Commun. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/CKMS.c170204
pISSN: 1225-1763 / eISSN: 2234-3024

ON PARTIAL SUMS OF NORMALIZED q-BESSEL FUNCTIONS

İbrahİm Aktaş and Halİt Orhan

Abstract. In the present investigation our main aim is to give lower bounds for the ratio of some normalized q-Bessel functions and their sequences of partial sums. Especially, we consider Jackson's second and third q-Bessel functions and we apply one normalization for each of them.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}.$$

We denote by $\mathcal{S}$ the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$.

The Jackson's second and third q-Bessel functions are defined by (see [4])

$$J_\nu^{(2)}(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{n\geq 0} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+\nu}}{(q;q)_n (q^{\nu+1};q)_n} q^{n(n+\nu)} \tag{2}$$

and

$$J_\nu^{(3)}(z;q) = \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \sum_{n\geq 0} \frac{(-1)^n z^{2n+\nu}}{(q;q)_n (q^{\nu+1};q)_n} q^{\frac{1}{2}n(n+1)}, \tag{3}$$

where $z \in \mathbb{C}, \nu > -1, q \in (0,1)$ and

$$(a;q)_0 = 1, (a;q)_n = \prod_{k=1}^{n} \left(1 - aq^{k-1}\right), (a,q)_\infty = \prod_{k\geq 1} \left(1 - aq^{k-1}\right).$$

Here we would like to say that Jackson's third q-Bessel function is also known as Hahn-Exton q-Bessel function.

Received May 15, 2017; Revised October 6, 2017; Accepted November 2, 2017.
2010 *Mathematics Subject Classification.* Primary 30C45.
Key words and phrases. analytic function, q-Bessel functions, univalent functions, partial sums.

Recently, the some geometric properties like univalence, starlikeness and convexity of the some special functions were investigated by many authors. Especially, in [1, 5, 6] authors have studied on the starlikeness and convexity of the some normalized special functions. In addition, the some lower bounds for the ratio of some special functions and their sequences of partial sums were given in [3, 7, 8, 10]. Moreover, results related with partial sums of analytic functions can be found in [2, 9, 11–13] etc.

Motivated by the previous works on analytic and some special functions, in this paper our aim is to present some lower bounds for the ratio of normalized q-Bessel functions to their sequences of partial sums.

Due to the functions defined by (2) and (3) do not belong to the class $\mathcal{A}$, we consider following normalized forms of the q-Bessel functions:

$$h_\nu^{(2)}(z;q) = 2^\nu c_\nu(q) z^{1-\frac{\nu}{2}} J_\nu^{(2)}(\sqrt{z};q) = \sum_{n\geq 0} K_n z^{n+1} \tag{4}$$

and

$$h_\nu^{(3)}(z;q) = c_\nu(q) z^{1-\frac{\nu}{2}} J_\nu^{(3)}(\sqrt{z};q) = \sum_{n\geq 0} T_n z^{n+1}, \tag{5}$$

where $K_n = \frac{(-1)^n q^{n(n+\nu)}}{4^n (q;q)_n (q^{\nu+1};q)_n}$, $T_n = \frac{(-1)^n q^{\frac{1}{2}n(n+1)}}{(q;q)_n (q^{\nu+1};q)_n}$ and $c_\nu(q) = (q;q)_\infty / (q^{\nu+1};q)_\infty$. As a result of the above normalizations, all of the above functions belong to the class $\mathcal{A}$.

Here we would like to mention that the following inequalities

$$q^{n(n+\nu)} \leq q^{n\nu}, \tag{6}$$

$$(1-q)^n \leq (q;q)_n, \tag{7}$$

$$(1-q^\nu)^n \leq (q^{\nu+1};q)_n \tag{8}$$

and

$$q^{\frac{1}{2}n(n+1)} \leq q^{\frac{1}{2}n} \tag{9}$$

are valid for $q \in (0,1)$ and $\nu > -1$. These inequalities and the well-known triangle inequality

$$|z_1 + z_2| \leq |z_1| + |z_2| \tag{10}$$

will be used frequently in the proofs of our main results.

2. Main results

The following lemmas will be required in order to derive our main results.

Lemma 2.1. *Let $q \in (0,1)$, $\nu > -1$ and $4(1-q)(1-q^\nu) > q^\nu$. Then the function $h_\nu^{(2)}(z;q)$ satisfies the next two inequalities for $z \in \mathbb{U}$:*

$$\left| h_\nu^{(2)}(z;q) \right| \leq \frac{4(1-q)(1-q^\nu)}{4(1-q)(1-q^\nu) - q^\nu}, \tag{11}$$

$$\left|\left(h_{\nu}^{(2)}(z;q)\right)'\right| \leq \left(\frac{4(1-q)(1-q^{\nu})}{4(1-q)(1-q^{\nu})-q^{\nu}}\right)^2. \tag{12}$$

