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REMARKS ON ISOMORPHISMS OF TRANSFORMATION SEMIGROUPS RESTRICTED BY AN EQUIVALENCE RELATION

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Abstract. Let $T(X)$ be the full transformation semigroup on a set X and σ be an equivalence relation on X. Denote

$$E(X,\sigma)=\{\alpha\in T(X):\forall x,y\in X,(x,y)\in\sigma \text{ implies } x\alpha=y\alpha\}.$$

Then $E(X,\sigma)$ is a subsemigroup of $T(X)$. In this paper, we characterize two semigroups of type $E(X,\sigma)$ when they are isomorphic.

1. Introduction and preliminaries

Let X be an arbitrary nonempty set. The semigroup $T(X)$ of all transformations on X consists of the mappings from X into itself with composition as the semigroup operation. In [4], H. Pei studied subsemigroups of $T(X)$ determined by an equivalence relation σ on X, defined by:

$$T(X,\sigma)=\{\alpha\in T(X):\forall x,y\in X,(x,y)\in\sigma \text{ implies } (x\alpha,y\alpha)\in\sigma\}.$$

It is clear that if $\sigma\in\{\triangle(X),X\times X\}$, where $\triangle(X)$ is the identity relation on X, then $T(X,\sigma)=T(X)$. He also discussed regularity of elements and Green's relations for $T(X,\sigma)$. Recently, R. P. Sullivan and S. Mendes-Gonçalves introduced a subsemigroup of $T(X)$ defined by

$$E(X,\sigma)=\{\alpha\in T(X):\forall x,y\in X,(x,y)\in\sigma \text{ implies } x\alpha=y\alpha\}$$

and called it the *semigroup of transformations restricted by the equivalence* σ in [3]. Then $E(X,\sigma)$ is a subsemigroup of $T(X,\sigma)$. The authors characterized Green's relations on the largest regular subsemigroup of $E(X,\sigma)$. They also showed that if $|X|\geq 2$ and $\sigma\neq\triangle(X)$, then $E(X,\sigma)$ is not isomorphic to $T(Z)$ for any set Z.

We easily get the following proposition which is a characterization of $E(X,\sigma)$.

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Proposition 1.1. *Let σ be an equivalence relation on a set X. Then the following statements hold.*

(1) *$id_X \in E(X,\sigma)$ if and only if $\sigma = \triangle(X)$ where id_X is the identity mapping on X.*
(2) *If σ and ρ are equivalence relations on X with $\rho \subseteq \sigma$, then $E(X,\sigma) \subseteq E(X,\rho)$.*
(3) *$E(X,\sigma) = T(X,\sigma)$ if and only if $\sigma = \triangle(X)$. If this is the case, then $E(X,\sigma) = T(X)$.*

J. Sanwong and W. Sommanee [6] introduced and studied the subsemigroup

$$T(X,Y) = \{\alpha \in T(X) : X\alpha \subseteq Y\}$$

of $T(X)$ where $\emptyset \neq Y \subseteq X$. We establish an embedding theorem for the semigroup $E(X,\sigma)$ into the semigroup $T(Y,Z)$.

Proposition 1.2. *Let σ be an equivalence relation on a set X. Every semigroup $E(X,\sigma)$ is embeddable in a semigroup $T(Y,Z)$ for some sets Y and Z with $Z \subseteq Y$.*

Proof. Let $Y = \sigma$ and $Z = \triangle(X)$. Then $Z \subseteq Y$. For each $\alpha \in E(X,\sigma)$, we define $\beta_\alpha \in T(Y)$ by

$$(x,y)\beta_\alpha = (x\alpha, y\alpha) \text{ for all } (x,y) \in Y.$$

Since $\alpha \in E(X,\sigma)$, it then follows that $Y\beta_\alpha \subseteq Z$. Hence $\beta_\alpha \in T(Y,Z)$. Define $\phi : E(X,\sigma) \to T(Y,Z)$ by

$$\alpha\phi = \beta_\alpha \text{ for all } \alpha \in E(X,\sigma).$$

Let $\alpha_1, \alpha_2 \in E(X,\sigma)$. To show that $\beta_{\alpha_1\alpha_2} = \beta_{\alpha_1}\beta_{\alpha_2}$, let $(x,y) \in Y$. Then

$$(x,y)\beta_{\alpha_1\alpha_2} = (x\alpha_1\alpha_2, y\alpha_1\alpha_2) = (x\alpha_1, y\alpha_1)\beta_{\alpha_2} = (x,y)\beta_{\alpha_1}\beta_{\alpha_2}.$$

Hence ϕ is a homomorphism. Suppose that $\alpha_1\phi = \alpha_2\phi$. Then $\beta_{\alpha_1} = \beta_{\alpha_2}$. If $x \in X$, then $(x,x) \in Y$ and

$$(x\alpha_1, x\alpha_1) = (x,x)\beta_{\alpha_1} = (x,x)\beta_{\alpha_2} = (x\alpha_2, x\alpha_2).$$

Hence $x\alpha_1 = x\alpha_2$ for all $x \in X$ which implies that ϕ is injective.

