D(T))$;
2. $\sigma(T) = \sigma_{gD}(T) \cup iso(\sigma(T))$;
3. $\sigma_{se}(T) = \sigma_{gK}(T) \cup iso(\sigma_{se}(T))$;
4. $\sigma_{es}(T) = \sigma_{gK}(T) \cup iso(\sigma_{es}(T))$;
5. $\sigma_e(T) = \sigma_{pBF}(T) \cup iso(\sigma_e(T))$;
6. $\sigma_W(T) = \sigma_{gD\mathcal{W}}(T) \cup iso(\sigma_W(T))$.

Proof. (1) Let $\lambda \in \sigma_D(T) \setminus \sigma_{gD}(T)$, then $T-\lambda$ is a generalized Drazin operator hence there exists an $\varepsilon > 0$ such that $T-\mu$ is Drazin invertible for all $\mu \in D(\lambda, \epsilon) \setminus \{\lambda\}$. Indeed, if $T-\lambda$ is a generalized Drazin operator, then there exist two closed T-invariant subspaces X_1 and X_2 of X such that $X = X_1 \oplus X_2$ and $T-\lambda = (T-\lambda)_{|X_1} \oplus (T-\lambda)_{|X_2}$ where $(T-\lambda)_{|X_1}$ is invertible and $(T-\lambda)_{|X_2}$ is quasi-nilpotent. If $X_1 = \{0\}$, $T-\lambda$ is quasi-nilpotent, then for all $\mu \neq \lambda$, $T-\mu$ is invertible, hence $T-\mu$ is Drazin invertible. If $X_1 \neq \{0\}$, then $(T-\lambda)_{|X_1}$ is invertible, hence there exists $\varepsilon > 0$ such that $(T-\mu)_{|X_1}$ is invertible for all $\mu \in D(\lambda, \epsilon)$, hence $T-\mu$ is Drazin invertible for all $\mu \in D(\lambda, \epsilon)$. As $(T-\lambda)_{|X_2}$ is quasi-nilpotent, then for all $\mu \neq \lambda$ $(T-\mu)_{|X_2}$ is invertible and hence $(T-\mu)_{|X_2}$ is Drazin invertible for all $\mu \in D(\lambda, \varepsilon) \setminus \{\lambda\}$. Since $(T-\mu)_{|X_2}$ and $(T-\mu)_{|X_1}$ are Drazin invertible for all $\mu \in D(\lambda, \varepsilon) \setminus \{\lambda\}$, then we get $T-\mu$ is Drazin invertible for all $\mu \in D(\lambda, \varepsilon) \setminus \{\lambda\}$. This implies that

$$D(\lambda, \epsilon) \setminus \{\lambda\} \cap \sigma_D(T) = \emptyset,$$

hence $\lambda \in iso(\sigma_D(T))$. Therefore,

$$\sigma_D(T) \subseteq \sigma_{gD}(T) \cup iso(\sigma_D(T)).$$

The reverse inclusion is always true.

The assertion (2) is clear, since $\sigma_{gD}(T) = acc\sigma(T)$.

For (3), let $\lambda \in \sigma_{se}(T) \setminus \sigma_{gK}(T)$, $T-\lambda$ is a pseudo Fredholm operator. By [12, Theorem 2.2], there exists an $\epsilon > 0$ such that $T-\mu$ is semi-regular for all $\mu \in D(\lambda, \epsilon) \setminus \{\lambda\}$, this implies that $D(\lambda, \epsilon) \setminus \{\lambda\} \cap \sigma_{se}(T) = \emptyset$, hence $\lambda \in iso(\sigma_{se}(T))$. Therefore, $\sigma_{se}(T) \subseteq \sigma_{gK}(T) \cup iso(\sigma_{se}(T))$, the opposite inclusion is always true.