Proof. Making use the inequalities (6), (7), (8) and the well-known triangle inequality which is given by (10), for $z \in \mathbb{U}$, we get

$$\begin{aligned}
\left|h_{\nu}^{(2)}(z;q)\right| &= \left|z + \sum_{n\geq 1} \frac{(-1)^n q^{n(n+\nu)}}{4^n (q;q)_n (q^{\nu+1};q)_n} z^{n+1}\right| \\
&\leq 1 + \sum_{n\geq 1} \frac{q^{n(n+\nu)}}{4^n (q;q)_n (q^{\nu+1};q)_n} \\
&\leq 1 + \sum_{n\geq 1} \left(\frac{q^{\nu}}{4(1-q)(1-q^{\nu})}\right)^n \\
&= 1 + \frac{q^{\nu}}{4(1-q)(1-q^{\nu})} \sum_{n\geq 1} \left(\frac{q^{\nu}}{4(1-q)(1-q^{\nu})}\right)^{n-1} \\
&= \frac{4(1-q)(1-q^{\nu})}{4(1-q)(1-q^{\nu})-q^{\nu}}
\end{aligned}$$

and

$$\begin{aligned}
\left|\left(h_{\nu}^{(2)}(z;q)\right)'\right| &= \left|1 + \sum_{n\geq 1} \frac{(-1)^n (n+1) q^{n(n+\nu)}}{4^n (q;q)_n (q^{\nu+1};q)_n} z^n\right| \\
&\leq 1 + \sum_{n\geq 1} \frac{(n+1) q^{n(n+\nu)}}{4^n (q;q)_n (q^{\nu+1};q)_n} \\
&\leq 1 + \sum_{n\geq 1} (n+1) \left(\frac{q^{\nu}}{4(1-q)(1-q^{\nu})}\right)^n \\
&= \left(\frac{4(1-q)(1-q^{\nu})}{4(1-q)(1-q^{\nu})-q^{\nu}}\right)^2.
\end{aligned}$$

Thus, the inequalities (11) and (12) are proved. □

Lemma 2.2. *Let $q \in (0,1), \nu > -1$ and $(1-q)(1-q^{\nu}) > \sqrt{q}$. Then the function $h_{\nu}^{(3)}(z;q)$ satisfies the inequalities*

$$\left|h_{\nu}^{(3)}(z;q)\right| \leq \frac{(1-q)(1-q^{\nu})}{(1-q)(1-q^{\nu})-\sqrt{q}}, \tag{13}$$

and

$$\left|\left(h_{\nu}^{(3)}(z;q)\right)'\right| \leq \left(\frac{(1-q)(1-q^{\nu})}{(1-q)(1-q^{\nu})-\sqrt{q}}\right)^2 \tag{14}$$

for $z \in \mathbb{U}$.

Proof. By using the inequalities (7), (8), (9) and the well-known triangle inequality which is given by (10) for $z \in \mathbb{U}$, we have

$$\begin{aligned}\left|h_\nu^{(3)}(z;q)\right| &= \left|z + \sum_{n\geq 1} \frac{(-1)^n q^{\frac{1}{2}n(n+1)}}{(q;q)_n (q^{\nu+1};q)_n} z^{n+1}\right| \\ &\leq 1 + \sum_{n\geq 1} \frac{q^{\frac{1}{2}n}}{(1-q)^n (1-q^\nu)^n} \\ &\leq 1 + \frac{\sqrt{q}}{(1-q)(1-q^\nu)} \sum_{n\geq 1} \left(\frac{\sqrt{q}}{(1-q)(1-q^\nu)}\right)^{n-1} \\ &= \frac{(1-q)(1-q^\nu)}{(1-q)(1-q^\nu)-\sqrt{q}}\end{aligned}$$

and

$$\begin{aligned}\left|\left(h_\nu^{(3)}(z;q)\right)'\right| &= \left|1 + \sum_{n\geq 1} \frac{(-1)^n (n+1) q^{\frac{1}{2}n(n+1)}}{(q;q)_n (q^{\nu+1};q)_n} z^n\right| \\ &\leq 1 + \sum_{n\geq 1} (n+1) \frac{q^{\frac{1}{2}n}}{(1-q)^n (1-q^\nu)^n} \\ &\leq 1 + \frac{\sqrt{q}}{(1-q)(1-q^\nu)} \sum_{n\geq 1} (n+1) \left(\frac{\sqrt{q}}{(1-q)(1-q^\nu)}\right)^{n-1} \\ &= \left(\frac{(1-q)(1-q^\nu)}{(1-q)(1-q^\nu)-\sqrt{q}}\right)^2.\end{aligned}$$

So, the inequalities (13) and (14) are proved. □

Let $w(z)$ denote an analytic function in $\mathbb{U}$. The following well-known result is important for our main results:

$$\Re\left\{\frac{1+w(z)}{1-w(z)}\right\} > 0, \text{ if and only if } |w(z)| < 1, z \in \mathbb{U}.$$

