

ON FRAMES FOR COUNTABLY GENERATED HILBERT MODULES OVER LOCALLY C^* -ALGEBRAS

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ABSTRACT. Let \mathcal{X} be a countably generated Hilbert module over a locally C^* -algebra \mathcal{A} in multiplier module $M(\mathcal{X})$ of \mathcal{X} . We propose the necessary and sufficient condition such that a sequence $\{h_n : n \in \mathbb{N}\}$ in $M(\mathcal{X})$ is a standard frame of multipliers in \mathcal{X} . We also show that if T in $b(L_{\mathcal{A}}(\mathcal{X}))$, the space of bounded maps in set of all adjointable maps on \mathcal{X} , is surjective and $\{h_n : n \in \mathbb{N}\}$ is a standard frame of multipliers in \mathcal{X} , then $\{T \circ h_n : n \in \mathbb{N}\}$ is a standard frame of multipliers in \mathcal{X} , too.

1. Introduction and preliminaries

Locally C^* -algebras are generalizations of C^* -algebras. Locally C^* -algebras were first introduced by A. Inoue [5] and were also studied more by N. C. Phillips (under the name of pro- C^* -algebra) [10].

A locally C^* -algebra is a complete Hausdorff complex topological $*$ -algebra \mathcal{A} , whose topology is determined by its continuous C^* -seminorms in the sense that the net $\{a_i\}_{i \in I}$ converges to 0 in \mathcal{A} if and only if the net $\{p(a_i)\}_{i \in I}$ converges to 0 for every continuous C^* -seminorm p in set $S(\mathcal{A})$ of all continuous C^* -seminorms on \mathcal{A} .

Hilbert modules are essentially objects like Hilbert spaces by allowing the inner product to take values in a (locally) C^* -algebra rather than the field of complex numbers. The notion of Hilbert module over locally C^* -algebras generalize the notion of Hilbert C^* -module. Hilbert modules over locally C^* -algebras were first considered by N. C. Phillips [10]. He showed that many properties of the Hilbert C^* -modules are valid for Hilbert modules over locally C^* -algebras. But the main body of the work on Hilbert modules over locally C^* -algebras is due to M. Joita, all her work on the subject can be found in her book, under the title “Hilbert modules over locally C^* -algebras” (See [7]).

Here we recall some results about Hilbert modules over locally C^* -algebras from [10] and [7].

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A (right) pre-Hilbert module over a locally C^* -algebra \mathcal{A} (or a pre-Hilbert \mathcal{A} -module) is a complex vector space \mathcal{X} which is also a right \mathcal{A} -module, compatible with the complex algebra structure, equipped with an \mathcal{A} -valued inner product $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{A}$ which is \mathbb{C} - and \mathcal{A} -linear in its second variable and satisfies the following relations:

- (1) $\langle y, x \rangle = \langle x, y \rangle^*$ for every $x, y \in \mathcal{X}$;
- (2) $\langle x, x \rangle \geq 0$ for every $x \in \mathcal{X}$;
- (3) $\langle x, x \rangle = 0$ if and only if $x = 0$.

A pre-Hilbert \mathcal{A} -module \mathcal{X} is a Hilbert \mathcal{A} -module if \mathcal{X} is complete with respect to the topology determined by the family of seminorms $\{\bar{p}_{\mathcal{X}}\}_{p \in S(\mathcal{A})}$ where $\bar{p}_{\mathcal{X}}(x) = \sqrt{p(\langle x, x \rangle)}$, $x \in \mathcal{X}$.

If \mathcal{A} is a locally C^* -algebra, then \mathcal{A} is a Hilbert \mathcal{A} -module with $\langle a, b \rangle = a^*b$, and the set $\mathcal{H}_{\mathcal{A}}$ of all sequences $(a_n)_n$ with $a_n \in \mathcal{A}$ such that $\sum_n a_n^* a_n$ converges in \mathcal{A} is a Hilbert \mathcal{A} -module with the action of \mathcal{A} on $\mathcal{H}_{\mathcal{A}}$ defined by $(a_n)_n b = (a_n b)_n$ and the inner product defined by $\langle (a_n)_n, (b_n)_n \rangle = \sum_n a_n^* b_n$.

