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NUMERICAL EXPERIMENTS OF THE LEGENDRE POLYNOMIAL BY GENERALIZED DIFFERENTIAL TRANSFORM METHOD FOR SOLVING THE LAPLACE EQUATION

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Abstract. Finding a solution for the Legendre equation is difficult. Especially if it is as a part of the Laplace equation solving in the electric fields. In this paper, first a problem of the generalized differential transform method (GDTM) is solved by the Sturm-Liouville equation, then the Legendre equation is solved by using it. To continue, the approximate solution is compared with the nth-degree Legendre polynomial for obtaining the inner and outer potential of a sphere. This approximate is more accurate than the previous solutions, and is closer to an ideal potential in the intervals.

1. Introduction

In recent years, the study of systems and fractional equations, with various methods, has helped a lot to improve physics and engineering [8, 13, 15, 16]. For example, Grunwald-Letnikov, Riemann-Liouville, and Caputo fractional derivatives have been introduced in [8, 13, 15]. The Laplace equation is one of the most important PDEs in Physics and Electronic [2, 6, 7, 9, 22]. It represents the equilibrium. For example, when the heat transfer in a body reaches the equilibrium, solving of the Laplace equation shows the temperature in different places. Also, the Laplace equation is used to experiment density of chemical material in equilibrium and in conditions of electric and gravitational fields. It is solved by using Legendre polynomials [2, 7, 9]. To continue, two practical examples of this equation are described. If $u(x)$ be the density of chemical material then its output flux in each region V is zero

$$\int_{\partial V} F\overrightarrow{n}\,ds = 0, \tag{1}$$

where F shows the flux which is proper to the gradient of function u.

$$F = -\alpha\nabla u, \quad \alpha > 0. \tag{2}$$

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According to the divergence theorem, we have

$$\int_{\partial V} F\overrightarrow{n}ds = \int_V div(F)dx = 0. \tag{3}$$

The region V is arbitrary. Hence, the Laplace equation is written in three dimensional space as Cartesian coordinates

$$div(F) = 0, \tag{4}$$

$$div(\nabla u) = 0, \tag{5}$$

$$\nabla^2 u = 0, \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{6}$$

For the second example, we suppose that E, ρ and ϵ_0 are field, density and the permittivity of vacuum, respectively. Hence, the mathematical model of mentioned values is as follow

$$\nabla.E = \frac{\rho}{\epsilon_0}, \tag{7}$$

$$\nabla \times E = 0. \tag{8}$$

When the nucleus of the field is zero, then according to the Liouville theorem, the field has the gradient of a scalar function as u. This function is called potential, which the electrostatic field is its gradient.

$$E = -\nabla u. \tag{9}$$

According to (7) and (9), we have

$$\nabla^2 u = -\frac{\rho}{\epsilon_0}. \tag{10}$$

Eq. (10) is called the Poisson equation. If there is no electric charge, that is $\rho = 0$, then the Poisson equation to be transformed into the Laplace equation.

$$\nabla^2 u = 0. \tag{11}$$

Using spherical coordinates in (11), we have

$$\nabla^2 u = \frac{1}{r^2}[\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r}) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial u}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}] = 0. \tag{12}$$

One of the methods to solve (12) is the separation of the variables. We suppose that the potential function is followed by

$$u = R(r).P(\theta).Q(\phi), \tag{13}$$

where r, θ and ϕ are radius, the angle between a vector and the z-axis and the angle of vector projection onto x-y plane with the positive x-axis, respectively. Substituting (13) into (12) and by using the direction symmetry condition as a boundary condition, the Eq. (12) is transformed into three ordinary differential equations in which direction the solution is symmetry.

$$\frac{d^2Q}{d\phi^2} = -m^2 Q, \tag{14}$$

$$r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}-n(n+1)R=0, \tag{15}$$

$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{dP}{d\theta})+n(n+1)P=0, \tag{16}$$

where $Q(\phi)$ is constant and $m^2=0$. Also, m and n are parameters for solving differential equations in spherical coordinates. Using a new variable $x=\cos\theta$, Eq. (16) is written as follow

$$\sin^2\theta\frac{d^2P}{dx^2}-2\cos\theta\frac{dP}{dx}+n(n+1)P=0. \tag{17}$$

The general solutions (15) and (17) are as follow

$$R(x)=c_1r^n+c_2r^{-(n+1)}, \tag{18}$$

$$P(x)=c_1P_1+c_2P_2, \tag{19}$$

where

$$P_1(x)=1+\sum_{q=1}^{\infty}(-1)^q\frac{n(n-2)\cdots(n-2q+2)(n+1)\cdots(n+2q-1)}{(2q)!}x^{2q}, \tag{20}$$

$$P_2(x)=x+\sum_{q=1}^{\infty}(-1)^q\frac{(n-1)(n-3)\cdots(n-2q+1)(n+2)\cdots(n+2q)}{(2q+1)!}x^{2q+1}. \tag{21}$$