Theorem 2.3. *Let $\nu > -1, q \in (0,1)$, the function $h_\nu^{(2)} : \mathbb{U} \to \mathbb{C}$ be defined by (4) and its sequences of partial sums by $(h_\nu^{(2)})_m(z;q) = z + \sum_{n=1}^m K_n z^{n+1}$. If the inequality $2(1-q)(1-q^\nu) \geq q^\nu$ is valid, then the following inequalities hold true for $z \in \mathbb{U}$:*

$$\Re\left\{\frac{h_\nu^{(2)}(z;q)}{(h_\nu^{(2)})_m(z;q)}\right\} \geq \frac{4(1-q)(1-q^\nu)-2q^\nu}{4(1-q)(1-q^\nu)-q^\nu}, \tag{15}$$

$$\Re\left\{\frac{(h_\nu^{(2)})_m(z;q)}{h_\nu^{(2)}(z;q)}\right\} \geq \frac{4(1-q)(1-q^\nu)-q^\nu}{4(1-q)(1-q^\nu)}. \tag{16}$$

Proof. From the inequality (11) we have that

$$1+\sum_{n\geq 1}|K_n| \leq \frac{4(1-q)(1-q^{\nu})}{4(1-q)(1-q^{\nu})-q^{\nu}}. \tag{17}$$

The inequality (17) is equivalent to

$$\frac{4(1-q)(1-q^{\nu})-q^{\nu}}{q^{\nu}}\sum_{n\geq 1}|K_n| \leq 1. \tag{18}$$

In order to prove the inequality (15), we consider the function $w(z)$ defined by

$$\frac{1+w(z)}{1-w(z)} = \frac{4(1-q)(1-q^{\nu})-q^{\nu}}{q^{\nu}}\left\{\frac{h_{\nu}^{(2)}(z;q)}{(h_{\nu}^{(2)})_m(z;q)} - \frac{4(1-q)(1-q^{\nu})-2q^{\nu}}{4(1-q)(1-q^{\nu})-q^{\nu}}\right\}$$

which is equivalent to

$$\frac{1+w(z)}{1-w(z)} = \frac{1+\sum_{n=1}^{m}K_n z^n + \frac{4(1-q)(1-q^{\nu})-q^{\nu}}{q^{\nu}}\sum_{n=m+1}^{\infty}K_n z^n}{1+\sum_{n=1}^{m}K_n z^n}. \tag{19}$$

By using the equality (19) we get

$$w(z) = \frac{\frac{4(1-q)(1-q^{\nu})-q^{\nu}}{q^{\nu}}\sum_{n=m+1}^{\infty}K_n z^n}{2+2\sum_{n=1}^{m}K_n z^n + \frac{4(1-q)(1-q^{\nu})-q^{\nu}}{q^{\nu}}\sum_{n=m+1}^{\infty}K_n z^n}$$

and

$$|w(z)| \leq \frac{\frac{4(1-q)(1-q^{\nu})-q^{\nu}}{q^{\nu}}\sum_{n=m+1}^{\infty}|K_n|}{2-2\sum_{n=1}^{m}|K_n| - \frac{4(1-q)(1-q^{\nu})-q^{\nu}}{q^{\nu}}\sum_{n=m+1}^{\infty}|K_n|}.$$

The inequality

$$\sum_{n=1}^{m}|K_n| + \frac{4(1-q)(1-q^{\nu})-q^{\nu}}{q^{\nu}}\sum_{n=m+1}^{\infty}|K_n| \leq 1 \tag{20}$$

implies that $|w(z)| \leq 1$. It suffices to show that the left hand side of (20) is bounded above by

$$\frac{4(1-q)(1-q^{\nu})-q^{\nu}}{q^{\nu}}\sum_{n\geq 1}|K_n|,$$

which is equivalent to

$$\frac{4(1-q)(1-q^{\nu})-2q^{\nu}}{q^{\nu}}\sum_{n\geq 1}|K_n| \geq 0.$$

The last inequality holds true for $2(1-q)(1-q^{\nu}) \geq q^{\nu}$.

The proof of the result (16) would run parallel to those of the result (15). Now, consider the function $p(z)$ given by

$$\frac{1+p(z)}{1-p(z)} = \left(1+\frac{4(1-q)(1-q^{\nu})-q^{\nu}}{q^{\nu}}\right)\left\{\frac{(h_{\nu}^{(2)})_m(z;q)}{h_{\nu}^{(2)}(z;q)} - \frac{4(1-q)(1-q^{\nu})-q^{\nu}}{4(1-q)(1-q^{\nu})}\right\}.$$

Then, from the last equality we get

$$p(z)=\frac{-\frac{4(1-q)(1-q^{\nu})}{q^{\nu}}\sum_{n=m+1}^{\infty}K_nz^n}{2+2\sum_{n=1}^{m}K_nz^n-\frac{4(1-q)(1-q^{\nu})}{q^{\nu}}\sum_{n=m+1}^{\infty}K_nz^n}$$

and

$$|p(z)|\leq\frac{\frac{4(1-q)(1-q^{\nu})}{q^{\nu}}\sum_{n=m+1}^{\infty}|K_n|}{2-2\sum_{n=1}^{m}|K_n|-\frac{4(1-q)(1-q^{\nu})}{q^{\nu}}\sum_{n=m+1}^{\infty}|K_n|}.$$

