Commun. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/CKMS.c170216
pISSN: 1225-1763 / eISSN: 2234-3024

# A NEW EXTENSION OF THE MITTAG-LEFFLER FUNCTION

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Abstract. Since Mittag-Leffler introduced the so-called Mittag-Leffler function in 1903, due to its usefulness and diverse applications, a variety and large number of its extensions (and generalizations) and variants have been presented and investigated. In this sequel, we aim to introduce a new extension of the Mittag-Leffler function by using a known extended beta function. Then we investigate ceratin useful properties and formulas associated with the extended Mittag-Leffler function such as integral representation, Mellin transform, recurrence relation, and derivative formulas. We also introduce an extended Riemann-Liouville fractional derivative to present a fractional derivative formula for a known extended Mittag-Leffler function, the result of which is expressed in terms of the new extended Mittag-Leffler functions.

## 1. Introduction and preliminaries

Gösta Mittag-Leffler (1846-1927) [13] presented an entire function

$$E_\rho(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n + 1)} \quad (z \in \mathbb{C};\ \Re(\rho) > 0), \tag{1}$$

which is called Mittag-Leffler function. Wiman [28, 29] gave and investigated a generalization of the Mittag-Leffler function (1)

$$E_{\rho,\sigma}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n + \sigma)} \quad (z,\ \sigma \in \mathbb{C};\ \Re(\rho) > 0). \tag{2}$$

Here and in the following, let $\mathbb{C}$, $\mathbb{R}^+$, $\mathbb{N}$, and $\mathbb{Z}_0^-$ be the sets of complex numbers, positive real numbers, positive integers, and non-positive integers, respectively, and let $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$. Since then, a variety of generalizations and

Received May 22, 2017; Accepted January 10, 2018.
2010 *Mathematics Subject Classification.* 33B20, 33C20, 33C45, 33C60, 33B15, 33C05, 26A33.
*Key words and phrases.* gamma function, beta function, extended beta function, Mittag-Leffler function, extended Mittag-Leffler functions, Fox-Wright function, generalized hypergeometric function, Mellin transform, Riemann-Liouville fractional derivative, extended Riemann-Liouville fractional derivative.

extensions of the Mittag-Leffler function (1) have been investigated (see, e.g, [2,3,5,8,9,11,12,15–17,19,20,27] and the references cited therein). The Mittag-Leffler function and its diverse generalizations are important, in particular, in connection with the theories of fractional calculus and special functions (see, e.g., [5], [21, 22, 24, 25]). Certain interesting applications of the Mittag-Leffler function and its generalizations have been carried out in a wide range of research subjects including physics and engineering such as kinetic equations, Lorenz system, random walk, Levy flights, complex system, fluid flow, electric network, probability, and statistical distribution theory.

Prabhakar [16] introduced and investigated the following generalization of the function $E_{\rho,\sigma}(z)$ (2)

$$E^{\gamma}_{\rho,\sigma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\rho n+\sigma)} \frac{z^n}{n!} \quad (z,\, \sigma,\, \gamma \in \mathbb{C};\, \Re(\rho) > 0), \tag{3}$$

where $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by (see, e.g., [23, p. 2 and pp. 4–6]):

$$\begin{aligned}(\lambda)_n :&= \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N}) \end{cases} \\ &= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad \left(\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-\right),\end{aligned} \tag{4}$$

and $\Gamma$ being the familiar gamma function (see, e.g., [23, Section 1.1]).

Shukla and Prajapati [20] (see also [27]) introduced a generalized Mittag-Leffler function $E^{\delta,q}_{\rho,\sigma}(z)$ defined by

$$E^{\delta,q}_{\rho,\sigma}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_{nq}}{\Gamma(\rho n+\sigma)} \frac{z^n}{n!} \tag{5}$$

$$(z,\, \rho,\, \sigma,\, \delta \in \mathbb{C};\, \min\{\Re(\rho),\, \Re(\sigma),\, \Re(\delta)\} > 0;\, q \in (0,1) \cup \mathbb{N})$$

and investigated to present many interesting properties and formulas involving the generalized Mittag-Leffler function such as Laplace transforms and Whittaker transforms, in a systematic way (see also [6, 7, 10, 18]).

