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SEMISIMPLE DIMENSION OF MODULES

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Abstract. In this paper we define and study a new kind of dimension called, semisimple dimension, that measures how far a module is from being semisimple. Like other kinds of dimensions, this is an ordinal valued invariant. We give some interesting and useful properties of rings or modules which have semisimple dimension. It is shown that a noetherian module with semisimple dimension is an artinian module. A domain with semisimple dimension is a division ring. Also, for a semiprime right non-singular ring R, if its maximal right quotient ring has semisimple dimension as a right R-module, then R is a semisimple artinian ring. We also characterize rings whose modules have semisimple dimension. In fact, it is shown that all right R-modules have semisimple dimension if and only if the free right R-module $\oplus_{i=1}^{\infty} R$ has semisimple dimension, if and only if R is a semisimple artinian ring.

1. Introduction

In the literature, many kinds of dimensions have been defined and investigated for modules and they have important roles in the study of ring and module theory. For example the *Krull dimension* has been first defined in 1928 by W. Krull for a commutative noetherian ring, motivated by E. Noether's studies about the relationship between the chain of prime ideals and dimension of algebraic varieties. After that this theory has been investigated and developed for non commutative rings and modules by many authors, such as W. Krull, G. Krause, A. V. Jategaonkar, and many other people. Also, *uniform module* and *uniform dimension* of a module have been introduced and studied by Goldie (1958-1960). It is also referred to as Goldie dimension. Recently [7] introduced uniserial dimension of a module. Furthermore, [3] introduced the dual version couniserial dimension. Each of these dimensions provides useful technical tools for the study of ring and module structure (see [7], [3], and [5]).

In this paper we define a notion of dimension of modules, called *semisimple dimension*. Similar to many other kinds of dimension, the value of this dimension is an ordinal. In fact, semisimple dimension of a module M shows in some

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sense that how far is M from being semisimple. In order to define semisimple dimension for modules over a ring R, we define, by transfinite induction, for every ordinal $\alpha \geq 1$ the class $\mathcal{X}_\alpha$ of R-modules. In the first step, let $\mathcal{X}_1$ be the class of all non-zero semisimple modules. Consider an ordinal $\alpha > 1$; and assume that $\mathcal{X}_\beta$ has been defined for ordinals $\beta < \alpha$. Now suppose that $\mathcal{X}_\alpha$ be the class of all R-modules M for which each of non-zero submodules N of M, which is not a direct summand of M, we have $N \in \bigcup_{\beta<\alpha} \mathcal{X}_\beta$. If there exists some $\mathcal{X}_\alpha$ containing the R-module M, then we say that *the semisimple dimension of M is defined* or *M has semisimple dimension* and the least such α is called the *semisimple dimension* of M, denoted by $\text{s.s.dim}(M)$. For $M = 0$, we define $\text{s.s.dim}(M) = 0$. If for a non-zero module M there is no $\mathcal{X}_\alpha$ containing M, then we say that $\text{s.s.dim}(M)$ *is not defined*, or *M has no semisimple dimension.*

In Section 2, we prove some basic properties of the semisimple dimension. We show that for an R-module M the existence of the semisimple dimension is equivalent to say that for each descending chain $M_1 \geq M_2 \geq \cdots$ of submodules of M, there is an integer $m \geq 1$ such that M_r is a direct summand of M_s for all $r \geq s \geq m$ (see Proposition 2.2). It is clear by definition that every submodule of a module with semisimple dimension must have semisimple dimension. It is proven in Proposition 2.5 that an R-module M has semisimple dimension and finite uniform dimension if and only if it is an artinian module. We also show with an example that an R-module with semisimple dimension need not be artinian (see Example 2.4). However, we prove in Proposition 2.7 that a noetherian module M with semisimple dimension has finite length and in this case $\text{length}(M)$ is an upper bound for semisimple dimension of M. Also, we study the relationship between the semisimple dimension and the uniform dimension of a module. In Section 3, we study the rings whose all modules have semisimple dimension. Proposition 3.1 shows that a domain D with semisimple dimension is a division ring. We show in Theorem 3.3 that if R is a semiprime right non-singular ring and its maximal right quotient ring Q has semisimple dimension as a right R-module, then R is a semisimple artinian ring. It is shown that for a ring R the right R-module $\oplus_{n=1}^{\infty} R$ has semisimple dimension if and only if; all right R-modules have semisimple dimension and this is also a necessary and sufficient on the ring R to be a semisimple artinian ring (see Proposition 3.5). This gives a new characterization of semisimple artinian rings.