Also, we can obtain the above term by the Legendre polynomial of degree n, known as Rodrigues' formula [7, 9, 14, 22]. The Laplace equation is solved by complicated and time-consuming methods. In section 3, by using the generalized differential transform, we can obtain better approximate than the previous methods. Solving the ordinary, partial and fractional differential equations is one of the advantages of this method. It obtains approximates of fractional model as well as ordinary and partial differential equations [3, 12, 17, 18, 19].

2. Method

In this section, we explain some definitions and theorems related to the Laplace equation and GDTM.

Theorem 2.1. *The solution of the Laplace equation is unique.*

Theorem 2.2. *The solution of Laplace equation is consistently depended on boundary conditions.*

For more details see [1, 20, 21].

Definition. The Caputo fractional derivative of order α is defined by

$$D^{\alpha}f(x)=\frac{1}{\Gamma(-\alpha+l)}\int_a^x(x-\tau)^{-\alpha+l-1}f^{(l)}(\tau)d\tau, \tag{22}$$

where $l-1<\alpha\leq l, l\in Z^+$. For more details see [8, 13, 15].

Definition. We define the generalized differential transform for the $k-th$ derivative of a function $f(x)$ as follow

$$F_\alpha(k) = \frac{1}{\Gamma(\alpha k+1)}[(D^\alpha)^k f(x)]_{x=x_0}, \tag{23}$$

where $0 < \alpha \leq 1$ and $(D^\alpha)^k = D^\alpha . D^\alpha . \cdots . D^\alpha (k - times)$.

Also, the inverse differential transform of $F_\alpha(k)$ is defined as

$$f(x) = \sum_{k=0}^{\infty} F_\alpha(k)(x-x_0)^{\alpha k}. \tag{24}$$

Substituting (23) into (24) and by using the generalized Taylor's formula [10], we obtain

$$f(x) = \sum_{k=0}^{\infty} F_\alpha(k)(x-x_0)^{\alpha k} = \sum_{k=0}^{\infty} \frac{(x-x_0)^{\alpha k}}{\Gamma(\alpha k+1)}((D^\alpha)^k f)(x_0). \tag{25}$$

Using theorem (4) in [10], we have

$$f(x) \cong \sum_{k=0}^{t} F_\alpha(k)(x-x_0)^{\alpha k}, \tag{26}$$

where t is sufficiently large. The following theorems help us to solve the fractional differential equations.

Theorem 2.3. *If* $f(x) = g(x) \pm h(x)$, *then* $F_\alpha(k) = G_\alpha(k) \pm H_\alpha(k)$, *where* $0 < \alpha \leq 1$.

Theorem 2.4. *If* $f(x) = cg(x)$ *and* $c \in \mathbb{R}$, *then* $F_\alpha(k) = cG_\alpha(k)$, *where* $0 < \alpha \leq 1$.

Theorem 2.5. *If* $f(x) = D^\beta g(x), l-1 < \beta \leq l$ *and the function* $g(x)$ *satisfies the conditions of Theorem 2-5 in* [11], *then* $F_\alpha(k) = \frac{\Gamma(\alpha k+\beta+1)}{\Gamma(\alpha k+1)} G_\alpha(k+\frac{\beta}{\alpha})$, *where* $0 < \alpha \leq 1$.

The proofs may be found in [11].

3. Discussion

In this section, Eq. (17) is solved by using Rodrigues' formula and other methods. Then we obtain the approximates of (17) by the Sturm-Liouville equation and GDTM.