Özarslan and Yilmaz [15] introduced to investigate the following extended Mittag-Leffler function $E^{\delta;c}_{\rho,\sigma}(z;p)$ (see also [14]):

$$E^{\delta;c}_{\rho,\sigma}(z;p) = \sum_{n=0}^{\infty} \frac{B_p(\delta+n, c-\delta)}{B(\delta, c-\delta)} \frac{(c)_n}{\Gamma(\rho n+\sigma)} \frac{z^n}{n!} \tag{6}$$

$$\left(p \in \mathbb{R}_0^+;\, \Re(c) > \Re(\delta) > 0;\, \Re(\rho) > 0\right),$$

where $B_p(x,y)$ is extended beta function defined by (see, e.g., [1,12])

$$B_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, \mathrm{e}^{-\frac{p}{t(1-t)}}\, dt \tag{7}$$

$$(\min\{\Re(p), \Re(x), \Re(y)\} > 0)$$

and $B_0(x,y) = B(x,y)$ is the familiar beta function (see, [23, Section 1.1]).

By giving a generalization of the extended beta function in (7)

$$B_p^{\lambda,\rho}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1\Big[\lambda;\rho;-\frac{p}{t(1-t)}\Big]dt \tag{8}$$

$$\big(p \in \mathbb{R}_0^+;\ \min\{\Re(x),\ \Re(y)\} > 0\big)\,,$$

we define a further extension of the extended Mittag-Leffler function in (6)

$$E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}(z;p) = \sum_{n=0}^{\infty} \frac{B_p^{\lambda,\rho}(\gamma+n, c-\gamma)}{B(\gamma, c-\gamma)} \frac{(c)_n}{\Gamma(\alpha n+\beta)} \frac{z^n}{n!} \tag{9}$$

$$\big(p \in \mathbb{R}_0^+;\ \Re(c) > \Re(\gamma) > 0;\ \Re(\alpha) > 0;\ \rho \in \mathbb{C}\setminus\mathbb{Z}_0^-\big)\,.$$

Then we investigate certain interesting properties and formulas involving the extended Mittag-Leffler function (9) such as integral representations, Mellin transform, and recurrence relation. We also introduce an extended Riemann-Liouville fractional derivative to present a fractional derivative formula (40) for the extended Mittag-Leffler function (3), the result of which is expressed in terms of the new extended Mittag-Leffler functions (9). The results presented here are sure to new and potentially useful.

It is remarked in passing that the special cases of (9) when $\rho = \lambda$ and $p = 0$ reduce to the extended Mittag-Leffler functions (6) and (3), respectively.

For our purpose, we recall the Fox-Wright function ${}_p\Psi_q$ (see [4] and [30–32]) (see also [26, p. 21])

$${}_p\Psi_q\left[\begin{matrix}(\alpha_1,\,A_1)\,,\,\ldots,\,(\alpha_p,\,A_p)\,;\\(\beta_1,\,B_1)\,,\,\ldots,\,(\beta_q,\,B_q)\,;\end{matrix}\ z\right] = \sum_{k=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma\left(\alpha_j + A_j\,k\right)}{\prod_{j=1}^{q}\Gamma\left(\beta_j + B_j\,k\right)}\frac{z^k}{k!}, \tag{10}$$

where the coefficients $A_j \in \mathbb{R}^+$ $(j = 1,\ldots,p)$ and $B_j \in \mathbb{R}^+$ $(j = 1,\ldots,q)$ such that

$$1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geqq 0. \tag{11}$$

A special case of (10) reduces to the generalized hypergeometric function ${}_pF_q$

$${}_p\Psi_q\left[\begin{matrix}(\alpha_1,\,1)\,,\,\ldots,\,(\alpha_p,\,1)\,;\\(\beta_1,\,1)\,,\,\ldots,\,(\beta_q,\,1)\,;\end{matrix}\ z\right] = \frac{\prod_{j=1}^{p}\Gamma\left(\alpha_j\right)}{\prod_{j=1}^{q}\Gamma\left(\beta_j\right)}\,{}_pF_q\left[\begin{matrix}\alpha_1,\,\ldots,\,\alpha_p\,;\\\beta_1,\,\ldots,\,\beta_q\,;\end{matrix}\ z\right]. \tag{12}$$