Let \mathcal{A} be a locally C^* -algebra and let \mathcal{X} be a Hilbert \mathcal{A} -module. A subset \mathcal{Y} of \mathcal{X} is a generating set for \mathcal{X} if the closed submodule of \mathcal{X} generated by \mathcal{Y} is the whole of \mathcal{X} . We say that \mathcal{X} is countably generated if it has a countable generating set.

Let \mathcal{A} be a locally C^* -algebra and let \mathcal{X} and \mathcal{Y} be Hilbert \mathcal{A} -modules. An \mathcal{A} -module map $T : \mathcal{X} \rightarrow \mathcal{Y}$ is called bounded if for all $p \in S(\mathcal{A})$, there is $M_p > 0$ such that $\bar{p}_{\mathcal{Y}}(Tx) \leq M_p \bar{p}_{\mathcal{X}}(x)$ for all $x \in \mathcal{X}$, and it is adjointable if there is a map $T^* : \mathcal{Y} \rightarrow \mathcal{X}$ such that $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x \in \mathcal{X}$, $y \in \mathcal{Y}$. It is easy to see that every adjointable map is a bounded \mathcal{A} -module map. The set of all adjointable maps from \mathcal{X} into \mathcal{Y} is denoted by $L_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$ and we write $L_{\mathcal{A}}(\mathcal{X})$ for $L_{\mathcal{A}}(\mathcal{X}, \mathcal{X})$. The vector space $L_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$ is a complete locally convex space with respect to the topology defined by the family of seminorms $\{\hat{p}_{\mathcal{X}, \mathcal{Y}}\}_{p \in S(\mathcal{A})}$, where $\hat{p}_{\mathcal{X}, \mathcal{Y}}$ defined by $\hat{p}_{\mathcal{X}, \mathcal{Y}}(T) = \sup\{\bar{p}_{\mathcal{Y}}(Tx) : x \in \mathcal{X} \text{ and } \bar{p}_{\mathcal{X}}(x) \leq 1\}$. In particular, $L_{\mathcal{A}}(\mathcal{X})$ becomes a locally C^* -algebra with respect to the topology defined by the family of seminorms $\{\hat{p}_{\mathcal{X}}\}_{p \in S(\mathcal{A})}$.

We say that an element T of $L_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$ is bounded in $L_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$ if there is $M > 0$ such that $\hat{p}_{\mathcal{X}, \mathcal{Y}}(T) \leq M$ for all $p \in S(\mathcal{A})$. The set of all bounded elements in $L_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$ is denoted by $b(L_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}))$. It is clear that the map $\|\cdot\|_{\infty}$ defined by $\|T\|_{\infty} = \sup\{\hat{p}_{\mathcal{X}, \mathcal{Y}}(T) : p \in S(\mathcal{A})\}$ is a norm on $b(L_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}))$. And $b(L_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}))$ is a Banach space with respect to the norm $\|\cdot\|_{\infty}$. So $b(L_{\mathcal{A}}(\mathcal{X}))$ is a C^* -algebra with respect to the norm $\|\cdot\|_{\infty}$.

Now we recall some fact about multiplier modules from [8] and [9].

Let \mathcal{A} be a locally C^* -algebra and let \mathcal{X} be a Hilbert \mathcal{A} -module. It is not difficult to check that $L_{\mathcal{A}}(\mathcal{A}, \mathcal{X})$ is a Hilbert $L_{\mathcal{A}}(\mathcal{A})$ -module with the action of $L_{\mathcal{A}}(\mathcal{A})$ on $L_{\mathcal{A}}(\mathcal{A}, \mathcal{X})$ defined by $t.s = t \circ s$, $t \in L_{\mathcal{A}}(\mathcal{A}, \mathcal{X})$ and $s \in L_{\mathcal{A}}(\mathcal{A})$ and the $L_{\mathcal{A}}(\mathcal{A})$ -valued inner-product defined by $\langle t, s \rangle = t^* \circ s$. Moreover, $\bar{p}_{L_{\mathcal{A}}(\mathcal{A}, \mathcal{X})}(s) = \hat{p}_{L_{\mathcal{A}}(\mathcal{A}, \mathcal{X})}(s)$ for all $s \in L_{\mathcal{A}}(\mathcal{A}, \mathcal{X})$ and for all $p \in S(\mathcal{A})$, the topology on $L_{\mathcal{A}}(\mathcal{A}, \mathcal{X})$ induced by the inner product coincides with the topology