The inequality

$$\sum_{n=1}^{m}|K_n|+\frac{4(1-q)(1-q^{\nu})}{q^{\nu}}\sum_{n=m+1}^{\infty}|K_n|\leq 1 \tag{21}$$

implies that $|p(z)|\leq 1$. Since the left hand side of (21) is bounded above by

$$\frac{4(1-q)(1-q^{\nu})-q^{\nu}}{q^{\nu}}\sum_{n=1}^{m}|K_n|\geq 0$$

the proof is completed. □

Theorem 2.4. *Let $\nu>-1, q\in(0,1)$, the function $h_{\nu}^{(2)}:\mathbb{U}\to\mathbb{C}$ be defined by (4) and its sequences of partial sums by $(h_{\nu}^{(2)})_m(z;q)=z+\sum_{n=1}^{m}K_nz^{n+1}$. If the inequality $(1-q)(1-q^{\nu})\geq q^{\nu}$ is valid, then the following inequalities hold true for $z\in\mathbb{U}$:*

$$\Re\left\{\frac{\left(h_{\nu}^{(2)}(z;q)\right)'}{\left((h_{\nu}^{(2)})_m(z;q)\right)'}\right\}\geq\frac{16(1-q)(1-q^{\nu})\left((1-q)(1-q^{\nu})-q^{\nu}\right)+2q^{2\nu}}{8(1-q)(1-q^{\nu})q^{\nu}-q^{2\nu}}, \tag{22}$$

$$\Re\left\{\frac{\left((h_{\nu}^{(2)})_m(z;q)\right)'}{\left(h_{\nu}^{(2)}(z;q)\right)'}\right\}\geq\frac{\left(4(1-q)(1-q^{\nu})-q^{\nu}\right)^2}{8(1-q)(1-q^{\nu})q^{\nu}-q^{2\nu}}. \tag{23}$$

Proof. From the inequality (12) we have that

$$1+\sum_{n\geq 1}(n+1)|K_n|\leq\left(\frac{4(1-q)(1-q^{\nu})}{4(1-q)(1-q^{\nu})-q^{\nu}}\right)^2. \tag{24}$$

The inequality (24) is equivalent to

$$\frac{\left(4(1-q)(1-q^{\nu})-q^{\nu}\right)^2}{8(1-q)(1-q^{\nu})q^{\nu}-q^{2\nu}}\sum_{n\geq 1}(n+1)|K_n|\leq 1. \tag{25}$$

In order to prove the inequality (22), we consider the function $h(z)$ defined by

$$\frac{1+h(z)}{1-h(z)}=\frac{(4(1-q)(1-q^{\nu})-q^{\nu})^2}{8(1-q)(1-q^{\nu})q^{\nu}-q^{2\nu}}\left\{\frac{\left(h_{\nu}^{(2)}(z;q)\right)'}{\left((h_{\nu}^{(2)})_m(z;q)\right)'}-\delta\right\},$$

where $\delta=\frac{16(1-q)(1-q^{\nu})((1-q)(1-q^{\nu})-q^{\nu})+2q^{2\nu}}{8(1-q)(1-q^{\nu})q^{\nu}-q^{2\nu}}$. The last equality is equivalent to

(26)
$$\frac{1+h(z)}{1-h(z)}=\frac{1+\sum_{n=1}^{m}(n+1)K_nz^n+\frac{(4(1-q)(1-q^{\nu})-q^{\nu})^2}{8(1-q)(1-q^{\nu})q^{\nu}-q^{2\nu}}\sum_{n=m+1}^{\infty}(n+1)K_nz^n}{1+\sum_{n=1}^{m}(n+1)K_nz^n}.$$

By using the equality (26) we get

$$h(z)=\frac{\frac{(4(1-q)(1-q^{\nu})-q^{\nu})^2}{8(1-q)(1-q^{\nu})q^{\nu}-q^{2\nu}}\sum_{n=m+1}^{\infty}(n+1)K_nz^n}{2+2\sum_{n=1}^{m}(n+1)K_nz^n+\frac{(4(1-q)(1-q^{\nu})-q^{\nu})^2}{8(1-q)(1-q^{\nu})q^{\nu}-q^{2\nu}}\sum_{n=m+1}^{\infty}(n+1)K_nz^n}$$

and

$$|h(z)|\leq\frac{\frac{(4(1-q)(1-q^{\nu})-q^{\nu})^2}{8(1-q)(1-q^{\nu})q^{\nu}-q^{2\nu}}\sum_{n=m+1}^{\infty}(n+1)|K_n|}{2-2\sum_{n=1}^{m}(n+1)|K_n|-\frac{(4(1-q)(1-q^{\nu})-q^{\nu})^2}{8(1-q)(1-q^{\nu})q^{\nu}-q^{2\nu}}\sum_{n=m+1}^{\infty}(n+1)|K_n|}.$$

The inequality

$$\sum_{n=1}^{m}(n+1)|K_n|+\frac{(4(1-q)(1-q^{\nu})-q^{\nu})^2}{8(1-q)(1-q^{\nu})q^{\nu}-q^{2\nu}}\sum_{n=m+1}^{\infty}(n+1)|K_n|\leq 1 \tag{27}$$

implies that $|h(z)|\leq 1$. It suffices to show that the left hand side of (27) is bounded above by

$$\frac{(4(1-q)(1-q^{\nu})-q^{\nu})^2}{8(1-q)(1-q^{\nu})q^{\nu}-q^{2\nu}}\sum_{n\geq 1}(n+1)|K_n|,$$

which is equivalent to

$$\delta\sum_{n\geq 1}(n+1)|K_n|\geq 0.$$

Thus, the result (22) is proved.