We also recall the generalized gamma function $\Gamma_p^{(\alpha,\beta)}$ (see [14, Eq. (3)])

$$\Gamma_p^{(\alpha,\beta)}(x) = \int_0^{\infty} t^{x-1}\,{}_1F_1\left(\alpha;\ \beta;\ -t-\frac{p}{t}\right)\,dt \tag{13}$$

$$\big(\min\{\Re(\alpha),\ \Re(\beta),\ \Re(x)\} > 0;\ p \in \mathbb{R}_0^+\big)$$

and (see [14, Eq. (5)])

$$\Gamma_0^{(\alpha,\beta)}(x) =: \Gamma^{(\alpha,\beta)}(x) = \frac{\Gamma(\beta)\,\Gamma(\alpha-x)\,\Gamma(x)}{\Gamma(\alpha)\,\Gamma(\beta-x)}. \tag{14}$$

The special case of (13) when $\alpha = \beta$ and $p = 0$ is seen to yield the familiar gamma function $\Gamma$.

## 2. Properties of the extended Mittag-Leffler function (9)

Here, we give an integral representation and the Mellin transform of the extended Mittag-Leffler function (9).

**Theorem 2.1.** *Let $c$, $\alpha$, $\beta$, $\gamma$, $\lambda \in \mathbb{C}$ with $\Re(c) > \Re(\gamma) > 0$ and $\Re(\alpha) > 0$. Also, let $p \in \mathbb{R}_0^+$ and $\rho \in \mathbb{C} \setminus \mathbb{Z}_0^-$. Then*

$$\begin{aligned} E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}(z;p) &= \frac{1}{B(\gamma, c-\gamma)} \\ &\quad \times \int_0^1 t^{\gamma-1}(1-t)^{c-\gamma-1}\,{}_1F_1\Big[\lambda;\rho;-\frac{p}{t(1-t)}\Big] E_{\alpha,\beta}^{c}(tz)\,dt. \end{aligned} \tag{15}$$

*Proof.* Using (8) in (9), we have

$$\begin{aligned} E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}(z;p) = \sum_{n=0}^{\infty} \bigg\{ &\int_0^1 t^{\gamma+n-1}(1-t)^{c-\gamma-1} \\ &\times {}_1F_1\Big[\lambda;\rho;-\frac{p}{t(1-t)}\Big] dt\, \frac{(c)_n}{B(\gamma,c-\gamma)} \frac{z^n}{\Gamma(\alpha n+\beta) n!} \bigg\}. \end{aligned}$$

Interchanging the order of summation and integration in the above equation, which is verified under the given conditions here, we get

$$\begin{aligned} E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}(z;p) = &\int_0^1 t^{\gamma-1}(1-t)^{c-\gamma-1} \\ &\times {}_1F_1\Big[\lambda;\rho;-\frac{p}{t(1-t)}\Big] \sum_{n=0}^{\infty} \frac{(c)_n}{B(\gamma,c-\gamma)} \frac{(tz)^n}{\Gamma(\alpha n+\beta) n!} dt. \end{aligned}$$

Using (3) in the above equation, we get the desired result. □

Setting $t = \frac{u}{1+u}$ and $t = \sin^2\theta$ in the result in Theorem 2.1, we get two interesting integral formulas for the extended Mittag-Leffler function (9), which are given in Corollaries 2.2 and 2.3, respectively.

**Corollary 2.2.** *Let $c$, $\alpha$, $\beta$, $\gamma$, $\lambda \in \mathbb{C}$ with $\Re(c) > \Re(\gamma) > 0$ and $\Re(\alpha) > 0$. Also, let $p \in \mathbb{R}_0^+$ and $\rho \in \mathbb{C} \setminus \mathbb{Z}_0^-$. Then*

$$\begin{aligned} E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}(z;p) = &\frac{1}{B(\gamma,c-\gamma)} \int_0^\infty \frac{u^{\gamma-1}}{(u+1)^c} \\ &\times {}_1F_1\Big[\lambda;\rho;-\frac{p(1+u)^2}{u}\Big] E_{\alpha,\beta}^{c}\Big(\frac{uz}{1+u}\Big) du. \end{aligned} \tag{16}$$