Example 3.1. Figure 1 shows the spherical capacitor consisting of two metallic hemispheres of radius 1 ft separated by a small slit for reasons of isolation, under this condition, the upper hemisphere is kept $110V$ and the lower is grounded. The boundary condition is as follow

$$f(\theta)=\begin{cases}110, & 0\le\theta<\dfrac{\pi}{2},\\ 0, & \dfrac{\pi}{2}<\theta\le\pi.\end{cases}\tag{27}$$

The inner and outer potential of sphere are written as follow, respectively:

$$u_n(r,\theta)=A_n r^n P_n(\cos\theta),\tag{28}$$

$$u_n(r,\theta)=\frac{B_n}{r^{n+1}}P_n(\cos\theta),\tag{29}$$

for $n=0,1,2,\ldots$. $P_n(\cos\phi)$ is the Legendre polynomials. We consider a series of terms Eq.(28)

$$u(r,\theta)=\sum_{n=0}^{\infty}A_n r^n P_n(\cos\theta),\quad r\le R.\tag{30}$$

Since the sphere S is given by $r=R$, the Dirichlet condition satisfies for (30). (see Eq.(9) in Sect. 12.11 in [9]). Hence, we have

$$u(R,\theta)=\sum_{n=0}^{\infty}A_n R^n P_n(\cos\theta)=f(\theta),\tag{31}$$

where (31) is the Fourier-Legendre series of $f(\theta)$. According to Eq.(7) in Sect. 11.9 in [9], we obtain

$$A_n R^n=\frac{2n+1}{2}\int_{-1}^{1}\tilde{f}(\omega)P_n(\omega)d\omega,\tag{32}$$

where $\tilde{f}(\theta)$ denotes $f(\theta)$. We suppose $\omega=\cos\theta$. Since the limits of integration -1 and 1 correspond to $\theta=\pi$ and $\theta=0$, respectively, we can write

$$A_n=\frac{2n+1}{2R^n}\int_0^{\pi}f(\theta)P_n(\cos\theta)\sin\theta\;d\theta,\quad n=0,1,2,\ldots.\tag{33}$$

Also, from Eq.(29) we have

$$u(r,\theta)=\sum_{n=0}^{\infty}\frac{B_n}{r^{n+1}}P_n(\cos\theta),\quad r\ge R.\tag{34}$$

According to (8), (9), and (10) in Sect. 12.11 in [9], we obtain

$$B_n=(\frac{2n+1}{2})R^{n+1}\int_0^{\pi}f(\theta)P_n(\cos\theta)\sin\theta\;d\theta,\quad n=0,1,2,\ldots.\tag{35}$$

Since $R=1$, we can write Eq.(33) as follow

$$A_n=(\frac{2n+1}{2})110\int_0^{\frac{\pi}{2}}P_n(\cos\theta)\sin\theta\;d\theta=(\frac{2n+1}{2})110\int_0^{1}P_n(\omega)d\omega.\tag{36}$$

According to Sect. 5.2 in [9], we obtain

$$A_n = 55(2n+1)\sum_{m=0}^{M} \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}\int_0^1 \omega^{n-2m}\, d\omega, \tag{37}$$

where $M = \frac{n}{2}$ for even n and $M = \frac{n-1}{2}$ for odd n. For $n = 0, 1, 2, \dots$ we have

$$A_0 = 55,\ A_1 = \frac{165}{2},\ A_2 = 0,\ A_3 = -\frac{385}{8}, \dots\ . \tag{38}$$

Substituting (38) into (30), we have

$$u(r,\theta) = 55 + (\frac{165}{2})rP_1(\cos\theta) - (\frac{385}{8})r^3P_3(\cos\theta) + \dots\ . \tag{39}$$

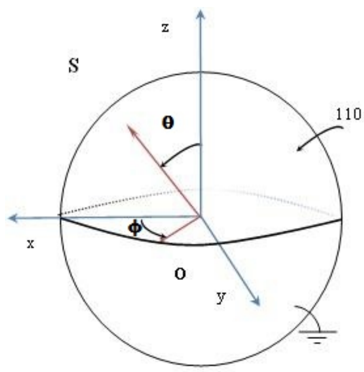

FIGURE 1. The spherical capacitor.

Note that the even coefficients of $A_n, n = 2, 4, 6, \dots$, are zero. Since $R = 1$ then $A_n = B_n$. Hence, the inner and outer potential of sphere are equal and it is as follow

$$\begin{aligned} u(r,\theta) = \frac{55}{r} &+ (\frac{165}{2r^2})P_1(\cos\theta) - (\frac{385}{8r^4})P_3(\cos\theta) + (\frac{605}{16r^6})P_5(\cos\theta) \\ &-(\frac{4125}{128r^8})P_7(\cos\theta) + (\frac{7315}{256r^{10}})P_9(\cos\theta) + \dots\ , \end{aligned} \tag{40}$$