Therefore the theorem is proved. □

Over the past, isomorphism theorems for semigroups have been widely considered, see [1, 2, 5, 7]. The purpose of this paper is to find necessary and sufficient conditions for two transformation semigroups restricted by a equivalence in order to be isomorphic.

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2. Main results

For the fixed equivalence relation σ on a set X and $a \in X$, we write $a\sigma$ for the set of all elements of X that are equivalent to a, that is, $a\sigma = \{x \in X : (a, x) \in \sigma\}$.

To obtain the main result, the following two lemmas are needed.

Lemma 2.1. *Let $\alpha \in E(X, \sigma)$. Then α is a right zero element of $E(X, \sigma)$ if and only if α is constant.*

Proof. It is clear that if α is constant, then $\beta\alpha = \alpha$ for all $\beta \in E(X, \sigma)$.

Suppose that α is nonconstant. Then there exist distinct elements $a, b \in X\alpha$. Thus $a'\alpha = a$ and $b'\alpha = b$ for some $a', b' \in X$. Since $\alpha \in E(X, \sigma)$ and $a'\alpha \neq b'\alpha$, we deduce that $(a', b') \notin \sigma$. Define $\beta \in T(X)$ by

$$x\beta = \begin{cases} a', & \text{if } x \in b'\sigma, \\ b', & \text{otherwise} \end{cases}$$

for all $x \in X$. It is clear that $\beta \in E(X, \sigma)$. Since $b'\beta\alpha = a'\alpha = a$ and $b'\alpha = b$, it follows that $\beta\alpha \neq \alpha$. Consequently, α is not a right zero element of $E(X, \sigma)$. □

Hence the corollary is an immediate consequence of Lemma 2.1.

Corollary 2.2. *$E(X, \sigma)$ is a right zero semigroup if and only if $\sigma = X \times X$.*

Proof. Suppose that $\sigma \neq X \times X$. Then there exist $a, b \in X$ such that $(a, b) \notin \sigma$. Thus $a \neq b$. Define $\alpha \in E(X, \sigma)$ by

$$x\alpha = \begin{cases} a, & \text{if } x \in a\sigma, \\ b, & \text{otherwise} \end{cases}$$

for all $x \in X$. Then α is nonconstant in $E(X, \sigma)$. By Lemma 2.1, α is not a right zero element of $E(X, \sigma)$.

Conversely, assume that $\sigma = X \times X$. Then the semigroup $E(X, \sigma)$ consists of all constant mappings in $T(X)$. By Lemma 2.1, $E(X, \sigma)$ is a right zero semigroup. □

Lemma 2.3. *Let $\alpha_1, \alpha_2 \in E(X, \sigma)$ and $a \in X$. If $a\alpha_1\beta = a\alpha_2\beta$ for all $\beta \in E(X, \sigma)$, then $(a\alpha_1, a\alpha_2) \in \sigma$.*

Proof. Suppose that $(a\alpha_1, a\alpha_2) \notin \sigma$. Then $a\alpha_1 \neq a\alpha_2$. Define $\beta \in T(X)$ by

$$x\beta = \begin{cases} a\alpha_1, & \text{if } x \in (a\alpha_1)\sigma, \\ a\alpha_2, & \text{otherwise} \end{cases}$$

for all $x \in X$. It is easy to see that $\beta \in E(X, \sigma)$ and $a\alpha_1\beta \neq a\alpha_2\beta$. □

From now on, suppose that σ_1 and σ_2 are equivalence relations on sets X and Y, respectively. In what follows, $|A|$ means the cardinality of the set A.

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Theorem 2.4. *$E(X, \sigma_1)$ and $E(Y, \sigma_2)$ are isomorphic as semigroups if and only if there exists a bijection $\theta : X \to Y$ such that $(x\sigma_1)\theta = (x\theta)\sigma_2$ for all $x \in X$.*

Proof. Assume that $E(X, \sigma_1)$ and $E(Y, \sigma_2)$ are isomorphic. Let $\varphi : E(X, \sigma_1) \to E(Y, \sigma_2)$ be an isomorphism.

