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ON [1, 2]-DOMINATION IN TREES

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Abstract. Chellai et al. [3] gave an upper bound on the [1, 2]-domination number of tree and posed an open question "how to classify trees satisfying the sharp bound?". Yang and Wu [5] gave a partial solution for tree of order n with ℓ-leaves such that every non-leaf vertex has degree at least 4. In this paper, we give a new upper bound on the [1, 2]-domination number of tree which extends the result of Yang and Wu. In addition, we design a polynomial time algorithm for solving the open question. By using this algorithm, we give a characterization on the [1, 2]-domination number for trees of order n with ℓ leaves satisfying $n - \ell$. Thereby, the open question posed by Chellai et al. is solved.

1. Introduction

Graph theory terminology not presented here can be found in [3]. Let $G = (V, E)$ be a graph with $|V| = n$. The neighborhood and closed neighborhood of a vertex v in the graph G are denoted by $N(v)$ and $N[v] = N(v) \cup \{v\}$, respectively. The graph induced by $S \subseteq V$ is denoted by $G[S]$. Let $G - S$ denote the induced subgraph $G[V - S]$. A *tree* is a connected graph that contains no cycles. A *leaf* of a tree T is a vertex of degree 1. We denote the set of leaves in tree T by $L(T)$.

A subset $D \subseteq V$ in a graph $G = (V, E)$ is a [1, 2]-set if, for every vertex $v \in V \setminus D$, $1 \leq |N(v) \cap D| \leq 2$. A [1, 2]-set D is a dominating set. The [1, 2]-domination number $\gamma_{[1,2]}(G)$ of G is the minimum cardinality of all [1, 2]-sets in G. The notions of [1, 2]-set and [1, 2]-domination were first investigated by Dejter [4]. For any two integers j and k, a subset $D \subseteq V$ in a graph $G = (V, E)$ is a $[j, k]$-set if, for every vertex $v \in V \setminus D$, $j \leq |N(v) \cap D| \leq k$. For $j \geq 1$, a $[j, k]$-set D is a dominating set. The notions of $[j, k]$-set and $[j, k]$-domination were recently introduced by Chellali et al. [3]. For more general concepts, called set-restricted dominating set and set-restricted domination number, we refer to Amin and Slater [1, 2].

Chellali et al. [3] gave the following open question.

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Question 1.1. If T is a tree of order n with ℓ leaves, then $\gamma_{[1,2]}(T) \leq n - \ell$. For which trees is this bound sharp?

Yang and Wu [5] gave the following result.

Theorem 1.1. *Let T be a tree of order n with ℓ leaves such that every non-leaf vertex has degree at least 4. Then $\gamma_{[1,2]}(T) = n - \ell$.*

In this paper, we give a new upper bound on the $[1,2]$-domination number of tree which extends the result of Yang and Wu. In addition, we design a polynomial time algorithm for solving the open question. By using this algorithm, we give a characterization on the $[1,2]$-domination number for trees of order n with ℓ leaves satisfying $n-\ell$. Thereby, the open question posed by Chellai et al. is solved.

2. Main results

View T as the rooted tree at vertex t. For a vertex v in a rooted tree T, let $C(v)$ and $D(v)$ denote the sets of children and descendants of v, respectively. Let $T_v = T[D(v) \cup \{v\}]$. Let T be a tree. For $1 \leq i \leq \Delta(T)$, let $S_i(T) = \{v \mid v \in V(T), d(v) = i\}$. If T has ℓ leaves, then $|S_1(T)| = |L(T)| = \ell$. Let $S(T)$ denote the set of support vertices of T. Let $I(T) = V(T) - S_1(T)$. By the definition of $[1,2]$-dominating set of T, if $v \in S(T)$ and $|N(v) \cap S_1(T)| \geq 3$, then v belongs to every $\gamma_{[1,2]}$-set of T. A new upper bound on the $[1,2]$-domination number of tree is given in the following.