To prove the result (23), consider the function $k(z)$ defined by

$$\frac{1+k(z)}{1-k(z)}=\left\{1+\frac{(4(1-q)(1-q^{\nu})-q^{\nu})^2}{8(1-q)(1-q^{\nu})q^{\nu}-q^{2\nu}}\right\}\left\{\frac{\left(h_{\nu}^{(2)}(z;q)\right)'}{\left((h_{\nu}^{(2)})_m(z;q)\right)'}-\frac{(4(1-q)(1-q^{\nu})-q^{\nu})^2}{8(1-q)(1-q^{\nu})q^{\nu}-q^{2\nu}}\right\}.$$

The last equality is equivalent to

(28)
$$\frac{1+k(z)}{1-k(z)}=\frac{1+\sum_{n=1}^{m}(n+1)K_nz^n-\frac{(4(1-q)(1-q^{\nu})-q^{\nu})^2}{8(1-q)(1-q^{\nu})q^{\nu}-q^{2\nu}}\sum_{n=m+1}^{\infty}(n+1)K_nz^n}{1+\sum_{n\geq 1}(n+1)K_nz^n}.$$

From the equality (28) we have

$$k(z)=\frac{-\frac{16(1-q)^2(1-q^\nu)^2}{8(1-q)(1-q^\nu)q^\nu-q^{2\nu}}\sum_{n=m+1}^{\infty}(n+1)K_nz^n}{2+2\sum_{n=1}^{m}(n+1)K_nz^n-\delta\sum_{n=m+1}^{\infty}(n+1)K_nz^n}$$

and

$$|k(z)|\leq\frac{\frac{16(1-q)^2(1-q^\nu)^2}{8(1-q)(1-q^\nu)q^\nu-q^{2\nu}}\sum_{n=m+1}^{\infty}(n+1)\left|K_n\right|}{2-2\sum_{n=1}^{m}(n+1)\left|K_n\right|-\delta\sum_{n=m+1}^{\infty}(n+1)\left|K_n\right|}.$$

The inequality

$$\sum_{n=1}^{m}(n+1)\left|K_n\right|+\frac{(4(1-q)(1-q^\nu)-q^\nu)^2}{8(1-q)(1-q^\nu)q^\nu-q^{2\nu}}\sum_{n=m+1}^{\infty}(n+1)\left|K_n\right|\leq 1 \tag{29}$$

implies that $|k(z)|\leq 1$. Since the left hand side of (29) is bounded above by

$$\frac{(4(1-q)(1-q^\nu)-q^\nu)^2}{8(1-q)(1-q^\nu)q^\nu-q^{2\nu}}\sum_{n\geq 1}(n+1)\left|K_n\right|,$$

which is equivalent to

$$\delta\sum_{n=m+1}^{\infty}(n+1)\left|K_n\right|\geq 0,$$

the proof of result (23) is completed. □

Theorem 2.5. *Let $\nu>-1, q\in(0,1)$, the function $h_\nu^{(3)}:\mathbb{U}\to\mathbb{C}$ be defined by (5) and its sequences of partial sums by $(h_\nu^{(3)})_m(z;q)=z+\sum_{n=1}^{m}T_nz^{n+1}$. If the inequality $(1-q)(1-q^\nu)\geq 2\sqrt{q}$ is valid, then the next two inequalities are valid for $z\in\mathbb{U}$:*

$$\Re\left\{\frac{h_\nu^{(3)}(z;q)}{(h_\nu^{(3)})_m(z;q)}\right\}\geq\frac{(1-q)(1-q^\nu)-2\sqrt{q}}{\sqrt{q}}, \tag{30}$$

$$\Re\left\{\frac{(h_\nu^{(3)})_m(z;q)}{h_\nu^{(3)}(z;q)}\right\}\geq\frac{(1-q)(1-q^\nu)-\sqrt{q}}{\sqrt{q}}. \tag{31}$$

Proof. From the inequality (13) we have that

$$1+\sum_{n\geq 1}\left|T_n\right|\leq\frac{(1-q)(1-q^\nu)}{(1-q)(1-q^\nu)-\sqrt{q}}. \tag{32}$$

The inequalitiy (32) is equivalent to

$$\frac{(1-q)(1-q^\nu)-\sqrt{q}}{\sqrt{q}}\sum_{n\geq 1}\left|T_n\right|\leq 1. \tag{33}$$

In order to prove the inequality (30), we consider the function $\phi(z)$ defined by

$$\frac{1+\phi(z)}{1-\phi(z)}=\frac{(1-q)(1-q^\nu)-\sqrt{q}}{\sqrt{q}}\left\{\frac{h_\nu^{(3)}(z;q)}{(h_\nu^{(3)})_m(z;q)}-\frac{(1-q)(1-q^\nu)-2\sqrt{q}}{\sqrt{q}}\right\},$$

which is equivalent to

$$\frac{1+\phi(z)}{1-\phi(z)}=\frac{1+\sum_{n=1}^{m}T_nz^n+\frac{(1-q)(1-q^\nu)-\sqrt{q}}{\sqrt{q}}\sum_{n=m+1}^{\infty}T_nz^n}{1+\sum_{n=1}^{m}T_nz^n}. \tag{34}$$