Now we recall some of basic necessary definitions in ring and module theory. An R-module M is called a *semisimple module* if M is a direct sum of its simple submodules. For an R-module M a submodule N of M is called an *essential submodule* and we write $N \lhd M$ if N intersects every non-zero submodule of M non-trivially. The submodule $Z(M) = \{x \in M : \text{ann}(x) \lhd R\}$ is called the *singular submodule* of M. If $Z(M) = 0$, then M is called a *non-singular* module. Taking $M = R_R$, the ring R is called a *right non-singular ring* if $Z(R_R) = 0$. A module M is of *finite uniform dimension* (or *finite Goldie dimension*) if M contains no infinite direct sum of its non-zero submodules, or equivalently, M is an essential extension of $\oplus_{i=1}^{n} U_i$ for some natural number n and independent uniform submodules $U_1, \ldots, U_n$ in M. Note that the natural number n is unique and it is called the *uniform dimension* of M and we write $\text{u.dim}(M) = n$.

Throughout this paper, R denotes always an arbitrary associative ring with unit element $1 \neq 0$ and all modules are unitary right modules. If N is a (proper) submodule of M we write $(N < M)$ $N \leq M$. Also, for an R-module M, $\oplus_{i=1}^{\infty} M$ denotes the countably infinite direct sum of copies of M. For a submodule N of M and an integer $k > 1$, $\oplus_{i=k}^{\infty} N$ is the submodule $\oplus_{i=1}^{\infty} N_i$ of $\oplus_{i=1}^{\infty} M$ with $N_1 = N_2 = \cdots = N_{k-1} = 0$ and $i \geq k$ $N_i = N$ for $i \geq k$.

2. Some basic and preliminary results

As we defined in the introduction, semisimple dimension of an R-module is an ordinal valued invariant. For basic concepts about ordinals the reader is refered to [8]. We start this section with a remark on the definition of semisimple dimension.

Remark 2.1. It can be easily seen from the definition that, if an R-module M has semisimple dimension and $N \leq M$, then N has also semisimple dimension and $\text{s.s.dim}(N) \leq \text{s.s.dim}(M)$. Moreover, if $\text{s.s.dim}(M) = \text{s.s.dim}(N)$, where N is a submodule of M, then N is a direct summand of M. Also, since every set of ordinal numbers has a supremum, it is an immediate consequence of the definition that M has semisimple dimension if and only if for every submodule N of M which N is not a direct summand of M, $\text{s.s.dim}(N)$ is defined. In the latter case, if

$$\alpha = \sup\{\text{s.s.dim}(N) : N \leq M,\ N \text{ is not a direct summand of } M\},$$

then $\text{s.s.dim}(M) \leq \alpha + 1$.

The following proposition helps us to determine that an R-module has semisimple dimension.

Proposition 2.2. *For an R-module M the following assertions are equivalent.*

(1) *M has semisimple dimension.*

(2) *For every descending chain of submodules $M_1 \geq M_2 \geq \cdots$, there exists $n \geq 1$ such that M_j is a direct summand of M_i for all $j \geq i \geq n$.*

Proof. (1) $\Rightarrow$ (2) Let $M_1 \geq M_2 \geq \cdots$ be a chain of submodules of M. Put $\gamma = \inf\{\text{s.s.dim}(M_n) : n \geq 1\}$. So $\gamma = \text{s.s.dim}(M_n)$ for some $n \geq 1$. By Remark 2.1, M_j is a direct summand of M_i, for all $j \geq i \geq n$, because γ is the infimum.

(2) $\Rightarrow$ (1) Let on contrary, M does not have semisimple dimension. Then there is a submodule M_1 of M which is not a direct summand of M and M_1 does not have semisimple dimension. So there exists a submodule M_2 of M_1 such that M_2 is not a direct summand of M_1 and M_2 does not have semisimple dimension. Continuing this process, we obtain a descending chain

of submodules $M_1 \geq M_2 \geq \cdots$, such that for every $j \geq i \geq 1$, M_j is not a direct summand of M_i and this is a contradiction. □

The following corollary is an immediate consequence of Proposition 2.2.