**Corollary 2.3.** *Let $c$, $\alpha$, $\beta$, $\gamma$, $\lambda \in \mathbb{C}$ with $\Re(c) > \Re(\gamma) > 0$ and $\Re(\alpha) > 0$. Also, let $p \in \mathbb{R}_0^+$ and $\rho \in \mathbb{C} \setminus \mathbb{Z}_0^-$. Then*

$$\begin{aligned} E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}(z;p) &= \frac{2}{B(\gamma, c-\gamma)} \int_0^{\frac{\pi}{2}} \sin^{2\gamma-1}\theta \cos^{2c-1}\theta \\ &\quad \times {}_1F_1\Big[\lambda;\rho;-\frac{p}{\sin^2\theta\cos^2\theta}\Big] E_{\alpha,\beta}^{c}\Big(z\sin^2\theta\Big)\,d\theta. \end{aligned} \tag{17}$$

Kurulay and Bayram [11] presented the following recurrence relation for the extended Mittag-Leffler function (3)

$$E_{\alpha,\beta}^{c}(tz) = \beta E_{\alpha,\beta+1}^{c}(tz) + \alpha z \frac{d}{dz} E_{\alpha,\beta+1}^{c}(tz). \tag{18}$$

We also present a recurrence relation for the extended Mittag-Leffler function (9) asserted by the following corollary.

**Corollary 2.4.** *Let $c$, $\alpha$, $\beta$, $\gamma$, $\lambda \in \mathbb{C}$ with $\Re(c) > \Re(\gamma) > 0$ and $\Re(\alpha) > 0$. Also, let $p \in \mathbb{R}_0^+$ and $\rho \in \mathbb{C} \setminus \mathbb{Z}_0^-$. Then*

$$E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}(z;p) = \beta E_{\alpha,\beta+1}^{\gamma,c;\lambda,\rho}(z;p) + \alpha\gamma z E_{\alpha,\alpha+\beta+1}^{\gamma+1,c+1;\lambda,\rho}(z;p). \tag{19}$$

*Proof.* Applying (18) to the integrand in (15), we obtain

$$E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}(z;p) = \beta E_{\alpha,\beta+1}^{\gamma,c;\lambda,\rho}(z;p) + \alpha z \frac{d}{dz} E_{\alpha,\beta+1}^{\gamma,c;\lambda,\rho}(z;p). \tag{20}$$

Then we use (37) with $n = 1$ in the derivative in (20) to yield (19). □

**Theorem 2.5.** *Let $\alpha \in \mathbb{R}^+$, $p \in \mathbb{R}_0^+$, $\Re(c) > \Re(\gamma) > 0$, $\min\{\Re(\beta), \Re(\lambda), \Re(\rho)\} > 0$. Then Mellin transform of the extended Mittag-Leffler function (9) is given by*

$$\begin{aligned} &\mathfrak{M}\Big\{E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}(z;p);s\Big\} \\ &= \frac{\Gamma^{(\lambda;\rho)}(s)\,\Gamma(c+s-\gamma)}{\Gamma(\gamma)\Gamma(c-\gamma)}\,{}_2\Psi_2\left[\begin{array}{c} (c,1),\ (\gamma+s,1); \\ (\beta,\alpha),(c+2s,1); \end{array} z\right]. \end{aligned} \tag{21}$$

*Proof.* Taking Mellin transform of the extended Mittag-Leffler function (9), we have

$$\mathfrak{M}\Big\{E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}(z;p);s\Big\} = \int_0^\infty p^{s-1}\, E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}(z;p)\,dp. \tag{22}$$

Applying (15) to (22), we have

$$\begin{aligned} &\mathfrak{M}\Big\{E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}(z;p);s\Big\} = \frac{1}{B(\gamma,c-\gamma)} \\ &\times \int_0^\infty p^{s-1}\Big\{\int_0^1 t^{\gamma-1}(1-t)^{c-\gamma-1}\,{}_1F_1\Big[\lambda;\rho;-\frac{p}{t(1-t)}\Big]\Big\} E_{\alpha,\beta}^{c}(tz)\,dt\,dp. \end{aligned} \tag{23}$$