From the equality (34) we obtain

$$\phi(z)=\frac{\frac{(1-q)(1-q^\nu)-\sqrt{q}}{\sqrt{q}}\sum_{n=m+1}^{\infty}T_nz^n}{2+2\sum_{n=1}^{m}T_nz^n+\frac{(1-q)(1-q^\nu)-\sqrt{q}}{\sqrt{q}}\sum_{n=m+1}^{\infty}T_nz^n}$$

and

$$|\phi(z)|\leq\frac{\frac{(1-q)(1-q^\nu)-\sqrt{q}}{\sqrt{q}}\sum_{n=m+1}^{\infty}|T_n|}{2-2\sum_{n=1}^{m}|T_n|-\frac{(1-q)(1-q^\nu)-\sqrt{q}}{\sqrt{q}}\sum_{n=m+1}^{\infty}|T_n|}.$$

The inequality

$$\sum_{n=1}^{m}|T_n|+\frac{(1-q)(1-q^\nu)-\sqrt{q}}{\sqrt{q}}\sum_{n=m+1}^{\infty}|T_n|\leq 1 \tag{35}$$

implies that $|\phi(z)|\leq 1$. It suffices to show that the left hand side of (35) is bounded above by

$$\frac{(1-q)(1-q^\nu)-\sqrt{q}}{\sqrt{q}}\sum_{n\geq 1}|T_n|,$$

which is equivalent to

$$\frac{(1-q)(1-q^\nu)-2\sqrt{q}}{\sqrt{q}}\sum_{n=1}^{m}|T_n|\geq 0.$$

The last inequality holds true for $(1-q)(1-q^\nu)\geq 2\sqrt{q}$.

In order to prove the result (31), we consider the function $\varphi(z)$ given by

$$\frac{1+\varphi(z)}{1-\varphi(z)}=\left(1+\frac{(1-q)(1-q^\nu)-\sqrt{q}}{\sqrt{q}}\right)\left\{\frac{(h_\nu^{(3)})_m(z;q)}{h_\nu^{(3)}(z;q)}-\frac{(1-q)(1-q^\nu)-\sqrt{q}}{\sqrt{q}}\right\}.$$

Then, from the last equality we get

$$\varphi(z)=\frac{-\frac{(1-q)(1-q^\nu)-\sqrt{q}}{\sqrt{q}}\sum_{n=m+1}^{\infty}T_nz^n}{2+2\sum_{n=1}^{m}T_nz^n-\frac{(1-q)(1-q^\nu)-\sqrt{q}}{\sqrt{q}}\sum_{n=m+1}^{\infty}T_nz^n}$$

and

$$|\varphi(z)|\leq\frac{\frac{(1-q)(1-q^\nu)-\sqrt{q}}{\sqrt{q}}\sum_{n=m+1}^{\infty}|T_n|}{2-2\sum_{n=1}^{m}|T_n|-\frac{(1-q)(1-q^\nu)-\sqrt{q}}{\sqrt{q}}\sum_{n=m+1}^{\infty}|T_n|}.$$

The inequality

$$\sum_{n=1}^{m}|T_n|+\frac{(1-q)(1-q^\nu)-\sqrt{q}}{\sqrt{q}}\sum_{n=m+1}^{\infty}|T_n|\leq 1 \tag{36}$$

implies that $|\varphi(z)| \leq 1$. Since the left hand side of (36) is bounded above by

$$\frac{(1-q)(1-q^{\nu})-\sqrt{q}}{\sqrt{q}}\sum_{n\geq 1}|T_n|,$$

which is equivalent to

$$\frac{(1-q)(1-q^{\nu})-2\sqrt{q}}{\sqrt{q}}\sum_{n=1}^{m}|T_n| \geq 0.$$

This completes the proof of the theorem. ◻

Theorem 2.6. *Let* $\nu > -1, q \in (0,1)$, *the function* $h_{\nu}^{(3)} : \mathbb{U} \to \mathbb{C}$ *be defined by* (5) *and its sequences of partial sums by* $(h_{\nu}^{(3)})_m(z;q) = z + \sum_{n=1}^{m} T_n z^{n+1}$. *If the inequality* $(1-q)(1-q^{\nu}) \geq 4\sqrt{q}$, *then the next two inequalities are valid for* $z \in \mathbb{U}$:

$$\Re\left\{\frac{\left(h_{\nu}^{(3)}(z;q)\right)'}{\left((h_{\nu}^{(3)})_m(z;q)\right)'}\right\} \geq \frac{(1-q)^2(1-q^{\nu})^2-4(1-q)(1-q^{\nu})\sqrt{q}+2q}{2(1-q)(1-q^{\nu})\sqrt{q}-q}, \tag{37}$$

$$\Re\left\{\frac{\left((h_{\nu}^{(3)})_m(z;q)\right)'}{\left(h_{\nu}^{(3)}(z;q)\right)'}\right\} \geq \frac{\left((1-q)(1-q^{\nu})-\sqrt{q}\right)^2}{2(1-q)(1-q^{\nu})\sqrt{q}-q}. \tag{38}$$