Corollary 2.3. *Let M be an artinian R-module. Then M has semisimple dimension.*

The following example shows that a ring with semisimple dimension need not be artinian.

Example 2.4. (1) Let R be a commutative local ring whose Jacobson radical J is not finitely generated and $J^2 = 0$. Then J is a semisimple R-module. We can easily see that, $\text{s.s.dim}(R_R) = 2$. But the R-module R_R is not artinian (see [2, Example 3.16]).

(2) Let R and S be two rings and ${}_RM_S$ be a bimodule such that R is right artinian and $(M \oplus S)_S$ is a semisimple module which is not artinian. Then the ring $T = \left(\begin{smallmatrix} R & M \\ 0 & S \end{smallmatrix}\right)$ as a right T-module has semisimple dimension but T is not a right artinian ring. This can be seen immediately from Proposition 2.2 and the fact that every right ideal of T is of the form $J_1 \oplus J_2$, where J_1 is right ideal of R, J_2 is a submodule of $(M \oplus S)_S$ and $J_1M \leq J_2$ (see [6, Proposition 1.17]).

In the following proposition, we characterize artinian modules in terms of having semisimple dimension and finite uniform dimension.

Proposition 2.5. *An R-module M has semisimple dimension and finite uniform dimension if and only if it is an artinian module.*

Proof. ($\Rightarrow$) Let M be an R-module with finite uniform dimension $\text{u.dim}(M) = m$ and consider a descending chain $M_1 \geq M_2 \geq \cdots$ of submodules of M. By Proposition 2.2, there exists $n \geq 1$ such that M_j is a direct summand of M_i for all $j \geq i \geq n$. Thus there exist submodules $K_1, \ldots, K_{m-1}$ such that $K_1 \oplus K_2 \oplus \cdots \oplus K_{m-1} \oplus M_{n+m} = M_n$. But since $\text{u.dim}(M_n) \leq m$, we have $M_t = M_{n+m}$ for all $t \geq n+m$ and so M is artinian.

($\Leftarrow$) The assertion holds by Corollary 2.3 and [9, Theorems 31.1 and 21.3]. □

In the following example, we show that finiteness of uniform dimension in the above proposition can not be removed.

Example 2.6. Let $\mathbb{P}$ be the set of all prime numbers and $M = \oplus_{p \in \mathbb{P}} \mathbb{Z}/p\mathbb{Z}$. Then $M_{\mathbb{Z}}$ has semisimple dimension 1, but it is not artinian.

For a right R-module M, we say that M is *of finite length* if it has a composition series. A right R-module M is of finite length if and only if M is both right noetherian and right artinian. The length of any two composition series of M_R are the same and it is said to be the *length of* M_R and is denoted by $\text{length}(M)$. A module of finite length obviously has semisimple dimension. The following proposition shows a relation between semisimple dimension of a module M with finite length and $\text{length}(M)$.

Proposition 2.7. *Let M be an R-module with semisimple dimension. If M is a noetherian module, then we have the following*:

(1) *M is an artinian module.*
(2) $\text{s.s.dim}(M) \leq \text{length}(M)$.
(3) *If $M \neq 0$, then* $\text{length}(M) - \text{u.dim}(M) + 1 \leq \text{s.s.dim}(M)$.
(4) *If M is uniform, then* $\text{s.s.dim}(M) = \text{length}(M)$.
(5) *For an exact sequence $0 \to N \to M \to M/N \to 0$ with $M \neq 0$,* $\text{s.s.dim}(N) + \text{s.s.dim}(M/N) \leq \text{s.s.dim}(M) + \text{u.dim}(M) - 1$. *In particular, if N and M/N are uniform, then* $\text{s.s.dim}(N) + \text{s.s.dim}(M/N) = \text{s.s.dim}(M)$.

Proof. (1) By Proposition 2.5, M is artinian.

(2) We prove this part by induction on $\text{length}(M) = n$. The case $n = 1$ is trivial. Now let $n > 1$ and K be a proper submodule of M. Then, by assumption, $\text{s.s.dim}(K) \leq \text{length}(K) < \text{length}(M)$. Now using Remark 2.1, $\text{s.s.dim}(M) \leq \text{length}(M)$.