Interchanging the order of integrations in (23), which is guaranteed under the conditions here, we have

$$\begin{aligned}\mathfrak{M}\Big\{E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}(z;p);s\Big\} = &\frac{1}{B(\gamma,c-\gamma)}\int_0^1 \Big[t^{\gamma-1}(1-t)^{c-\gamma-1}E_{\alpha,\beta}^{c}(tz)\Big]\\ &\int_0^\infty p^{s-1}\,{}_1F_1\Big[\lambda;\rho;-\frac{p}{t(1-t)}\Big]dp\,dt.\end{aligned} \tag{24}$$

Substituting $u=\frac{p}{t(1-t)}$ in the inner integral in (24), we get

$$\begin{aligned}\int_0^\infty p^{s-1}\,{}_1F_1\Big[\lambda;\rho;-\frac{p}{t(1-t)}\Big]dp &= \int_0^\infty u^{s-1}t^s(1-t)^s\,{}_1F_1\Big[\lambda;\rho;-u\Big]\,du\\ &= t^s(1-t)^s\int_0^\infty u^{s-1}\,{}_1F_1\Big[\lambda;\rho;-u\Big]\,du\\ &= t^s(1-t)^s\,\Gamma^{(\lambda,\rho)}(s),\end{aligned} \tag{25}$$

where $\Gamma_p^{(\lambda,\rho)}(s)$ is the extended gamma function given in (13). Using (25) and (24), we find

$$\begin{aligned}\mathfrak{M}\Big\{E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}(z;p);s\Big\} &= \frac{\Gamma^{(\lambda,\rho)}(s)}{B(\gamma,c-\gamma)}\int_0^1 t^{\gamma+s-1}(1-t)^{c+s-\gamma-1}\\ &\quad\times\sum_{n=0}^\infty \frac{(c)_n z^n}{\Gamma(\alpha n+\beta)n!}\int_0^1 t^{\gamma+n+s-1}(1-t)^{c+s-\gamma-1}dt\\ &= \frac{\Gamma^{(\lambda,\rho)}(s)}{B(\gamma,c-\gamma)}\sum_{n=0}^\infty \frac{(c)_n z^n}{\Gamma(\alpha n+\beta)n!}\frac{\Gamma(\gamma+n+s)\Gamma(c+s-\gamma)}{\Gamma(c+n+2s)}\\ &= \frac{\Gamma^{(\lambda,\rho)}(s)\,\Gamma(c+s-\gamma)}{\Gamma(\gamma)\Gamma(c-\gamma)}\sum_{n=0}^\infty \frac{\Gamma(c+n)}{\Gamma(\alpha n+\beta)}\frac{\Gamma(\gamma+n+s)}{\Gamma(c+n+2s)}\,\frac{z^n}{n!},\end{aligned}$$

which, in terms of (10), leads to the right side of (21). □

Setting $s=1$ in the result in Theorem 2.5 and using (14), we obtain an integral formula involving the extended Mittag-Leffler function (9), which is asserted by the following corollary.

**Corollary 2.6.** *Let* $\alpha\in\mathbb{R}^+$, $p\in\mathbb{R}_0^+$, $\Re(c)>\Re(\gamma)>0$, $\Re(\beta)>0$, $\min\{\Re(\lambda),\Re(\rho)\}>1$. *Then*

$$\begin{aligned}&\int_0^\infty E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}(z;p)\,dp\\ &= \frac{(\rho-1)\,\Gamma(c+1-\gamma)}{(\lambda-1)\,\Gamma(\gamma)\Gamma(c-\gamma)}\,{}_2\Psi_2\left[\begin{array}{l}(c,1),(\gamma+1,1);\\(\beta,\alpha),(c+2,1);\end{array}\,z\right].\end{aligned} \tag{26}$$

Setting $\lambda=\rho$ in Corollary 2.6, we get an integral formula involving the extended Mittag-Leffler function (6), which is asserted by the following corollary.