Proof. From the inequality (14) we have that

$$1+\sum_{n\geq 1}(n+1)|T_n| \leq \left(\frac{(1-q)(1-q^{\nu})}{(1-q)(1-q^{\nu})-\sqrt{q}}\right)^2. \tag{39}$$

The inequality (39) is equivalent to

$$\frac{\left((1-q)(1-q^{\nu})-\sqrt{q}\right)^2}{2(1-q)(1-q^{\nu})\sqrt{q}-q}\sum_{n\geq 1}(n+1)|T_n| \leq 1. \tag{40}$$

In order to prove the inequality (37), we consider the function $\psi(z)$ defined by

$$\frac{1+\psi(z)}{1-\psi(z)} = \frac{\left((1-q)(1-q^{\nu})-\sqrt{q}\right)^2}{2(1-q)(1-q^{\nu})\sqrt{q}-q}\left\{\frac{\left(h_{\nu}^{(3)}(z;q)\right)'}{\left((h_{\nu}^{(3)})_m(z;q)\right)'}-\lambda\right\},$$

where $\lambda = \frac{(1-q)^2(1-q^{\nu})^2-4(1-q)(1-q^{\nu})\sqrt{q}+2q}{2(1-q)(1-q^{\nu})\sqrt{q}-q}$. The last equality is equivalent to

$$\frac{1+\psi(z)}{1-\psi(z)} = \frac{1+\sum_{n=1}^{m}(n+1)T_n z^n + \frac{\left((1-q)(1-q^{\nu})-\sqrt{q}\right)^2}{2(1-q)(1-q^{\nu})\sqrt{q}-q}\sum_{n=m+1}^{\infty}(n+1)T_n z^n}{1+\sum_{n=1}^{m}(n+1)T_n z^n}. \tag{41}$$

By using the equality (41) we get

$$\psi(z)=\frac{\frac{\left((1-q)(1-q^{\nu})-\sqrt{q}\right)^2}{2(1-q)(1-q^{\nu})\sqrt{q}-q}\sum_{n=m+1}^{\infty}(n+1)\,T_n z^n}{2+2\sum_{n=1}^{m}(n+1)\,T_n z^n+\frac{\left((1-q)(1-q^{\nu})-\sqrt{q}\right)^2}{2(1-q)(1-q^{\nu})\sqrt{q}-q}\sum_{n=m+1}^{\infty}(n+1)\,T_n z^n}$$

and

$$|\psi(z)|\leq\frac{\frac{\left((1-q)(1-q^{\nu})-\sqrt{q}\right)^2}{2(1-q)(1-q^{\nu})\sqrt{q}-q}\sum_{n=m+1}^{\infty}(n+1)\,|T_n|}{2-2\sum_{n=1}^{m}(n+1)\,|T_n|-\frac{\left((1-q)(1-q^{\nu})-\sqrt{q}\right)^2}{2(1-q)(1-q^{\nu})\sqrt{q}-q}\sum_{n=m+1}^{\infty}(n+1)\,|T_n|}.$$

The inequality

$$\sum_{n=1}^{m}(n+1)\,|T_n|+\frac{\left((1-q)(1-q^{\nu})-\sqrt{q}\right)^2}{2(1-q)(1-q^{\nu})\sqrt{q}-q}\sum_{n=m+1}^{\infty}(n+1)\,|T_n|\leq 1 \tag{42}$$

implies that $|\psi(z)|\leq 1$. It suffices to show that the left hand side of (42) is bounded above by

$$\frac{\left((1-q)(1-q^{\nu})-\sqrt{q}\right)^2}{2(1-q)(1-q^{\nu})\sqrt{q}-q}\sum_{n\geq 1}(n+1)\,|T_n|\,,$$

which is equivalent to

$$\lambda\sum_{n=1}^{m}(n+1)\,|T_n|\geq 0.$$

Thus, the result (37) is proved.

To prove the result (38), consider the function $\rho(z)$ defined by

$$\frac{1+\rho(z)}{1-\rho(z)}=\left\{1+\frac{\left((1-q)(1-q^{\nu})-\sqrt{q}\right)^2}{2(1-q)(1-q^{\nu})\sqrt{q}-q}\right\}\left\{\frac{\left(h_{\nu}^{(3)}(z;q)\right)'}{\left((h_{\nu}^{(3)})_m(z;q)\right)'}-\frac{\left((1-q)(1-q^{\nu})-\sqrt{q}\right)^2}{2(1-q)(1-q^{\nu})\sqrt{q}-q}\right\}.$$

The last equality is equivalent to

(43)

$$\frac{1+\rho(z)}{1-\rho(z)}=\frac{1+\sum_{n=1}^{m}(n+1)T_n z^n-\frac{\left((1-q)(1-q^{\nu})-\sqrt{q}\right)^2}{2(1-q)(1-q^{\nu})\sqrt{q}-q}\sum_{n=m+1}^{\infty}(n+1)T_n z^n}{1+\sum_{n=1}^{\infty}(n+1)T_n z^n}.$$