To prove (4), let $\lambda \in \sigma_{es}(T) \setminus \sigma_{gK}(T)$, $T-\lambda$ is a pseudo Fredholm operator. By [12, Theorem 2.2], there exists an $\epsilon > 0$ such that $T-\mu$ is semi-regular for all $\mu \in D(\lambda, \epsilon) \setminus \{\lambda\}$, hence $T-\mu$ is essentially semi-regular for all $\mu \in D(\lambda, \epsilon) \setminus \{\lambda\}$, this implies that $D(\lambda, \epsilon) \setminus \{\lambda\} \cap \sigma_{es}(T) = \emptyset$, thus $\lambda \in iso(\sigma_{se}(T))$. Therefore, $\sigma_{es}(T) \subseteq \sigma_{gK}(T) \cup iso(\sigma_{es}(T))$, since $\sigma_{gK}(T) \subseteq \sigma_{es}(T)$, we have

$$\sigma_{es}(T) = \sigma_{gK}(T) \cup iso(\sigma_{es}(T)).$$

For the assertion (5), let $\lambda \in \sigma_e(T) \setminus \sigma_{pBF}(T)$, then $T-\lambda$ is a pseudo B-Freholm operator, hence there exists an $\epsilon > 0$ such that $T-\mu$ is Fredholm for all $\mu \in D(\lambda, \epsilon) \setminus \{\lambda\}$. Indeed, without loss of generality we can assume that $\lambda = 0$. If T is pseudo B-Fredholm, then there exist two closed T-invariant subspaces X_1 and X_2 such that $X = X_1 \oplus X_2$; $T_{|X_1}$ is Fredholm, $T_{|X_2}$ is quasi-nilpotent and $T = T_{|X_1} \oplus T_{|X_2}$.

If $X_1 = \{0\}$, T is quasi-nilpotent, hence $\mu I - T$ is invertible for all $\mu \neq 0$, that is $\mu I - T$ is Fredholm for all $\mu \neq 0$.

If $X_1 \neq \{0\}$, then $T_{|X_1}$ is Fredholm, hence there exists $\varepsilon > 0$ such that $(\mu I - T)_{|X_1}$ is Fredholm for all $\mu \in D(0, \epsilon)$. As $T_{|X_2}$ is quasi-nilpotent, then for all $\mu \neq 0$, $(\mu I - T)_{|X_2}$ is invertible, then $(\mu I - T)_{|X_2}$ is Fredholm for all $\mu \in D^*(0, \varepsilon)$. Since $(\mu I - T)_{|X_2}$ and $(\mu I - T)_{|X_1}$ are Fredholm for all $\mu \in D^*(0, \varepsilon)$, we have $\mu I - T$ is Fredholm for all $\mu \in D^*(0, \varepsilon)$.

This implies that $D(\lambda, \epsilon) \setminus \{\lambda\} \cap \sigma_e(T) = \emptyset$, hence $\lambda \in iso(\sigma_e(T))$. Therefore,

$$\sigma_e(T) \subseteq \sigma_{pBF}(T) \cup iso(\sigma_e(T)).$$

Since the opposite inclusion is true, then we conclude (5).

By a similar argument as in (5), we can prove (6). □

Theorem 3.1. *Let $T \in \mathcal{B}(X)$ and $R \in \mathcal{B}(X)$ be a Riesz operator which commutes with T. Then the following statements hold:*

(1) *If $iso(\sigma_e(T)) = \emptyset$, then $\sigma_{pBF}(T+R) = \sigma_{pBF}(T)$;*

(2) *If* $iso(\sigma_W(T)) = \emptyset$, *then* $\sigma_{gD\mathcal{W}}(T+R) = \sigma_{gD\mathcal{W}}(T)$.

Proof. To prove (1), we have $\sigma_e(T+R) = \sigma_e(T)$ and since $iso(\sigma_e(T)) = \emptyset$, then by Lemma 3.1, we get $\sigma_{pBF}(T) = \sigma_e(T)$, hence $\sigma_{pBF}(T+R) = \sigma_{pBF}(T)$. For the assertion (2), we have $\sigma_W(T+R) = \sigma_W(T)$ and since $iso(\sigma_W(T)) = \emptyset$, then by Lemma 3.1, we have $\sigma_{gD\mathcal{W}}(T) = \sigma_W(T)$, hence $\sigma_{gD\mathcal{W}}(T+R) = \sigma_{pBW}(T)$. □

Note that the essential quasi-Fredholm spectrum is not stable under commuting quasi-nilpotent and compact perturbations, hence it is not stable under commuting Riesz perturbation, see [20].