**Corollary 2.7.** *Let* $\alpha \in \mathbb{R}^+$, $p \in \mathbb{R}_0^+$, $\Re(c) > \Re(\gamma) > 0$, *and* $\Re(\beta) > 0$. *Then*

$$\int_0^\infty E_{\alpha,\beta}^{\gamma,c}(z;p)\,dp = \frac{c-\gamma}{\Gamma(\gamma)}\,{}_2\Psi_2\left[\begin{array}{c}(c,1),(\gamma+1,1);\\ (\beta,\alpha),(c+2,1);\end{array} z\right]. \tag{27}$$

## 3. Derivative formulas of the extended Mittag-Leffler function (9)

Here, we define a further extension of a known extended Riemann-Liouville fractional derivative and present a derivative formula for the extended Mittag-Leffler function (3) involving this extended fractional derivative, which is expressed in terms of the extended Mittag-Leffler functions (9). To do this, we recall the well-known Riemann-Liouville fractional derivative of order $\mu \in \mathbb{C}$ $(\Re(\mu) \geq 0)$ defined by

$$\mathfrak{D}_x^{\mu}\Big\{f(x)\Big\} := \frac{1}{\Gamma(m-\mu)}\frac{d^m}{dx^m}\int_0^x f(t)(x-t)^{-\mu+m-1}\,dt \tag{28}$$

$$(m \in \mathbb{N};\ m-1 < \Re(\mu) < m;\ x > 0).$$

In particular,

$$\mathfrak{D}_x^{\mu}\Big\{f(x)\Big\} = \frac{1}{\Gamma(1-\mu)}\frac{d}{dx}\int_0^x f(t)(x-t)^{-\mu}\,dt \tag{29}$$

$$(0 < \Re(\mu) < 1;\ x > 0).$$

Özarslan and Yilmaz [15] extended the Riemann-Liouville fractional derivative (28) as follows:

$$\begin{aligned}&\mathfrak{D}_x^{\mu,p}\Big\{f(x)\Big\}\\ &:= \frac{1}{\Gamma(m-\mu)}\frac{d^m}{dx^m}\int_0^x f(t)(x-t)^{-\mu+m-1}\exp\Big(-\frac{p\,x^2}{t(x-t)}\Big)\,dt\end{aligned} \tag{30}$$

$$\big(p \in \mathbb{R}_0^+;\ m \in \mathbb{N};\ m-1 < \Re(\mu) < m;\ x > 0\big).$$

In particular,

$$\mathfrak{D}_x^{\mu,p}\Big\{f(x)\Big\} := \frac{1}{\Gamma(1-\mu)}\frac{d}{dx}\int_0^x f(t)(x-t)^{-\mu}\exp\Big(-\frac{p\,x^2}{t(x-t)}\Big)\,dt \tag{31}$$

$$\big(p \in \mathbb{R}_0^+;\ 0 < \Re(\mu) < 1;\ x > 0\big).$$

Here, we define a further extension of the extended Riemann-Liouville fractional derivative (30) by

(32)

$$\mathfrak{D}_{x;\lambda,\rho}^{\mu,p}\Big\{f(x)\Big\} := \frac{1}{\Gamma(m-\mu)}\frac{d^m}{dx^m}\int_0^x f(t)(x-t)^{-\mu+m-1}\,{}_1F_1\Big[\lambda;\rho;-\frac{px^2}{t(x-t)}\Big]\,dt$$

$$\big(p \in \mathbb{R}_0^+;\ m \in \mathbb{N};\ m-1 < \Re(\mu) < m;\ x \in \mathbb{R}^+;\ \lambda \in \mathbb{C},\ \rho \in \mathbb{C}\setminus\mathbb{Z}_0^-\big).$$

In particular,

$$\mathfrak{D}_{x;\lambda,\rho}^{\mu,p}\Big\{f(x)\Big\} := \frac{1}{\Gamma(1-\mu)}\frac{d}{dx}\int_0^x f(t)(x-t)^{-\mu}\,{}_1F_1\Big[\lambda;\rho;-\frac{p\,x^2}{t(x-t)}\Big]\,dt \tag{33}$$

$$\left(p \in \mathbb{R}_0^+;\ 0 < \Re(\mu) < 1;\ x \in \mathbb{R}^+;\ \lambda \in \mathbb{C},\ \rho \in \mathbb{C} \setminus \mathbb{Z}_0^-\right).$$

It is obvious that the special cases of (32) when $\lambda = \rho$ and $p = 0$ reduce to the extended Riemann-Liouville fractional derivative (30) and the Riemann-Liouville fractional derivative (28), respectively.