where $P_0, P_1, P_3, \dots$ are the Legendre polynomials of degree n and we can obtain them by using the Rodrigues'forumla

$$P_n(\omega) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n. \tag{41}$$

Therefore, we have

$$P_0(\omega) = 1, P_1(\omega) = \omega, P_3(\omega) = \frac{1}{2}(5\omega^3 - 3\omega), P_5(\omega) = \frac{63}{8}\omega^5 - \frac{35}{4}\omega^3 + \frac{15}{8}\omega,$$

$$P_7(\omega)=\frac{429}{16}\omega^7-\frac{693}{16}\omega^5+\frac{315}{16}\omega^3-\frac{35}{16}\omega,$$

$$P_9(\omega)=\frac{12155}{128}\omega^9-\frac{6435}{32}\omega^7+\frac{9009}{64}\omega^5-\frac{1155}{32}\omega^3+\frac{315}{128}\omega,\ldots. \tag{42}$$

Substituting (42) into (40) and setting $r=1$ and $\omega=\cos(\theta)$ we obtain the potential of sphere by Rodrigues' formula as follow

$$\begin{aligned}u(1,\theta)=55+(\frac{165}{2})\omega-(\frac{385}{8})(\frac{1}{2}(5\omega^3-3\omega))+(\frac{605}{16})(\frac{63}{8}\omega^5-\frac{35}{4}\omega^3+\frac{15}{8}\omega)\\ -(\frac{4125}{128})(\frac{429}{16}\omega^7-\frac{693}{16}\omega^5+\frac{315}{16}\omega^3-\frac{35}{16}\omega)\\ +(\frac{7315}{256})(\frac{12155}{128}\omega^9-\frac{6435}{32}\omega^7+\frac{9009}{64}\omega^5-\frac{1155}{32}\omega^3+\frac{315}{128}\omega).\end{aligned} \tag{43}$$

By setting the generalized differential transform $\sin^2\theta$ in the denominator, GDTM is unable to solve of Eq.(17) because this transform is zero in one of the steps. Therefore, we consider the Sturm-Liouville equation to solve the problem.

$$\frac{d}{d\omega}[h(\omega)\frac{dP}{d\omega}]+[i(\omega)+\lambda j(\omega)]P=0, \tag{44}$$

where $i=0, j=1$. We suppose $\omega=\cos\theta$ and $h(\omega)=1-\omega^2$, then Eq.(17) is transformed by the Sturm-Liouville equation as follow

$$h(\omega)\frac{d^2P}{d\omega^2}+\lambda P=0, \quad \lambda=n(n+1). \tag{45}$$

We consider the below initial conditions by using Rodrigues' formula for $n=0,1,3,\ldots,9$, respectively:

$$P_0(1)=1, P_0'(1)=0, \tag{46}$$

$$P_1(1)=1, P_1'(1)=1, \tag{47}$$

$$P_3(1)=1, P_3'(1)=6, \tag{48}$$

$$P_5(1)=1, P_5'(1)=15, \tag{49}$$

$$P_7(1)=1, P_7'(1)=28, \tag{50}$$

$$P_9(1)=1, P_9'(1)=45. \tag{51}$$

We suppose $\alpha=0.5$ and $\beta=2$. According to definition of Caputo fractional derivative and Eq.(23), the generalized differential transform Eq.(45) and (46) are as follow, respectively:

$$P^0_{0.5}(k+2)=0, \tag{52}$$

$$P^0_{0.5}(0)=1, P^0_{0.5}(1)=0. \tag{53}$$

Hence, we have the solution $P_0(\omega)$ up to $O((\omega-1)^0)$

$$P_0(\omega)=1, \tag{54}$$

where $O((\omega-1)^0)$ and $(\omega-1)^0$ are truncation error and the first term of solution series of GDTM, respectively. By using theorem 2.5 for $n = 1, 3, \ldots, 9$ and considering theorems 2.3 and 2.4 to perform the operation of addition, than multiplication λ by P in Eq. (45), respectively, we obtain the generalized differential transform of Eq. (45) and initial conditions (47),(48),(49),(50) and (51) as follow