determined by the family of seminorms $\{\bar{p}_{L_{\mathcal{A}}(\mathcal{A}, \mathcal{X})}\}_{p \in S(\mathcal{A})}$. Therefore $L_{\mathcal{A}}(\mathcal{A}, \mathcal{X})$ is a Hilbert $L_{\mathcal{A}}(\mathcal{A})$ -module and since $L_{\mathcal{A}}(\mathcal{A})$ can be identified with the multiplier algebra $M(\mathcal{A})$ of \mathcal{A} (see [7] and [10]), $L_{\mathcal{A}}(\mathcal{A}, \mathcal{X})$ becomes a Hilbert $M(\mathcal{A})$ -module. The Hilbert $M(\mathcal{A})$ -module $L_{\mathcal{A}}(\mathcal{A}, \mathcal{X})$ is called the multiplier module of \mathcal{X} , and it is denoted by $M(\mathcal{X})$.

The map $i_{\mathcal{X}} : \mathcal{X} \rightarrow M(\mathcal{X})$ defined by $i_{\mathcal{X}}(x)(a) = xa$, $x \in \mathcal{X}$ and $a \in \mathcal{A}$ embeds \mathcal{X} as a closed submodule of $M(\mathcal{X})$. Moreover, if $t \in M(\mathcal{X})$, then $t.a = t(a)$ for all $a \in \mathcal{A}$ and $\langle t, x \rangle = t^*(x)$ for all $x \in \mathcal{X}$.

A Hilbert \mathcal{A} -module \mathcal{X} is countably generated in $M(\mathcal{X})$ if there is a countable set $\{h_n : h_n \in M(\mathcal{X}), n = 1, 2, \dots\}$ such that the closed submodule of $M(\mathcal{X})$ generated by $\{h_n.a : a \in \mathcal{A}, n = 1, 2, \dots\}$ is the whole of \mathcal{X} .

If \mathcal{X} is a countably generated Hilbert \mathcal{A} -module, then \mathcal{X} is countably generated in $M(\mathcal{X})$. In general, \mathcal{X} is not always countably generated when \mathcal{X} is countably generated in $M(\mathcal{X})$.

Frames for Hilbert spaces were introduced by R. J. Duffin and A. C. Schaeffer [3] in 1952 as part of their research in non-harmonic Fourier series. They were reintroduced and developed by I. Daubechies, A. Grossmann and Y. Meyer [2] in 1986. Many generalizations of frames were introduced, meanwhile, M. Frank and D. R. Larson [4] presented a general approach to the frame theory in Hilbert C^* -modules. A frame for a countably generated Hilbert C^* -module \mathcal{X} is a sequence $\{x_n : n \in \mathbb{N}\}$ for which there are constants $C, D > 0$ such that

$$C\langle x, x \rangle \leq \sum_n \langle x, x_n \rangle \langle x_n, x \rangle \leq D\langle x, x \rangle, \quad x \in \mathcal{X}.$$

M. Joita [8] generalized this definition to the situation Hilbert modules over locally C^* -algebras. A frame of multipliers for a countably generated Hilbert module \mathcal{X} over a locally C^* -algebra \mathcal{A} is a sequence $\{h_n : n \in \mathbb{N}\}$ in $M(\mathcal{X})$ for which there are constants $C, D > 0$ such that

$$(1.1) \quad C\langle x, x \rangle \leq \sum_n \langle x, h_n \rangle \langle h_n, x \rangle \leq D\langle x, x \rangle, \quad x \in \mathcal{X}.$$

The numbers C and D are called lower and upper frame bounds, respectively. We consider standard frames of multipliers for which the sum in the middle of (1.1) converges in \mathcal{A} for every $x \in \mathcal{X}$.