Theorem 2.1. *Let T be a tree with ℓ leaves. If $S_2(T) \setminus S(T) \neq \emptyset$, then $\gamma_{[1,2]}(T) \leq n - \ell - \lceil \frac{2}{3}|S_2(T) \setminus S(T)| \rceil$.*

Proof. Let $T_1, T_2, \ldots, T_j$ be the components of $T[S_2(T) \setminus S(T)]$. Then each T_i is a path. Assume $v_1, v_2, \ldots, v_{a_i}$ denote the vertices of T_i. Define

$$S_{T_i} = \begin{cases} \{v_{3k+1}, v_{3k+2} | k = 0, 1, \ldots, \frac{a_i - 3}{3}\} & \text{if } a_i \equiv 0 \pmod 3 \\ \{v_{3k+1}, v_{3k+2} | k = 0, 1, \ldots, \frac{a_i - 4}{3}\} \cup \{v_{a_i}\} & \text{if } a_i \equiv 1 \pmod 3 \\ \{v_{3k+1}, v_{3k+2} | k = 0, 1, \ldots, \frac{a_i - 5}{3}\} \cup \{v_{a_i - 1}, v_{a_i}\} & \text{if } a_i \equiv 2 \pmod 3 \end{cases}$$

Then $I(T) \setminus (\bigcup_{i=1}^{j} S_{T_i})$ is a $[1,2]$-dominating set of T. It is obvious that $|S_{T_i}| = \lceil \frac{2a_i}{3} \rceil$. Hence, $\gamma_{[1,2]}(T) \leq |I(T) \setminus (\bigcup_{i=1}^{j} S_{T_i})| = |I(T)| - |(\bigcup_{i=1}^{j} S_{T_i})| = n - \ell - \sum_{i=1}^{j} |S_{T_i}| = n - \ell - \sum_{i=1}^{j} \lceil \frac{2a_i}{3} \rceil \leq n - \ell - \lceil \frac{2}{3}|S_2(T) \setminus S(T)| \rceil$. □

Corollary 2.1. *Let T be a tree with ℓ leaves. If $S_2(T) \setminus S(T) \neq \emptyset$, then $\gamma_{[1,2]}(T) < n - \ell$.*

By Corollary 2.1, we will assume that $S_2(T) \setminus S(T) = \emptyset$. Theorem 1.1 is extended by the following result.

Theorem 2.2. *Let T be a tree of order n with ℓ leaves and $S_2(T) \setminus S(T) = \emptyset$. If $|S_2(T) \cup S_3(T)| \leq 1$, then $\gamma_{[1,2]}(T) = n - \ell$.*

Proof. By Theorem 1.1, if $|S_2(T) \cup S_3(T)| = 0$, then the theorem holds. So, we can assume that $|S_2(T) \cup S_3(T)| = 1$. Suppose that $\gamma_{[1,2]}(T) < n - \ell$. Among all $\gamma_{[1,2]}$-sets of T, let D be a $\gamma_{[1,2]}$-set of T such that $|D \cap I(T)|$ is maximized. Since $\gamma_{[1,2]}(T) < n - \ell = |I(T)|$, it follows that there exists a vertex $u \in I(T)$ such that $u \in V(T) - D$. Set $W = \{w \,|\, w$ is reachable by a path from u, all vertices of which belong to $V(T) - D\}$.

Case 1. $|W| = 1$. Then $W = \{u\}$. Since D is a $\gamma_{[1,2]}$-set of T and $u \in I(T)$, it follows that $d(u) = 2$. Since $S_2(T) \setminus S(T) = \emptyset$, it follows that $u \in S(T)$. Say $v \in N(u) \cap S_1(T)$. Let $D^{'} = (D \setminus \{v\}) \cup \{u\}$. Then $D^{'}$ is a $\gamma_{[1,2]}$-set of T such that $|D' \cap I(G)| > |D \cap I(G)|$, which is a contradiction.

Case 2. $|W| \geq 2$. Then $T[W]$ is a subtree of T with at least two leaves. Let v and t be two leaves of $T[W]$. Then $d(v) \geq 2$ and $d(t) \geq 2$. Since $|S_2(T) \cup S_3(T)| = 1$, it follows that $d(v) \geq 4$ or $d(t) \geq 4$. Without loss of generality, we can assume that $d(v) \geq 4$. Then v is dominated by D at least three times, a contradiction. □

Lemma 2.1. *Let T be a tree with ℓ leaves. Suppose that $v \in S(T)$ and $|N(v) \cap S_1(T)| \geq 3$. Say $N(v) \setminus S_1(T) = \{v_1, v_2, \ldots, v_k\}$. For $i = 1, 2, \ldots, k$, let T_i denote the component of $T - v$ containing v_i, and let $T_i^{'} = T - \bigcup_{j=1, j\neq i}^{k} T_j$. Then*

$\gamma_{[1,2]}(T) = n - \ell$ *if and only if* $\gamma_{[1,2]}(T_i^{'}) = n(T_i^{'}) - |S_1(T_i^{'})|$ *for* $i = 1, 2, \ldots, k$.