Theorem 3.2. *Let* $T \in \mathcal{B}(X)$.

(1) *If* $iso(\sigma_{es}(T)) = \emptyset$ *and* R *is a Riesz operator such that* $TR = RT$, *then*

$$\sigma_{gK}(T+R) = \sigma_{gK}(T) \text{ and } \sigma_{eq}(T+R) = \sigma_{eq}(T).$$

(2) *If* $iso(\sigma_{se}(T)) = \emptyset$ *and* Q *is a quasi-nilpotent operator such that* $QT = TQ$, *then*

$$\sigma_{gK}(T+Q) = \sigma_{gK}(T) \text{ and } \sigma_{eq}(T+Q) = \sigma_{eq}(T).$$

Proof. To prove (1), since $\sigma_{gK}(T) \subseteq \sigma_{eq}(T) \subseteq \sigma_{es}(T)$, then by part (4) of Lemma 3.1, we have

$$\sigma_{eq}(T) \cup iso(\sigma_{es}(T)) = \sigma_{es}(T).$$

According to [15, Corollary 17], if R is a Riesz operator commutes with T, then we have $\sigma_{es}(T+R) = \sigma_{es}(T)$. By hypothesis $iso(\sigma_{es}(T)) = \emptyset$, $\sigma_{es}(T) = \sigma_{eq}(T)$ and by Lemma 3.1, we have $\sigma_{es}(T) = \sigma_{gK}(T)$. This gives the result.

To prove (2), from [20], if Q is a quasi-nilpotent operator, then we have $\sigma_{se}(T+Q) = \sigma_{se}(T)$. By hypothesis $iso(\sigma_{se}(T)) = \emptyset$, hence by Lemma 3.1, we have that $\sigma_{se}(T) = \sigma_{gK}(T)$. This gives the result. □

Example 1. Let T be an unilateral weighted right shift on $l^p(\mathbb{N})$, $1 \leq p < \infty$, with weight sequence $(w_n)_{n\in\mathbb{N}}$. If $\lim_{n\to\infty} \inf(w_1 \cdots w_n)^{1/n} = 0$, then T and T^* have the SVEP and by [1, Corollary 3.118]:

$$\sigma_{su}(T) = \sigma_{ap}(T) = \sigma_{se}(T) = \sigma_e(T) = \sigma_W(T) = \sigma(T) = \mathbf{D}(0, r(T)),$$

where $\mathbf{D}(0, r(T))$ the closed disc, hence $iso(\sigma(T)) = iso(\sigma_W(T)) = iso(\sigma_e(T)) = \emptyset$. If R is a Riesz operator which commutes with T and Q a quasi-nilpotent operator commutes with T, then:

$$\sigma_{gK}(T+Q) = \sigma_{gK}(T);\ \sigma_{pBF}(T+R) = \sigma_{pBF}(T);\ \sigma_{pBW}(T+R) = \sigma_{pBW}(T).$$

4. Commutator and pseudo B-Fredholm perturbations

Let $T, S \in \mathcal{B}(X)$, denote by $[T, S]$ the commutator of T and S.

In what follows, we prove that we can perturb a pseudo B-Fredholm operator $T \in \mathcal{B}(X)$ by a bounded operator S satisfying $[T, S] = 0$ to obtain a Fredholm operator $T + S$.