For each $a \in X$, we define $\alpha_a \in E(X, \sigma_1)$ by $x\alpha_a = a$ for all $x \in X$. By Lemma 2.1, α_a is a right zero element of $E(X, \sigma_1)$ and hence

$$\alpha_a\varphi = (\beta\alpha_a)\varphi = (\beta\varphi)(\alpha_a\varphi) \text{ for all } \beta \in E(X, \sigma_1).$$

Since φ is a bijection, we deduce that $\alpha_a\varphi$ is a right zero element of $E(Y, \sigma_2)$. Then from Lemma 2.1, there exists a unique $y_a \in Y$ such that $y(\alpha_a\varphi) = y_a$ for all $y \in Y$.

Define $\theta : X \to Y$ by

$$x\theta = y_x \text{ for all } x \in X.$$

Clearly, θ is well-defined. Let $x_1, x_2 \in X$ be such that $x_1\theta = x_2\theta$. Then $y_{x_1} = y_{x_2}$ which implies that $\alpha_{x_1}\varphi = \alpha_{x_2}\varphi$. Since φ is injective, it follows that $\alpha_{x_1} = \alpha_{x_2}$ and hence $x_1 = x_2$. This shows that α is injective.

To show that θ is surjective, let $y \in Y$. Then there exists $\beta_y \in E(Y, \sigma_2)$ such that $z\beta_y = y$ for all $z \in Y$. Since φ^{-1} is an isomorphism and β_y is a right zero of $E(Y, \sigma_2)$, it follows that $\beta_y\varphi^{-1}$ is a right zero of $E(X, \sigma_1)$. Then there exists an element $x' \in X$ such that $w(\beta_y\varphi^{-1}) = x' = w\alpha_{x'}$ for all $w \in X$. Since $\alpha_{x'}\varphi = \beta_y\varphi^{-1}\varphi = \beta_y$, we have $y_{x'} = y$. Therefore $x'\theta = y$ and whence θ is surjective.

Finally, we will show that $(x\sigma_1)\theta = (x\theta)\sigma_2$ for all $x \in X$. Let $x \in X$ and $a \in (x\sigma_1)\theta$. Then $a = b\theta$ for some $b \in x\sigma_1$ and thus $(x, b) \in \sigma_1$. It follows that $\alpha_x\beta = \alpha_b\beta$ for all $\beta \in E(X, \sigma_1)$. Since φ is a homomorphism,

$$(\alpha_x\varphi)(\beta\varphi) = (\alpha_x\beta)\varphi = (\alpha_b\beta)\varphi = (\alpha_b\varphi)(\beta\varphi)$$

for all $\beta \in E(X, \sigma_1)$. Since φ is a bijection, it follows that

$$(\alpha_x\varphi)\gamma = (\alpha_b\varphi)\gamma \text{ for all } \gamma \in E(Y, \sigma_2).$$

We note here that if $y \in Y$, then $y(\alpha_x\varphi)\gamma = y(\alpha_b\varphi)\gamma$ for all $\gamma \in E(Y, \sigma_2)$. By Lemma 2.3, we obtain that $(y(\alpha_x\varphi), y(\alpha_b\varphi)) \in \sigma_2$. Since $(y(\alpha_x\varphi), y(\alpha_b\varphi)) = (y_x, y_b) = (x\theta, b\theta) = (x\theta, a)$, we deduce $a \in (x\theta)\sigma_2$. This proves that $(x\sigma_1)\theta \subseteq (x\theta)\sigma_2$. For the reverse inclusion, let $c \in (x\theta)\sigma_2$. Then $(c, x\theta) \in \sigma_2$. Since θ is surjective, $c = d\theta$ for some $d \in X$. It follows that $(\alpha_x\varphi)\beta = (\alpha_d\varphi)\beta$ for all $\beta \in E(Y, \sigma_2)$. Since φ^{-1} is a homomorphism,

$$((\alpha_x\varphi)\varphi^{-1})(\beta\varphi^{-1}) = (\alpha_x\varphi\beta)\varphi^{-1} = (\alpha_d\varphi\beta)\varphi^{-1} = ((\alpha_d\varphi)\varphi^{-1})(\beta\varphi^{-1})$$

for all $\beta \in E(Y, \sigma_2)$. It follows from the bijection of φ^{-1} that

$$d\alpha_x\gamma = d(\alpha_x\varphi)\varphi^{-1}\gamma = d(\alpha_d\varphi)\varphi^{-1}\gamma = d\alpha_d\gamma$$

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for all $\gamma \in E(X, \sigma_1)$. By Lemma 2.3, we deduce that $(x, d) = (d\alpha_x, d\alpha_d) \in \sigma_1$, thus $d \in x\sigma_1$. This means that $c = d\theta \in (x\sigma_1)\theta$. Hence $(x\theta)\sigma_2 \subseteq (x\sigma_1)\theta$ and the equality holds.