Proof. It is obvious that $\bigcap_{j=1}^{k}(I(T_j^{'})) = \{v\}$ and $\bigcup_{j=1}^{k}(I(T_j^{'})) = I(T)$. Suppose that $\gamma_{[1,2]}(T) = n - \ell = |I(T)|$. Then $I(T)$ is a $\gamma_{[1,2]}$-set of T. If there exists i such that $\gamma_{[1,2]}(T_i^{'}) < n(T_i^{'}) - |S_1(T_i^{'})|$, assume that $D_i^{'}$ is a $\gamma_{[1,2]}$-set of $T_i^{'}$, then $|D_i^{'}| < |I(T_i^{'})|$. Since v is adjacent to at least three leaves in $T_i^{'}$, $v \in D_i^{'}$. Hence, $D_i^{'} \cup \bigcup_{j=1, j\neq i}^{k} I(T_j^{'})$ is a $[1,2]$-dominating set of T with cardinality less than $\gamma_{[1,2]}(T)$, which is a contradiction. Hence, $\gamma_{[1,2]}(T_i^{'}) = n(T_i^{'}) - |S_1(T_i^{'})|$.

Conversely, let D be a $\gamma_{[1,2]}$-set of T. It is obvious that $v \in D$. Then $D \cap V(T_i^{'})$ is a $[1,2]$-dominating set of $T_i^{'}$. If $\gamma_{[1,2]}(T) < n - \ell$, there exists i such that $|D \cap V(T_i^{'})| < |I(T_i^{'})|$. Then $\gamma_{[1,2]}(T_i^{'}) < n(T_i^{'}) - |S_1(T_i^{'})|$, which is a contradiction. □

Lemma 2.2. *Let T be a tree of order n. Assume that $|N(u) \cap L(T)| \geq 4$. Say $w \in N(u) \cap L(T)$. Let $T' = T - w$. Then*

$\gamma_{[1,2]}(T) = |V(T)| - |L(T)|$ *if and only if* $\gamma_{[1,2]}(T') = |V(T')| - |L(T')|$.

Let T be a tree with $n \geq 3$. If $diam(T) = 2, 3$, it is obvious that $\gamma_{[1,2]}(T) = |V(T)| - |L(T)|$. So we can assume that $diam(T) \geq 4$. By Corollary 2.1, Lemma 2.1 and Lemma 2.2, in order to give a characterization of tree with $\gamma_{[1,2]}(T) = n - \ell$, we define a family of trees. Let Γ' be a family of trees T satisfying the following properties.

(1) $diam(T) \geq 4$.

(2) For each vertex $u \in I(T) \setminus S(T)$, $d(u) \geq 3$.

(3) For each vertex $u \in V(T)$, $|N(v) \cap L(T)| \leq 3$.
(4) If $|N(v) \cap L(T)| = 3$, then $|N(v) \cap I(T)| = 1$.

If $|N(v) \cap L(T)| = 3$, v is called a strong support vertex. Define $A(T) = \{u||N(u) \cap L(T)| = 3\}$. Let P be the longest path in T. Let t denote the third vertex in the path P. View T as a tree rooted at t. For $i = 0, 1, 2, \ldots, diam(T)-2$, define $L_i = \{u|d(u,t) = i, u \in V(T)\}$. For each $v \in V(T)$, define

$$h(v) = \begin{cases} 1 & \text{if } |N(v) \cap L(T)| = 2 \\ 0 & \text{if} |N(v) \cap L(T)| = 1 \\ -1 & \text{if } |N(v) \cap L(T)| = 0 \\ +\infty & \text{if } v \in A(T) \cup L(T) \end{cases}$$