From the equality (43) we get

$$\rho(z)=\frac{-\frac{\left((1-q)(1-q^{\nu})-\sqrt{q}\right)^2}{2(1-q)(1-q^{\nu})\sqrt{q}-q}\sum_{n=m+1}^{\infty}(n+1)T_n z^n}{2+2\sum_{n=1}^{m}(n+1)T_n z^n-\frac{\left((1-q)(1-q^{\nu})-\sqrt{q}\right)^2}{2(1-q)(1-q^{\nu})\sqrt{q}-q}\sum_{n=m+1}^{\infty}(n+1)T_n z^n}$$

and

$$|\rho(z)|\leq\frac{\frac{\left((1-q)(1-q^{\nu})-\sqrt{q}\right)^2}{2(1-q)(1-q^{\nu})\sqrt{q}-q}\sum_{n=m+1}^{\infty}(n+1)\,|T_n|}{2-2\sum_{n=1}^{m}(n+1)\,|T_n|-\frac{\left((1-q)(1-q^{\nu})-\sqrt{q}\right)^2}{2(1-q)(1-q^{\nu})\sqrt{q}-q}\sum_{n=m+1}^{\infty}(n+1)\,|T_n|}.$$

The inequality

$$\sum_{n=1}^{m}(n+1)\left|T_n\right| + \frac{\left((1-q)(1-q^{\nu})-\sqrt{q}\right)^2}{2(1-q)(1-q^{\nu})\sqrt{q}-q}\sum_{n=m+1}^{\infty}(n+1)\left|T_n\right| \leq 1 \tag{44}$$

implies that $|\rho(z)| \leq 1$. Since the left hand side of (44) is bounded above by

$$\frac{\left((1-q)(1-q^{\nu})-\sqrt{q}\right)^2}{2(1-q)(1-q^{\nu})\sqrt{q}-q}\sum_{n\geq 1}(n+1)\left|T_n\right|,$$

which is equivalent to

$$\frac{(1-q)(1-q^{\nu})\left((1-q)(1-q^{\nu})-4\sqrt{q}\right)+2q}{2(1-q)(1-q^{\nu})\sqrt{q}-q}\sum_{n=1}^{m}(n+1)\left|T_n\right| \geq 0,$$

the proof of result (38) is completed. □

References

[1] İ. Aktaş and Á. Baricz, *Bounds for radii of starlikeness of some q-Bessel functions*, Results Math. **72** (2017), no. 1-2, 947–963.

[2] İ. Aktaş and H. Orhan, *Distortion bounds for a new subclass of analytic functions and their partial sums*, Bull. Transilv. Univ. Braşov Ser. III **8(57)** (2015), no. 2, 1–11 pp.

[3] ———, *Partial sums of normalized Dini functions*, J. Class. Anal. **9** (2016), no. 2, 127–135.

[4] M. H. Annaby and Z. S. Mansour, *q-fractional calculus and equations*, Lecture Notes in Mathematics, **2056**, Springer, Heidelberg, 2012.

[5] Á. Baricz, D. K. Dimitrov, and I. Mező *Radii of starlikeness and convexity of some q-Bessel functions*, J. Math. Anal. Appl. **435** (2016), no. 1, 968–985.

[6] Á. Baricz, D. K. Dimitrov, H. Orhan, and N. Yağmur, *Radii of starlikeness of some special functions*, Proc. Amer. Math. Soc. **144** (2016), no. 8, 3355–3367.

[7] M. Çağlar and E. Deniz, *Partial sums of the normalized Lommel functions*, Math. Inequal. Appl. **18** (2015), no. 3, 1189–1199.

[8] H. Orhan and N. Yağmur, *Partial sums of generalized Bessel functions*, J. Math. Inequal. **8** (2014), no. 4, 863–877.

[9] S. Owa, H. M. Srivastava, and N. Saito, *Partial sums of certain classes of analytic functions*, Int. J. Comput. Math. **81** (2004), no. 10, 1239–1256.

[10] D. Răducanu, *On partial sums of normalized Mittag-Leffler functions*, An. Şt. Univ. Ovidius Constanta, **25** (2017), no. 2, 123–133.

[11] T. Sheil-Small, *A note on the partial sums of convex schlicht functions*, Bull. Lond. Math. Soc. **2** (1970), 165–168.

[12] H. Silverman, *Partial sums of starlike and convex functions*, J. Math. Anal. Appl. **209** (1997), no. 1, 221–227.

[13] E. M. Silvia, *On partial sums of convex functions of order α*, Houston J. Math. **11** (1985), no. 3, 397–404.

İbrahim Aktaş
Department of Mathematical Engineering
Gümüşhane University
Gümüşhane 29100, Turkey
E-mail address: aktasibrahim38@gmail.com

Halİt Orhan
Department of Mathematics
Atatürk University
Erzurum 25240, Turkey
E-mail address: orhanhalit607@gmail.com