Any countably generated Hilbert \mathcal{A} -module \mathcal{X} in $M(\mathcal{X})$ admits a standard frame of multipliers [8, Proposition 3.6].

In this paper we extend some results from [1] in the context of Hilbert modules over locally C^* -algebras.

2. Main results

First, we investigate some properties of bounded \mathcal{A} -linear maps between Hilbert \mathcal{A} -modules.

Definition 2.1. An element T of $L_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$ is bounded below in $L_{\mathcal{A}}(\mathcal{X}, \mathcal{Y})$ if there is $M > 0$ such that $M\bar{p}_{\mathcal{X}}(x) \leq \bar{p}_{\mathcal{Y}}(Tx)$ for all $p \in S(\mathcal{A})$ and $x \in \mathcal{X}$.

Our first result is a generalization of [1, Proposition 2.1].

Proposition 2.2. *Let \mathcal{A} be a locally C^* -algebra, \mathcal{X} and \mathcal{Y} Hilbert \mathcal{A} -modules and $T \in b(L_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}))$. The following statements are mutually equivalent:*

- (1) T is surjective;
- (2) T^* is bounded below in $L_{\mathcal{A}}(\mathcal{Y}, \mathcal{X})$, i.e., there is $m > 0$ such that $m\bar{p}_{\mathcal{Y}}(x) \leq \bar{p}_{\mathcal{X}}(T^*x)$ for all $p \in S(\mathcal{A})$ and $x \in \mathcal{Y}$;
- (3) T^* is bounded below with respect to inner product, i.e., there is $m' > 0$ such that $m'\langle x, x \rangle \leq \langle T^*x, T^*x \rangle$ for all $x \in \mathcal{Y}$.

Proof. If (1) holds, then $\text{Im}T = \mathcal{Y}$ is closed. It follows from [7, Remark 3.2.5] that $\text{Im}T^*$ is also closed, $\text{Ker}T \oplus \text{Im}T^* = \mathcal{X}$ and $\text{Ker}T^* \oplus \text{Im}T = \mathcal{Y}$. We shall prove that TT^* is bijective.

If $TT^*x = 0$ for some $x \in \mathcal{Y}$, then $T^*x \in \text{Ker}T \cap \text{Im}T^* = \{0\}$, hence $T^*x = 0$. Now $x \in \text{Ker}T^* = (\text{Im}T)^\perp = \mathcal{Y}^\perp = \{0\}$, implies $x = 0$. This proves that TT^* is injective.

Let z be an arbitrarily chosen element of \mathcal{Y} . T is surjective, then we have $z = Ty$ for some $y \in \mathcal{X}$. There are $y_1 \in \text{Ker}T$ and $x \in \mathcal{Y}$ such that $y = y_1 \oplus T^*x$. Then $z = Ty = T(y_1 \oplus T^*x) = Ty_1 + TT^*x = TT^*x$; therefore TT^* is surjective.

Since TT^* is a positive invertible element of the C^* -algebra $b(L_{\mathcal{A}}(\mathcal{Y}))$, then $0 \leq (TT^*)^{-1} \leq \|(TT^*)^{-1}\|_\infty id_{\mathcal{Y}}$, so $TT^* \geq \|(TT^*)^{-1}\|_\infty^{-1} id_{\mathcal{Y}}$, where $id_{\mathcal{Y}}$ stands for the identity operator on \mathcal{Y} . Denoting $m' = \|(TT^*)^{-1}\|_\infty^{-1}$ we get $TT^* - m'id_{\mathcal{Y}} \geq 0$. This is equivalent to $\langle (TT^* - m'id_{\mathcal{Y}})x, x \rangle \geq 0$ for all $x \in \mathcal{Y}$, i.e., $\langle T^*x, T^*x \rangle \geq m'\langle x, x \rangle$ for all $x \in \mathcal{Y}$. So (3) holds.