Algorithm 1::
Input:: A tree $T \in \Gamma'$ and a root vertex t.
Output:: $\gamma_{[1,2]}(T) < n - \ell$ or $\gamma_{[1,2]}(T) = n - \ell$.
Step 0:: For each vertex $v \in \{u|u \in S(T), C(u) \subseteq L(T)\} \cup L(T)$, define $g(v) = 0$ and label v with $(h(v), g(v))$.
Step 1:: **while** there exists a vertex $v \in I(T)\backslash\{t\}$ such that v is unlabeled **do**

Choose an unlabeled vertex $v \in V(T)$ such that each vertex of $C(v)$ has been labeled. Say $C(v) = \{v_1, v_2, \ldots, v_{d_v-1}\}$ and $h(v_1) + g(v_1) \geq h(v_2) + g(v_2) \geq \cdots \geq h(v_{d_v-1}) + g(v_{d_v-1})$.

(1) Case 1. $|C(v) \cap (A(T) \cup L(T))| = 2$.
 (a) Define $g(v) = \sum_{w \in C(v)\backslash(A(T)\cup L(T))}(h(w) + g(w))$.
 (b) Label v with $(h(v), g(v))$.

(2) Case 2. $|C(v) \cap (A(T) \cup L(T))| \leq 1$.
If $h(v)+\sum_{w\in C(v)}(h(w)+g(w)) < 0$ or $h(v)+\sum_{w\in C(v)\backslash\{v_1\}}(h(w)+g(w)) < 0$ **then** output $r_{[1,2]}(T) < n - \ell$ **else**
$g(v) = \sum_{w\in C(v)\backslash\{v_1,v_2\}}(h(w) + g(w))$ (// If $d(v) = 3$, then $g(v) = 0$.)
Label v with $(h(v), g(v))$.
End-while

Step 2: Suppose that every vertex $v \in I(T) \setminus \{t\}$ has been labeled. Say $C(t) = \{v_1, v_2, \ldots, v_{d_t}\}$ and $h(v_1) + g(v_1) \geq h(v_2) + g(v_2) \geq \cdots \geq h(v_{d_t}) + g(v_{d_t})$.
If $h(t) + \sum_{w\in C(t)\backslash\{v_1\}}(h(w) + g(w)) < 0$ or
$h(t) + \sum_{w\in C(v)\backslash\{v_1,v_2\}}(h(w) + g(w)) < 0$ **then** output $r_{[1,2]}(T) < n - \ell$ **else**
 (a) Define $g(t) = \sum_{w\in C(t)\backslash\{v_1,v_2\}}(h(w) + g(w))$
 (b) Label t with $(h(t), g(t))$.

Theorem 2.3. *Let T be the input tree of Algorithm 1. If there exists $v \in I(T)\setminus\{t\}$ with $|C(v)\cap(A(T)\cup L(T))| \leq 1$ such that $h(v)+\sum_{w\in C(v)}(h(w)+g(w)) < 0$ or $h(v) + \sum_{w\in C(v)\backslash\{v_1\}}(h(w) + g(w)) < 0$, then $r_{[1,2]}(T) < n - \ell$.*

Proof. In order to prove this Theorem, we design Algorithm 2 as follows.

Algorithm 2::
Input:: Tree T and a root vertex t.
Output:: $S \subseteq V(T)$
Step 0:: Say $v \in L_i$. Define $S = \{v\}$.
Step 1:: If $h(v) + \sum_{w \in C(v)}(h(w) + g(w)) < 0$, $S = S \cup C(v)$; If $h(v) + \sum_{w \in C(v) \setminus \{v_1\}}(h(w)+g(w)) < 0$, $S = (S \cup (C(v) \setminus \{v_1\})) \cup (\{v_1\} \cap L(T))$.
Step 2:: **For** $j = i+1$ to $diam(T) - 3$
  For each $v \in S \cap (L_j \setminus L(T))$
  Say $C(v) = \{v_1, v_2, \ldots, v_{d_v - 1}\}$ and $h(v_1) + g(v_1) \geq h(v_2) + g(v_2) \geq \cdots \geq h(v_{d_v-1}) + g(v_{d_v-1})$.
  If $g(v) = \sum_{w \in C(v) \setminus (v_1, v_2)}(h(w) + g(w))$ **then** $S = (S \cup (C(v) \setminus \{v_1, v_2\})) \cup (C(v) \cap L(T))$ **else** $S = S \cup (C(v) \cap L(T))$
  End-for
End-for

By Algorithm 2, we have a subset $S \subseteq V(T)$.