$$P^1_{0.5}(k+2) = -2\frac{\Gamma(k+1)}{\Gamma(k+3)}P^1_{0.5}(k), \tag{55}$$

$$P^1_{0.5}(0) = 1, P^1_{0.5}(1) = 0, \tag{56}$$

$$P^3_{0.5}(k+2) = -12\frac{\Gamma(k+1)}{\Gamma(k+3)}P^3_{0.5}(k), \tag{57}$$

$$P^3_{0.5}(0) = 1, P^3_{0.5}(1) = 0, \tag{58}$$

$$P^5_{0.5}(k+2) = -20\frac{\Gamma(k+1)}{\Gamma(k+3)}P^5_{0.5}(k), \tag{59}$$

$$P^5_{0.5}(0) = 1, P^5_{0.5}(1) = 0, \tag{60}$$

$$P^7_{0.5}(k+2) = -56\frac{\Gamma(k+1)}{\Gamma(k+3)}P^7_{0.5}(k), \tag{61}$$

$$P^7_{0.5}(0) = 1, P^7_{0.5}(1) = 0, \tag{62}$$

and

$$P^9_{0.5}(k+2) = -90\frac{\Gamma(k+1)}{\Gamma(k+3)}P^9_{0.5}(k), \tag{63}$$

$$P^9_{0.5}(0) = 1, P^9_{0.5}(1) = 0. \tag{64}$$

Considering $k = 0, 1, 2, \ldots$ for each n in Eq. (55),(57),(59), (61), and (63) at once and substituting the above coefficients instead of $F_\alpha(k)$ and $\omega_0 = 1$ (because of initial $\theta = 0$) instead of x_0 in Eq.(26) at the second, we obtain the Legendre polynomials $P_1(\omega), P_3(\omega), \ldots, P_9(\omega)$ as follow

$$P_1(\omega) = 1 - (\frac{1}{h})(\omega-1)^2 + (\frac{0.1666666667}{h^2})(\omega-1)^4, \tag{65}$$

$$P_3(\omega) = 1 - (\frac{6}{h})(\omega-1)^2 + (\frac{6}{h^2})(\omega-1)^4, \tag{66}$$

$$P_5(\omega) = 1 - (\frac{15}{h})(\omega-1)^2 + (\frac{37.5000000000}{h^2})(\omega-1)^4, \tag{67}$$

$$P_7(\omega) = 1 - (\frac{28}{h})(\omega-1)^2 + (\frac{130.6666666666}{h^2})(\omega-1)^4, \tag{68}$$

$$P_9(\omega) = 1 - (\frac{45}{h})(\omega - 1)^2 + (\frac{337.5000000000}{h^2})(\omega - 1)^4. \tag{69}$$

It should be noted that including more components of the series solution results in increasing errors. Therefore, we consider the solution $P_n(\omega)$ up to $O((\omega - 1)^4)$. Also, setting $\theta = 0$ results in changing of Eq. (45). In fact, we can't consider it as Strum-Liouville equation. Setting the above equations in Eq.(40) we obtain the solutions of the Laplace equation by using GDTM for $\theta \in [0.1745329252, 1.570796327]$ as follow

$$\begin{aligned} u(1, \theta) = 55 &+ \frac{165}{2}[1 - (\frac{1}{h})(\omega - 1)^2 + (\frac{0.1666666667}{h^2})(\omega - 1)^4] \\ &- \frac{385}{8}[1 - (\frac{6}{h})(\omega - 1)^2 + (\frac{6}{h^2})(\omega - 1)^4] \\ &+ \frac{605}{16}[1 - (\frac{15}{h})(\omega - 1)^2 + (\frac{37.5000000000}{h^2})(\omega - 1)^4] \\ &- \frac{4125}{128}[1 - (\frac{28}{h})(\omega - 1)^2 + (\frac{130.6666666666}{h^2})(\omega - 1)^4] \\ &+ \frac{7315}{256}[1 - (\frac{45}{h})(\omega - 1)^2 + (\frac{337.5000000000}{h^2})(\omega - 1)^4], \end{aligned} \tag{70}$$

where θ is shown in Radian. Table 1 shows a comparison of the approximate of the Legendre polynomial of degree 9 by using GDTM, Rodrigues' formula, RKF45 and Taylor' series methods. RKF45 is a Fehlberg fourth-fifth order by using Runge-Kutta method [4, 5]. Figure 2 shows the approximate and error of the methods for $\theta \in [0.01745329252, 0.1745329252]$. GDTM is not suitable in the first limited interval. Also, figure 3 and 4 show that the values of GDTM are closer to the ideal potential that is $110V$ and includes the least error. As we know, the potential reduces considerably when we approach to isolation and error increases subsequently. Figure 4 shows that the potential of GDTM decreases as same as RKF45 and Taylor's series.