The implication (3) \Rightarrow (2) is trivial.

Suppose that (2) holds. Then T^* is clearly injective, and it is easy to see that $\text{Im}T^*$ is closed. Then T has the closed range, again by [7, Remark 3.2.5], and $\mathcal{Y} = \text{Ker}T^* \oplus \text{Im}T = \{0\} \oplus \text{Im}T = \text{Im}T$. This gives (1). \square

In view of [6, Theorem 3.7], we notice that if $T \in b(L_{\mathcal{A}}(\mathcal{X}))$, $\langle Tx, Tx \rangle \leq \|T\|_\infty^2 \langle x, x \rangle$ for all $x \in \mathcal{X}$. So we have the following corollary:

Corollary 2.3. *Let \mathcal{A} be a locally C^* -algebra, \mathcal{X} a Hilbert \mathcal{A} -module and $T \in b(L_{\mathcal{A}}(\mathcal{X}))$ such that $T^* = T$. The following statements are mutually equivalent:*

- (1) T is surjective;
- (2) There are $m, M > 0$ such that $m\bar{p}_{\mathcal{X}}(x) \leq \bar{p}_{\mathcal{X}}(Tx) \leq M\bar{p}_{\mathcal{X}}(x)$ for all $p \in S(\mathcal{A})$ and $x \in \mathcal{X}$;
- (3) There are $m', M' > 0$ such that $m'\langle x, x \rangle \leq \langle Tx, Tx \rangle \leq M'\langle x, x \rangle$ for all $x \in \mathcal{X}$.

Remark 2.4. Let \mathcal{X} be a countably generated Hilbert \mathcal{A} -module in $M(\mathcal{X})$ and let $\{h_n\}_n$ be a standard frame of multipliers in \mathcal{X} . The module morphism $\theta : \mathcal{X} \rightarrow \mathcal{H}_{\mathcal{A}}$ defined by $\theta(x) = (\langle h_n, x \rangle)_n$ is called the frame transform for $\{h_n\}_n$. The frame transform θ is an injective adjointable module morphism from \mathcal{X} to $\mathcal{H}_{\mathcal{A}}$ with closed range. Moreover, $\theta \in b(L_{\mathcal{A}}(\mathcal{X}, \mathcal{H}_{\mathcal{A}}))$ which realizes an embedding of \mathcal{X} onto an orthogonal summand of $\mathcal{H}_{\mathcal{A}}$. The adjoint operator

θ^* is surjective. Moreover, $\theta^* \circ \theta$ is an invertible element in $b(L_{\mathcal{A}}(\mathcal{X}))$. For details we refer the reader to [8].

Theorem 2.5. *Let \mathcal{A} be a locally C^* -algebra, \mathcal{X} a countably generated Hilbert \mathcal{A} -module in $M(\mathcal{X})$, $\{h_n : n \in \mathbb{N}\}$ a sequence in $M(\mathcal{X})$ and $\theta(x) = (\langle h_n, x \rangle)_{n \in \mathbb{N}}$ for $x \in \mathcal{X}$. The following statements are mutually equivalent:*

- (1) $\{h_n : n \in \mathbb{N}\}$ is a standard frame of multipliers in \mathcal{X} .
- (2) $\theta \in b(L_{\mathcal{A}}(\mathcal{X}, \mathcal{H}_{\mathcal{A}}))$ and θ is bounded below in $L_{\mathcal{A}}(\mathcal{X}, \mathcal{H}_{\mathcal{A}})$.
- (3) $\theta \in b(L_{\mathcal{A}}(\mathcal{X}, \mathcal{H}_{\mathcal{A}}))$ and θ^* is surjective.

Proof. (1) \Rightarrow (3): It was proved in [8, Theorem 3.11].

(2) \Leftrightarrow (3): It follows from Proposition 2.2.