It is easy to prove that $T[S]$ is a subtree of T_v. Furthermore, $|V(T[S]) \cap L(T)| - |V(T[S]) \cap I(T)| = h(v) + \sum_{w \in C(v)}(h(w) + g(w)) < 0$ or $|V(T[S]) \cap L(T)| - |V(T[S]) \cap I(T)| = h(v) + \sum_{w \in C(v) \setminus \{v_1\}}(h(w) + g(w)) < 0$. It is obvious that $(I(T) \setminus (V(T[S]) \cap I(T))) \cup (V(T[S]) \cap L(T))$ is a $[1,2]$-dominating set of T. Hence, $r_{[1,2]}(T) \leq |(I(T) \setminus (V(T[S]) \cap I(T))) \cup (V(T[S]) \cap L(T))| = |I(T)| - |(V(T[S]) \cap I(T))| + |(V(T[S]) \cap L(T))| < I(T) = n - \ell$. □

By a similar proof as above, the following result holds.

Theorem 2.4. *Let T be the input tree of Algorithm 1. Suppose that every vertex $v \in I(T) \setminus \{t\}$ has been labeled. If $h(t) + \sum_{w \in C(t) \setminus \{v_1\}}(h(w) + g(w)) < 0$ or $h(t) + \sum_{w \in C(v) \setminus \{v_1, v_2\}}(h(w) + g(w)) < 0$, then $r_{[1,2]}(T) < n - \ell$.*

Theorem 2.5. *Suppose that t is labeled by Algorithm 1. Let S be a $\gamma_{[1,2]}$-set of T. Define $|A(w)| = |S \cap V(T_w)| - |I(T) \cap V(T_w)|$ for any $w \in V(T)$.*
For any $v \in I(T)$, we have
(1) *If $v \notin S$, then $|A(v)| \geq h(v) + g(v)$.*
(2) *If $v \in S$, then $|A(v)| \geq 0$.*

Proof. Suppose $v \in L_i$. We will prove it by induction on i.

Suppose that $i = diam(T) - 3$. If $v \notin S$, then $v \notin A(T)$ and $C(v) \subseteq S$. By Algorithm 1, $g(v) = 0$. Then $|A(v)| = h(v) + g(v)$. If $v \in S$, then it is obvious that $|A(v)| = 0$.

Suppose that the two results hold for $i = diam(T) - 3, \ldots, \ell + 1$. We will prove that the theorem holds for $i = l$. We will discuss it from the following two cases.

Case 1 $v \notin S$. Then

$$|A(v)| = (-1) + \sum_{w \in C(v) \cap S} |A(w)| + \sum_{w \in C(v) \setminus S} |A(w)|$$

$$= (-1) + \sum_{w\in(C(v)\cap S)\cap L(T)} |A(w)| + \sum_{w\in(C(v)\cap S)\setminus L(T)} |A(w)|$$
$$+ \sum_{w\in C(v)\setminus S} |A(w)|$$
$$= h(v) + \sum_{w\in(C(v)\cap S)\setminus L(T)} |A(w)| + \sum_{w\in C(v)\setminus S} |A(w)|.$$

Since $w \in C(v)$ and $v \in L_i$, it follows that $w \in L_{i+1}$. By inductive hypothesis, it follows that

$$\sum_{w\in C(v)\setminus S} |A(w)| \geq \sum_{w\in C(v)\setminus S} (h(w) + g(w))$$

and

$$\sum_{w\in(C(v)\cap S)\setminus L(T)} |A(w)| \geq \sum_{w\in(C(v)\cap S)\setminus L(T)} 0 = 0.$$

Hence,

$$|A(v)| \geq h(v) + \sum_{w\in C(v)\setminus S} (h(w) + g(w)).$$

Since S is a $\gamma_{[1,2]}$-set of T, it follows that $|C(v) \cap S| \leq 2$. That is $|C(v) \setminus S| \geq |C(v)| - 2$.