(3) Set $\text{length}(M) = n$ and $\text{u.dim}(M) = m$. Since M is artinian, we have $\text{soc}(M) \leq_e M$. Hence $\text{length}(M/\text{soc}(M)) + \text{u.dim}(M) = \text{length}(M)$. Now we consider the chain $\text{soc}(M) = M_1 \lneq M_2 \lneq \cdots \lneq M_{n-m-1} \lneq M_{n-m} = M$ such that M_i/M_{i-1} is simple for every $2 \leq i \leq n-m$. Thus we have $M_2 \in \mathcal{X}_2, \ldots, M_{n-m} \in \mathcal{X}_{n-m}$. This shows that $\text{s.s.dim}(M) \geq n - m + 1$, and so the assertion holds.

(4) follows immediately from (2) and (3).

(5) It is well-known that $\text{length}(N) + \text{length}(M/N) = \text{length}(M)$. On the other hand, by (2) and (3), we have $\text{s.s.dim}(M) \leq \text{length}(M) \leq \text{s.s.dim}(M) + \text{u.dim}(M) - 1$. Thus we have

$$\begin{aligned}\text{s.s.dim}(N) + \text{s.s.dim}(M/N) &\leq \text{length}(N) + \text{length}(M/N)\\ &= \text{length}(M)\\ &\leq \text{s.s.dim}(M) + \text{u.dim}(M) - 1,\end{aligned}$$

as desired. The last statement follows immediately from (4). □

Now we give an example showing that the converse of Proposition 2.7 (part (1)) does not hold.

Example 2.8. For $\mathbb{Z}$-module $M = \mathbb{Z}_{p^\infty}$, it can be easily seen that $\text{s.s.dim}(M) = \omega$ since every non-zero submodule of M is of the form $K_i = \{m/p^i + \mathbb{Z} : m \in \mathbb{Z}\}$ for some $i \in \mathbb{N}$. Therefore M is artinian as a $\mathbb{Z}$-module. However, it is not noetherian.

Corollary 2.9. *If a non-zero R-module M is of finite length and $M = M_1 \oplus \cdots \oplus M_n$, then*

$$\text{s.s.dim}(M_1) + \cdots + \text{s.s.dim}(M_n) \leq \text{s.s.dim}(M) + \text{u.dim}(M) - 1.$$

In particular, in case M is a uniform or a semisimple module, then the equality holds.

Proof. The proof follows immediately from part (5) of Proposition 2.7. □

Lemma 2.10. *If a finitely generated module M has semisimple dimension and $M \cong M \oplus N$ for some module N, then N has finite length.*

Proof. First we show that N is noetherian. If N_1 is a submodule of N that is not finitely generated, then M contains a submodule K_1 isomorphic to $M \oplus N_1$ which is not finitely generated. Since $K_1 \cong M \oplus N_1$, there is a submodule K_2 of K_1 isomorphic to M. Continuing this process in a similar way, we obtain a chain $M \geq K_1 \geq K_2 \geq \cdots$ such that $K_i \cong M \oplus N_1$ is not finitely generated for i odd, and $K_i \cong M$ for i even. But M is finitely generated with semisimple dimension, which gives us a contradiction. Thus N is noetherian. Now, by Proposition 2.7, N is also artinian, and so N has finite length. This completes the proof. □

We know that every artinian module has non-zero socle. In the following lemma we show that this is the case for every module with semisimple dimension.

Lemma 2.11. *For a non-zero R-module M with semisimple dimension the following assertions hold*:

(1) *If* $\text{s.s.dim}(M) = \gamma$, *then for any ordinal β with* $0 \leq \beta \leq \gamma$, *there is a submodule L of M such that* $\text{s.s.dim}(L) = \beta$.

(2) *M has non-zero socle.*

Proof. (1) We prove this part using transfinite induction on $\text{s.s.dim}(M) = \gamma$. The assertion is trivial for the case $\gamma = 1$. For $\gamma \geq 1$ let $0 \leq \beta < \gamma$. Then by Remark 2.1, there exists a submodule K of M which is not a direct summand of M and $\beta \leq \text{s.s.dim}(K)$. As $\beta \leq \text{s.s.dim}(K) < \gamma$, using induction hypothesis, there is a submodule L of K such that $\text{s.s.dim}(L) = \beta$.