(2) \Rightarrow (1): Since

$$\langle \theta x, \theta x \rangle = \sum_n \langle x, h_n \rangle \langle h_n, x \rangle, \quad x \in \mathcal{X},$$

from (2) it follows that $\{h_n : n \in \mathbb{N}\}$ is a standard frame of multipliers in \mathcal{X} . \square

Another direct consequence of Proposition 2.2 is that if $T \in b(L_{\mathcal{A}}(\mathcal{X}))$ is surjective and $\{h_n : n \in \mathbb{N}\}$ is a standard frame of multipliers in \mathcal{X} , then $\{T \circ h_n : n \in \mathbb{N}\}$ is a standard frame of multipliers in \mathcal{X} , too.

Theorem 2.6. *Let \mathcal{A} be a locally C^* -algebra, \mathcal{X} a countably generated Hilbert \mathcal{A} -module in $M(\mathcal{X})$, and $T \in b(L_{\mathcal{A}}(\mathcal{X}))$ surjective. If $\{h_n : n \in \mathbb{N}\}$ is a standard frame of multipliers in \mathcal{X} with frame bounds C and D , then $\{T \circ h_n : n \in \mathbb{N}\}$ is a standard frame of multipliers in \mathcal{X} with frame bounds $C\|(TT^*)^{-1}\|_{\infty}^{-1}$ and $D\|T\|_{\infty}^2$.*

Proof. Let $x \in \mathcal{X}$. Since

$$\sum_{k=1}^n \langle x, T \circ h_k \rangle \langle T \circ h_k, x \rangle = \sum_{k=1}^n \langle T^* x, h_k \rangle \langle h_k, T^* x \rangle$$

and since $\{h_n : n \in \mathbb{N}\}$ is a standard frame of multipliers in \mathcal{X} and $T^* x \in \mathcal{X}$, $\sum_n \langle x, T \circ h_n \rangle \langle T \circ h_n, x \rangle$ converges in \mathcal{A} , and

$$C \langle T^* x, T^* x \rangle \leq \sum_n \langle T^* x, h_n \rangle \langle h_n, T^* x \rangle \leq D \langle T^* x, T^* x \rangle.$$

From the proof of Proposition 2.2 we have $\langle T^* x, T^* x \rangle \geq \|(TT^*)^{-1}\|_{\infty}^{-1} \langle x, x \rangle$, since T is surjective. It follows that

$$C\|(TT^*)^{-1}\|_{\infty}^{-1} \langle x, x \rangle \leq \sum_n \langle x, T \circ h_n \rangle \langle T \circ h_n, x \rangle \leq D\|T\|_{\infty}^2 \langle x, x \rangle.$$

From these facts we conclude that $\{T \circ h_n : n \in \mathbb{N}\}$ is a standard frame of multipliers in \mathcal{X} with frame bounds $C\|(TT^*)^{-1}\|_{\infty}^{-1}$ and $D\|T\|_{\infty}^2$. \square

The next result shows that the condition (1.1) from the definition of standard frames can be replaced with a weaker one.

Theorem 2.7. *Let \mathcal{A} be a locally C^* -algebra, \mathcal{X} a countably generated Hilbert \mathcal{A} -module in $M(\mathcal{X})$, and $\{h_n : n \in \mathbb{N}\}$ a sequence in $M(\mathcal{X})$ such that $\sum_n \langle x, h_n \rangle \langle h_n, x \rangle$ converges in \mathcal{A} for every $x \in \mathcal{X}$. Then $\{h_n : n \in \mathbb{N}\}$ is a standard frame of multipliers in \mathcal{X} if and only if there are constants $C, D > 0$ such that*

$$(2.1) \quad C\bar{p}_{\mathcal{X}}(x)^2 \leq p\left(\sum_n \langle x, h_n \rangle \langle h_n, x \rangle\right) \leq D\bar{p}_{\mathcal{X}}(x)^2, \quad x \in \mathcal{X}, \quad p \in S(\mathcal{A}).$$

Proof. Obviously, every standard frame of multipliers in \mathcal{X} satisfies (2.1).