θ	**GDTM**	**Rodrigues**	**RKF45**	**Taylor's series**
0.1745329252	118.2223267	115.0677953	108.3734725	108.3734725
0.3490658504	106.7464309	108.7032745	82.2353712	82.2353714
0.5235987758	103.9841612	120.7591281	78.0034333	84.1951037
0.6981317008	99.4310173	122.3741211	62.2939333	68.4800347
0.8726646262	102.7453736	141.8542600	77.2680049	83.3120615
1.047197551	127.5694444	161.4808985	154.8091511	160.3441611
1.221730477	124.3915965	169.3100614	137.5806397	142.1331072
1.396263402	98.2646396	159.5602254	95.5078245	98.5605815

TABLE 1. Comparison GDTM with other methods in $\theta \in [0.1745329252, 1.396263402]$.

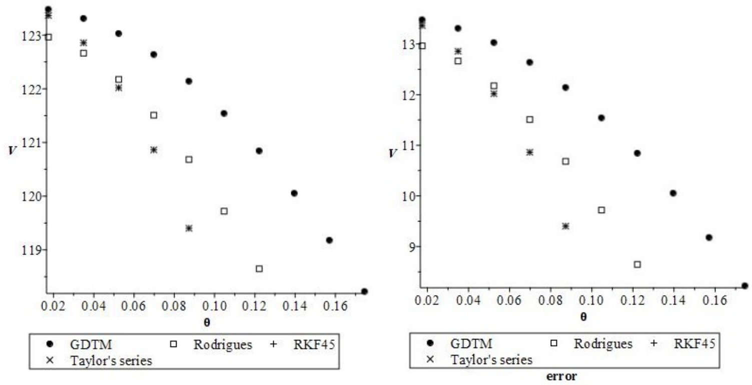

FIGURE 2. Comparison GDTM with other methods (b) error in $\theta \in [0.01745329252, 0.1745329252]$.

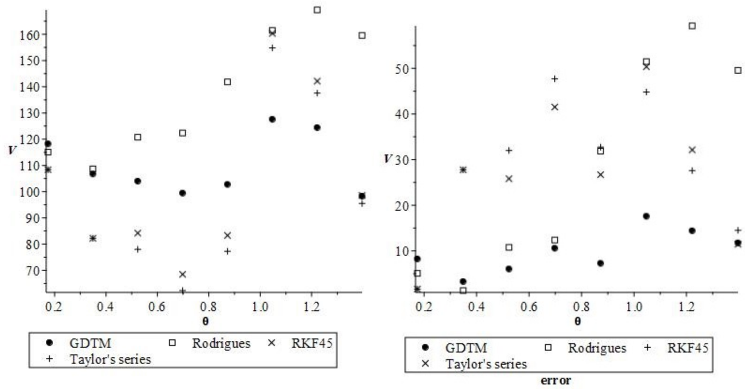

FIGURE 3. Comparison GDTM with other methods (b) error in $\theta \in [0.1745329252, 1.396263402]$.

Conclusion

In this paper, GDTM has been used to obtain the Lagrange polynomials. Although, variable h causes to increase the computations, the approximate of GDTM has more accurate than previous methods. The error showed that recent results of the Laplace equation are far from the ideal potential but the approximate of GDTM is closer to it in the most intervals.

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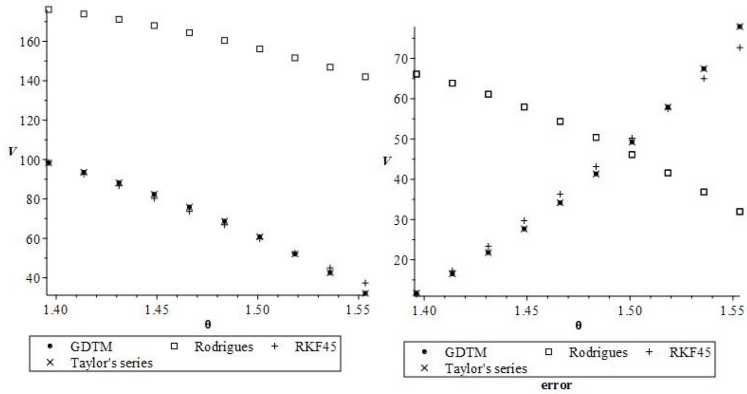

Figure 4. Comparison GDTM with other methods (b) error in $\theta \in [1.396263402, 1.553343034]$.

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