We provide some formulas for the $B_p^{\lambda,\rho}(x,y)$ function in (8) and the beta function $B(x,y)$, which are easily derivable and given in the following lemma.

**Lemma 3.1.** *The following formulas hold.*

$$B(x, y+1) = \frac{y}{x+y}\, B(x,y); \tag{34}$$

$$B(x+1, y) = \frac{x}{x+y}\, B(x,y); \tag{35}$$

$$B_p^{\lambda,\rho}(x,y) = B_p^{\lambda,\rho}(x+1,y) + B_p^{\lambda,\rho}(x,y+1). \tag{36}$$

We give a derivative formula for the extended Mittag-Leffler function (9), which is asserted by the following theorem.

**Theorem 3.2.** *Let* $p \in \mathbb{R}_0^+$, $\Re(c) > \Re(\gamma) > 0$, $\Re(\alpha) > 0$, *and* $\rho \in \mathbb{C} \setminus \mathbb{Z}_0^-$. *Then*

$$\frac{d^n}{dz^n}\, E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}(z;p) = (\gamma)_n\, E_{\alpha,n\alpha+\beta}^{\gamma+n,c+n;\lambda,\rho}(z;p) \quad (n \in \mathbb{N}_0). \tag{37}$$

*Proof.* We find from (9) that

$$\begin{aligned}
\frac{d}{dz} E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}(z;p) &= \sum_{n=1}^{\infty} \frac{B_p^{\lambda,\rho}(\gamma+n, c-\gamma)}{B(\gamma, c-\gamma)} \frac{(c)_n}{\Gamma(\alpha n+\beta)} \frac{z^{n-1}}{(n-1)!} \\
&= \sum_{n=0}^{\infty} \frac{B_p^{\lambda,\rho}(\gamma+1+n, c-\gamma)}{B(\gamma, c-\gamma)} \frac{(c)_{n+1}}{\Gamma(\alpha n+\alpha+\beta)} \frac{z^n}{n!} \\
&= c \sum_{n=0}^{\infty} \frac{B_p^{\lambda,\rho}(\gamma+1+n, c-\gamma)}{B(\gamma, c-\gamma)} \frac{(c+1)_n}{\Gamma(\alpha n+\alpha+\beta)} \frac{z^n}{n!},
\end{aligned}$$

which, upon using (35), leads to

$$\frac{d}{dz} E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}(z;p) = \gamma \sum_{n=0}^{\infty} \frac{B_p^{\lambda,\rho}(\gamma+1+n, c-\gamma)}{B(\gamma+1, c-\gamma)} \frac{(c+1)_n}{\Gamma(\alpha n+\alpha+\beta)} \frac{z^n}{n!}.$$

In terms of (9), we obtain

$$\frac{d}{dz} E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}(z;p) = \gamma\, E_{\alpha,\alpha+\beta}^{\gamma+1,c+1;\lambda,\rho}(z;p). \tag{38}$$

Similarly, we have

$$\frac{d^2}{dz^2}\, E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}(z;p) = \gamma(\gamma+1)\, E_{\alpha,2\alpha+\beta}^{\gamma+2,c+2;\lambda,\rho}(z;p).$$

Then, continuing this process $n-2$ times on the last formula is seen to yield the desired result (37). □

**Theorem 3.3.** *Let $p \in \mathbb{R}_0^+$, $\Re(c) > \Re(\gamma) > 0$, $\Re(\alpha) > 0$, $\rho \in \mathbb{C} \setminus \mathbb{Z}_0^-$, and $\beta$, $\mu \in \mathbb{C}$. Then*

$$\frac{d^n}{dz^n}\left\{z^{\beta-1}\,E_{\alpha,\beta}^{\gamma,c;\lambda,\rho}\,(\mu\,z^{\alpha};p)\right\} = z^{\beta-n-1}\,E_{\alpha,\beta-n}^{\gamma,c;\lambda,\rho}\,(\mu\,z^{\alpha};p) \quad (n \in \mathbb{N}_0)\,. \tag{39}$$

*Proof.* By using (9), it is easy to get the desired result. The detailed account of the proof is omitted. □