For the converse we suppose that a sequence $\{h_n : n \in \mathbb{N}\}$ in $M(\mathcal{X})$ fulfills (2.1). For an arbitrary $x \in \mathcal{X}$ and a finite $J \subseteq \mathbb{N}$ we define $x_J = \sum_{n \in J} h_n \langle h_n, x \rangle$. Then

$$\begin{aligned} \bar{p}_{\mathcal{X}}(x_J)^4 &= p(\langle x_J, x_J \rangle)^2 = p\left(\langle x_J, \sum_{n \in J} h_n \langle h_n, x \rangle \rangle\right)^2 \\ &= p\left(\sum_{n \in J} \langle x_J, h_n \rangle \langle h_n, x \rangle\right)^2 \\ &\leq p\left(\sum_{n \in J} \langle x_J, h_n \rangle \langle h_n, x_J \rangle\right) p\left(\sum_{n \in J} \langle x, h_n \rangle \langle h_n, x \rangle\right) \\ &\leq D\bar{p}_{\mathcal{X}}(x_J)^2 p\left(\sum_{n \in J} \langle x, h_n \rangle \langle h_n, x \rangle\right), \end{aligned}$$

therefore

$$\bar{p}_{\mathcal{X}}\left(\sum_{n \in J} h_n \langle h_n, x \rangle\right)^2 = \bar{p}_{\mathcal{X}}(x_J)^2 \leq Dp\left(\sum_{n \in J} \langle x, h_n \rangle \langle h_n, x \rangle\right).$$

Since J is arbitrary, the series $\sum_n h_n \langle h_n, x \rangle$ converges in \mathcal{X} and since

$$\begin{aligned} \bar{p}_{\mathcal{X}}\left(\sum_{n \in J} h_n \langle h_n, x \rangle\right)^2 &\leq Dp\left(\sum_{n \in J} \langle x, h_n \rangle \langle h_n, x \rangle\right) \leq D^2\bar{p}_{\mathcal{X}}(x)^2, \\ \bar{p}_{\mathcal{X}}\left(\sum_{n \in J} h_n \langle h_n, x \rangle\right) &\leq D\bar{p}_{\mathcal{X}}(x). \end{aligned}$$

Since $x \in \mathcal{X}$ is arbitrarily chosen, the operator

$$T : \mathcal{X} \longrightarrow \mathcal{X}, \quad x \longmapsto \sum_n h_n \langle h_n, x \rangle$$

is well defined, bounded and \mathcal{A} -linear. It is easy to check that $\langle Tx, y \rangle = \langle x, Ty \rangle$ for all $x, y \in \mathcal{X}$, so $T \in L_{\mathcal{A}}(\mathcal{X})$ and $T = T^*$. From $\langle Tx, x \rangle = \sum_n \langle x, h_n \rangle \langle h_n, x \rangle \geq 0$ for all $x \in \mathcal{X}$, it follows that $T \geq 0$. Now (2.1) and

$\langle T^{\frac{1}{2}}x, T^{\frac{1}{2}}x \rangle = \sum_n \langle x, h_n \rangle \langle h_n, x \rangle$ imply $C^{\frac{1}{2}}\bar{p}_{\mathcal{X}}(x) \leq \bar{p}_{\mathcal{X}}(T^{\frac{1}{2}}x) \leq D^{\frac{1}{2}}\bar{p}_{\mathcal{X}}(x)$ for all $x \in \mathcal{X}$. By Corollary 2.3, there are constants $C', D' > 0$ such that

$$C'\langle x, x \rangle \leq \langle T^{\frac{1}{2}}x, T^{\frac{1}{2}}x \rangle = \sum_n \langle x, h_n \rangle \langle h_n, x \rangle \leq D'\langle x, x \rangle, \quad x \in \mathcal{X}.$$

This proves that $\{h_n : n \in \mathbb{N}\}$ is a standard frame of multipliers in \mathcal{X} . \square

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