(2) It is clear by part (1). □

Lemma 2.12. *For each ordinal γ, being of semisimple dimension γ is a Morita invariant property for modules.*

Proof. This can be easily seen using the definition of semisimple dimension and [1, Propositions 21.7 and 21.8]. □

3. Rings whose modules have semisimple dimension

In this section we investigate the rings whose modules have semisimple dimension using our basic results in the last section. The following proposition gives us examples of modules which do not have semisimple dimension.

Proposition 3.1. *Let D be a ring. Then D is a division ring if and only if D is a domain and has semisimple dimension as a right D-module .*

Proof. It is enough to prove the "if" part. We claim that D is a principal right ideal domain. Let on contrary that there is a non-cyclic right ideal I of

D. For a non-zero element $x \in I$, let $J_1 = xD$. It is clearly isomorphic to D. Therefore, there is a right ideal J_2 of D with $J_2 \cong I$ and $J_2 \leq J_1$. Now consider again a cyclic right ideal J_3 contained in J_2 and by continuing this process in a similar way, we have a chain $J_1 \geq J_2 \geq \cdots$ of right ideals of D where for each odd integer i, J_i is cyclic and for each even integer i, J_i is not cyclic. Now using Proposition 2.2, we see that, for some n, J_{n+1} is a direct summand of J_n, a contradiction. This means that D is a principal right ideal domain and so, by Proposition 2.7, D is right artinian. Now by [5, Corollary 4.18], D is a division ring. □

For a ring R, $Q = Q_{\max}(R)$ denotes the maximal right quotient ring of R. In case R is a right non-singular ring, the maximal right quotient ring Q of R is just the injective hull of R_R, $E(R_R)$ (see [4, Corollary 2.31]).

Lemma 3.2. *For a right non-singular ring R let Q be its maximal right quotient ring and M be a right Q-module. If M is a non-singular right R-module, such that M has semisimple dimension as a right R-module, then M has also semisimple dimension as a right Q-module.*

Proof. Every chain $M \geq M_1 \geq M_2 \geq \cdots$ of Q-submodules of M, it is also a chain of R-submodules of M and thus, for some n, M_j is a direct summand of M_i for all $j \geq i \geq n$. Fix i, j for $j \geq i \geq n$. Hence there is an R-submodule K of M_i such that $M_j \oplus K = M_i$. It is enough to show that K is a Q-module. If $q \in Q$ and $t \in K$ there is an essential right ideal E of R such that $qE \leq R$. On the other hand, $tq = m_j + k$ with $m_j \in M_j$ and $k \in K$. Now we have $(tq - k)E = 0$. Since M is right non-singular, $tq = k$. Thus K is a Q-module, and so by Proposition 2.2, the assertion holds. □

Theorem 3.3. *Let Q be the maximal right quotient ring of a semiprime right non-singular ring R. If Q has semisimple dimension as an R-module, then R is a semisimple artinian ring.*