**Theorem 3.4.** *Let $p \in \mathbb{R}_0^+$, $x \in \mathbb{R}^+$, and $\rho \in \mathbb{C} \setminus \mathbb{Z}_0^-$. Also, let $m-2 < \Re(\delta - c) < m-1$ for $m \in \mathbb{N}$. Then*

$$\begin{aligned} &\mathfrak{D}_{x;\lambda,\rho}^{\delta+1-c,p}\left\{x^{\delta-1}\,E_{\alpha,\beta}^{c+m-1}(x)\right\} \\ &= \sum_{\ell=0}^{m}\binom{m}{\ell}\frac{\Gamma(\delta+\ell)}{\Gamma(c+\ell-1)}\,x^{c+\ell-2}\,E_{\alpha,\ell\alpha+\beta}^{\delta+\ell,c+\ell+m-1;\lambda,\rho}(x;p) \quad (m \in \mathbb{N}). \end{aligned} \tag{40}$$

*Proof.* Let $\mathcal{L}_1$ be the left side of (40). By using (32), we have

$$\begin{aligned} \mathcal{L}_1 = \frac{1}{\Gamma(m+c-\delta-1)}\,\frac{d^m}{dx^m}\int_0^x t^{\delta-1}\,(x-t)^{c+m-\delta-2} \\ \times\,E_{\alpha,\beta}^{c+m-1}(t)\,{}_1F_1\left[\lambda;\rho;-\frac{p\,x^2}{t(x-t)}\right]dt, \end{aligned}$$

which, upon taking $t = u\,x$, becomes

$$\begin{aligned} \mathcal{L}_1 = \frac{1}{\Gamma(m+c-\delta-1)}\,\frac{d^m}{dx^m}\bigg\{x^{c+m-2}\int_0^1 u^{\delta-1}(1-u)^{c+m-\delta-2} \\ \times E_{\alpha,\beta}^{c+m-1}(xu)\,{}_1F_1\left[\lambda;\rho;-\frac{p}{u(1-u)}\right]du\bigg\}. \end{aligned} \tag{41}$$

Applying (15) to the integral in (41), we obtain

$$\mathcal{L}_1 = \frac{\Gamma(\delta)}{\Gamma(c+m-1)}\,\frac{d^m}{dx^m}\left\{x^{c+m-2}\,E_{\alpha,\beta}^{\delta,c+m-1;\lambda,\rho}(x;p)\right\}.$$

By using the Leibniz's generalized rule for differentiation of product of two functions, we have

$$\mathcal{L}_1 = \frac{\Gamma(\delta)}{\Gamma(c+m-1)}\sum_{\ell=0}^{m}\binom{m}{\ell}\left\{\frac{d^{m-\ell}}{dx^{m-\ell}}x^{c+m-2}\right\}\left\{\frac{d^{\ell}}{dx^{\ell}}\,E_{\alpha,\beta}^{\delta,c+m-1;\lambda,\rho}(x;p)\right\}. \tag{42}$$

Using (37) and the following easily derivable formula

$$\frac{d^{\ell}}{dx^{\ell}}\,x^{\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\ell+1)}\,x^{\alpha-\ell} \quad (\ell \in \mathbb{N}_0) \tag{43}$$

in (42), we get the desired result. □

The particular case of Theorem 3.4 when $m = 1$ is given in the following corollary.

**Corollary 3.5.** *Let* $p \in \mathbb{R}_0^+$, $x \in \mathbb{R}^+$, $\rho \in \mathbb{C} \setminus \mathbb{Z}_0^-$, $0 < \Re(\delta) < \Re(c)$ *and* $\Re(\delta - c) > -1$. *Then*

$$
\begin{aligned}
&\mathfrak{D}^{\delta+1-c,p}_{x;\lambda,\rho}\left\{x^{\delta-1}\,E^{c}_{\alpha,\beta}(x)\right\} \\
&= \frac{\Gamma(\delta)}{\Gamma(c)}\,x^{c-2}\Big\{(c-1)\,E^{\delta,c;\lambda,\rho}_{\alpha,\beta}(x;p) + \delta x\,E^{\delta+1,c+1;\lambda,\rho}_{\alpha,\alpha+\beta}(x;p)\Big\}.
\end{aligned}
\tag{44}
$$

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