Conversely, suppose that $\theta : X \to Y$ is a bijection such that $(x\sigma_1)\theta = (x\theta)\sigma_2$ for all $x \in X$. Define $\varphi : E(X, \sigma_1) \to E(Y, \sigma_2)$ by

$$\alpha\varphi = \theta^{-1}\alpha\theta \text{ for all } \alpha \in E(X, \sigma_1).$$

Let $\alpha \in E(X, \sigma_1)$. To show that $\alpha\varphi \in E(Y, \sigma_2)$, let $(x, y) \in \sigma_2$. Since θ is surjective, we have $x'\theta = x$ and $y'\theta = y$ for some $x', y' \in X$. By assumption, we then have $y'\theta \in (x'\theta)\sigma_2 = (x'\sigma_1)\theta$ which implies that $(y', x') \in \sigma_1$. Since $\alpha \in E(X, \sigma_1)$, it follows that $y'\alpha = x'\alpha$. Therefore

$$x\alpha\varphi = x\theta^{-1}\alpha\theta = x'\alpha\theta = y'\alpha\theta = y\theta^{-1}\alpha\theta = y\alpha\varphi.$$

This shows that $\alpha\varphi \in E(Y, \sigma_2)$, whence φ is well-defined. Let $\alpha_1, \alpha_2 \in E(X, \sigma_1)$. We see that

$$\begin{aligned}(\alpha_1\alpha_2)\varphi &= \theta^{-1}(\alpha_1\alpha_2)\theta \\ &= (\theta^{-1}\alpha_1\theta)(\theta^{-1}\alpha_2\theta) \\ &= (\alpha_1\varphi)(\alpha_2\varphi).\end{aligned}$$

Therefore φ is a homomorphism. It is easy to verify that φ is bijective.

The theorem is thereby proven. □

Corollary 2.5. *For positive integers m and n, let X and Y be sets such that $|X| = |Y| = n$ and $|X/\sigma_1| = |Y/\sigma_2| = m$. If $m \in \{1, n-1, n\}$, then $E(X, \sigma_1) \cong E(Y, \sigma_2)$.*

Proof. Suppose that $m \in \{1, n-1, n\}$. Since $|X| = |Y|$, there exists a bijection $\theta : X \to Y$.

Case 1. $m = 1$. Then $\sigma_1 = X \times X$ and $\sigma_2 = Y \times Y$. Thus $(x\sigma_1)\theta = X\theta = Y = (x\theta)\sigma_2$ for all $x \in X$.

Case 2. $m = n$. Then $\sigma_1 = \triangle(X)$ and $\sigma_2 = \triangle(Y)$. Thus $(x\sigma_1)\theta = (x\theta)\sigma_2$ for all $x \in X$.

Case 3. $m = n - 1$. Then there exists a unique $a_1\sigma_1 \in X/\sigma_1$ such that $|a_1\sigma_1| = 2$ for some $a_1 \in X$, say that $a_1\sigma_1 = \{a_1, a_2\}$ for some $a_2 \in X$. Similarly, $\{b_1, b_2\} \in Y/\sigma_2$ for some $b_1, b_2 \in Y$. Thus

$$x\sigma_1 = \{x\} \text{ for all } x \in X \setminus \{a_1, a_2\}$$

and

$$y\sigma_2 = \{y\} \text{ for all } y \in Y \setminus \{b_1, b_2\}.$$

Since $|X \setminus \{a_1, a_2\}| = |Y \setminus \{b_1, b_2\}|$, there exists $\varphi : X \setminus \{a_1, a_2\} \to Y \setminus \{b_1, b_2\}$ is a bijection. Define $\theta : X \to Y$ by

$$x\theta = \begin{cases} b_i, & \text{if } x = a_i, \\ x\varphi, & \text{otherwise} \end{cases}$$

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for all $x \in X$. It is clear that θ is a bijection and each element x in X, $(x\sigma_1)\theta = (x\theta)\sigma_2$.

From the three cases above, $E(X, \sigma_1) \cong E(Y, \sigma_2)$ by Theorem 2.4. □

Note that if $|X| \leq 3$ and σ is an equivalence on X, then $|X/\sigma| \in \{1, 2, 3\}$. The following corollary is a direct consequence of Corollary 2.5 and Theorem 2.4.

Corollary 2.6. *Let X and Y be sets such that $|X| = |Y| \leq 3$. Then $E(X, \sigma_1) \cong E(Y, \sigma_2)$ if and only if $|X/\sigma_1| = |Y/\sigma_2|$.*

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