Proposition 4.1. *Let $T \in \mathcal{B}(X)$ be a pseudo B-Fredholm operator. Then there exists $S \in \mathcal{B}(X)$ such that:*

$$T + S \text{ is Fredholm, } TS \text{ is quasi-nilpotent and } [T, S] = 0.$$

Proof. If T is pseudo B-Fredholm, then there exist two closed T-invariant subspaces X_1 and X_2 of X such that $X = X_1 \oplus X_2$ and $T_1 = T_{|X_1}$ is upper semi-Fredholm and $T_2 = T_{|X_2}$ is quasi-nilpotent. Let $S = 0 \oplus (I_2 - T_2)$, $I_2 = I_{|X_2}$. Since T_1 is Fredholm, then $T + S = T_1 \oplus I_2$ is a Fredholm operator. We have:

$$\begin{aligned} TS &= [T_1 \oplus T_2][0 \oplus (I_2 - T_2)] \\ &= T_2(I_2 - T_2) = (I_2 - T_2)T_2 \\ &= [0 \oplus (I_2 - T_2)][T_1 \oplus T_2] = ST. \end{aligned}$$

From the well known spectral radius formula

$$r(TS) = r((I_2 - T_2)T_2) \leq r(I_2 - T_2)r(T_2) = 0.$$

Therefore TS is quasinilpotent. □

In what follows, we prove that we can perturb a pseudo Fredholm operator $T \in \mathcal{B}(X)$ by a bounded operator S satisfying $[T, S] = 0$ to obtain a semi-regular operator $T + S$.

Proposition 4.2. *Let $T \in \mathcal{B}(X)$ be a pseudo Fredholm operator. Then there exists $S \in \mathcal{B}(X)$ such that:*

$$T + S \text{ is semi-regular, } TS \text{ is quasi-nilpotent and } [T, S] = 0.$$

Proof. Since T is a pseudo Fredholm operator then there exist subsets M and N of X such that

$$X = M \oplus N \text{ and } T = T_1 \oplus T_2$$

with $T_1 = T_{|M}$ is a semi-regular operator and $T_2 = T_{|N}$ is a quasinilpotent. Let $S = 0 \oplus (I_2 - T_2)$, $I_2 = I_{|N}$. Since T_1 is semi-regular then $T + S = T_1 \oplus I_2$ is a semi-regular operator. We have:

$$\begin{aligned} TS &= [T_1 \oplus T_2][0 \oplus (I_2 - T_2)] \\ &= T_2(I_2 - T_2) = (I_2 - T_2)T_2 \\ &= [0 \oplus (I_2 - T_2)][T_1 \oplus T_2] = ST. \end{aligned}$$

□

In the following, we give a generalization of [21, Theorem 2.1] and [17, Proposition 1.1].

Theorem 4.1. *Let $T \in \mathcal{B}(X)$ be a pseudo B-Weyl operator. Then there exists an operator $F \in \mathcal{B}(X)$ such that $T + \lambda F$ is invertible for all $\lambda \in \mathbb{C} \setminus \{0\}$ and*

$$[T, F]q(T, F)[T, F] = 0.$$

Where $q(T, F)$ is any polynomial in T and F.

Proof. T is a pseudo B-Weyl operator, then there exist two closed T-invariant subspaces X_1 and X_2 of X such that $X = X_1 \oplus X_2$ and $T_1 = T_{|X_1}$ is pseudo B-Weyl and $T_2 = T_{|X_2}$ is quasi-nilpotent. Since T_1 is a Weyl operator $ind(T_1) = 0$, according to [21, Theorem 1.2], there exists $F_1 \in \mathcal{B}(M)$ such that $T_1 + \lambda F_1$ is invertible and $[T_1, F_1]q(T_1, F_1)[T_1, F_1] = 0$, where $q(T_1, F_1)$ is any polynomial in T_1 and F_1. Set $F = F_1 \oplus I_2$ where I_2 is the restriction of I to X_2. T_2 is quasi-nilpotent this implies that $T_2 + \lambda I_2$ is invertible for all $\lambda \neq 0$, hence $T + \lambda F = T_2 \oplus T_2 + \lambda(F_1 \oplus I_2) = (T_1 + \lambda F_1) \oplus (T_2 + \lambda I_2)$ is invertible. On the other hand, $[T_2, I_2] = 0$ and $[T_1, F_1]q(T_1, F_1)[T_1, F_1] = 0$ therefore $[T, F]q(T, F)[T, F] = 0$ where $q(T, F)$ is any polynomial in T and F. □

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