Proof. By Proposition 2.5, it is sufficient to prove that R_R has finite uniform dimension. As Q_R has semisimple dimension, R_R itself and every right ideal I of R have semisimple dimension. Therefore, every non zero right ideal contains a minimal submodule (see Lemma 2.11). Now [4, Theorem 3.29] implies that the maximal right quotient ring Q of R is a product of endomorphism rings of some right vector spaces, say $Q = \prod_{i \in I} Q_i$, where $Q_i = \mathrm{End}(V_i)$. As R_R is right non-singular, Q_R is also non-singular and so, using Lemma 3.2, Q_Q has semisimple dimension. At first we claim that each V_i is a finite dimensional vector space. Let on contrary V_j be an infinite dimensional vector space and $Q_j = \mathrm{End}(V_j)$ for some j. Hence, $Q_j \cong Q_j \times Q_j$. Now let $\iota : Q_j \longrightarrow Q$ be the canonical embedding. Then $\iota(Q_j)$ is a right ideal of Q and we obtain an isomorphism $Q \cong \iota(Q_j) \times Q$ of Q-modules. Thus there are right ideals T_1 and T of Q such that $Q = T_1 \oplus T$ and we have Q-isomorphisms $T_1 \cong Q$ and $T \cong \iota(Q_j)$. Now since V_j is an infinite dimensional vector space, then its endomorphism ring $Q_j = \text{End}(V_j)$ has a right ideal which is not principal, namely its socle. So $\iota(Q_j)$ and thus T contains a non-cyclic right ideal of Q and since $T \cong Q/T_1$, there exists a non-cyclic right ideal of Q, say K_1 such that $Q \geq K_1 \geq T_1$. Now T_1 is isomorphic to Q. So we can have a descending chain $Q > K_1 > T_1 > K_2 > T_2 > \cdots$ of right ideals of Q such that T_i are cyclic but K_i are not cyclic. This is a contradiction. So all Q_i are endomorphism ring of finite dimensional vector spaces. Now to show that R has finite uniform dimension it is enough to show that the index set I is finite. If I is infinite, there exist infinite subsets I_1 and I_2 of I such that $I = I_1 \cup I_2$. and $I_1 \cap I_2$ is empty. Let $T_1 = \prod_{i\in I} N_i$ such that $N_i = Q_i$ for all $i \in I_1$ and $N_i = 0$ for all $i \in I_2$. Similarly let $T = \prod_{i\in I} M_i$ such that $M_i = Q_i$ for all $i \in I_2$ and $M_i = 0$ for all $i \in I_1$. Then T_1 and T are right ideals of Q and $Q = T_1 \oplus T$. T contains a right ideal of Q which is not cyclic, for example $\oplus_{i\in I} M_i$. Since $T \cong Q/T_1$, there exists a non-cyclic right ideal K_1 of Q such that $Q \geq K_1 \geq T_1$. Note that T_1 is a cyclic Q-module and because I_1 is infinite, the structure of T_1 is similar to that of Q. We can continue in this manner and find a descending chain of right ideals of Q such that K_i are non cyclic Q-modules and T_i are cyclic Q modules, which is a contradiction. Therefore I is finite and R must have finite uniform dimension. Now by Proposition 2.5 and [5, Corollary 4.18], R is a semisimple artinian ring. □

We finish this paper with the following interesting results.

Proposition 3.4. *Let an R-module M has semisimple dimension. Then for every descending chain of submodules $M_1 \geq M_2 \geq \cdots$, there exists $n \geq 1$ such that M_i/M_j is a semisimple R-module for all $j \geq i \geq n$.*

Proof. By Remark 2.1, there exists $n \geq 1$ such that M_j is a direct summand of M_i for all $j \geq i \geq n$. Fix i, j with $j \geq i \geq n$. We show that every submodule K/M_j of M_i/M_j is a direct summand. Let n be the same in proof of the Proposition 2.2. Thus $\text{s.s.dim}(K) = \text{s.s.dim}(M_i)$, and so by Remark 2.1 there exists a submodule T of M_i such that $K \oplus T = M_i$. By property of modularity, $K \cap (T + M_j) = M_j$ and hence we have $K/M_j \oplus (T + M_j)/M_j = M_i/M_j$. Now by [9, Theorem 20.2], M_i/M_j is semisimple for all $j \geq i \geq n$ and hence the assertion holds. □

Corollary 3.5. *The following assertions are equivalent for a ring R.*

(1) *R is semisimple artinian.*
(2) *Every right R-module has semisimple dimension.*
(3) *Every left R-module has semisimple dimension.*
(4) *The right R-module $\oplus_{i=1}^{\infty} R$ has semisimple dimension.*
(5) *The left R-module $\oplus_{i=1}^{\infty} R$ has semisimple dimension.*

Proof. (1) ⇒ (2) ⇒ (4) are clear.

(4) ⇒ (1) Consider the descending chain

$$\oplus_{i=2}^{\infty} R \geq \oplus_{i=3}^{\infty} R \geq \cdots$$

of submodules of $\oplus_{i=1}^{\infty} R$. Since the right R-module $\oplus_{i=1}^{\infty} R$ has semisimple dimension, by Proposition 3.5 there exists n such that, $\oplus_{i=n}^{\infty} R / \oplus_{i=n+1}^{\infty} R$ is a semisimple module. This shows that R is a semisimple artinian ring.

(1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (5) are obtained symmetrically to (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4). □

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