Suppose that $|C(v)\cap(L(T)\cup A(T))| = 2$. Then $\sum_{w\in C(v)\setminus S}(h(w)+g(w)) = g(v)$. Hence, $|A(v)| \geq h(v) + g(v)$.

Suppose that $|C(v) \cap (L(T) \cup A(T))| = 1$. Then $h(v) \leq 0$. By Algorithm 1, $h(v) + \sum_{w\in C(v)\setminus\{v_1\}} (h(w) + g(w)) \geq 0$. So, $h(v_2) + g(v_2) \geq 0$. Hence, $|A(v)| \geq h(v)+\sum_{w\in C(v)\setminus S}(h(w)+g(w)) \geq h(v)+\sum_{w\in C(v)\setminus\{v_1,v_2\}}(h(w)+g(w)) = h(v)+g(v)$.

Suppose that $|C(v)\cap(L(T)\cup A(T))| = 0$. Then $h(v) = -1$. By Algorithm 1, $h(v) + \sum_{w\in C(v)} (h(w) + g(w)) \geq 0$ and $h(v) + \sum_{w\in C(v)\setminus\{v_1\}} (h(w) + g(w)) \geq 0$. So, $h(v_1)+g(v_1) \geq 0$ and $h(v_2)+g(v_2) \geq 0$. Hence, $h(v)+\sum_{w\in C(v)\setminus S}(h(w)+g(w)) \geq h(v) + \sum_{w\in C(v)\setminus\{v_1,v_2\}} (h(w) + g(w)) = h(v) + g(v)$.

Hence, if $v \notin S$, then $|A(v)| \geq h(v) + g(v)$.

Case 2 $v \in S$. Then $|A(v)| = 0 + \sum_{w\in C(v)\cap S} |A(w)| + \sum_{w\in C(v)\setminus S} |A(w)|$.

Since $w \in C(v)$ and $v \in L_i$, it follows that $w \in L_{i+1}$. By inductive hypothesis, it follows that $\sum_{w\in C(v)\cap S} |A(w)| \geq \sum_{w\in C(v)\cap S} 0 = 0$.

Hence,

$$|A(v)| \geq \sum_{w\in C(v)\setminus S} |A(w)|$$

$$= \sum_{w\in C(v)\setminus(S\cup L(T))} [(-1) + \sum_{w'\in C(w)\cap S} |A(w')| + \sum_{w'\in C(w)\setminus S} |A(w')|]$$
$$= \sum_{w\in C(v)\setminus(S\cup L(T))} [h(w) + \sum_{w'\in (C(w)\cap S)\setminus L(T)} |A(w')| + \sum_{w'\in C(w)\setminus S} |A(w')|].$$

By inductive hypothesis, $|A(w')| \geq 0$ for any $w' \in (C(w)\cap S)\setminus L(T)$ and $|A(w')| \geq h(w') + g(w')$ for any $w' \in C(w)\setminus S$. So

$$|A(v)| \geq \sum_{w\in C(v)\setminus(S\cup L(T))} [h(w) + \sum_{w'\in C(w)\setminus S} (h(w') + g(w'))].$$

Since $v \in S$ and $w \notin S$, it follows that $|C(w)\cap (A(T)\cup L(T))| \leq 1$ and $|C(w)\setminus S| \geq |C(w)| - 1$. Since t is labeled by Algorithm 1, it follows that $h(w) + \sum_{w'\in C(w)\setminus S} (h(w') + g(w')) \geq 0$. So $|A(v)| \geq \sum_{w\in C(v)\setminus S} 0 \geq 0$.

Hence, if $v \in S$, then $|A(v)| \geq 0$. □

Theorem 2.6. *Let $T \in \Gamma'$ be the tree rooted at vertex t. Then $\gamma_{[1,2]}(T) = n - \ell$ if and only if vertex t is labeled by Algorithm 1.*

Proof. Suppose that $\gamma_{12}(T) = n - \ell$. By Theorem 2.3, Theorem 2.4 and Algorithm 1, vertex t is labeled by Algorithm 1.

Conversely, we assume that vertex t is labeled by Algorithm 1. Let S be a $\gamma_{[1,2]}$-set of T. Suppose that $t \in S$. By Theorem 2.5, it follows that $|A(T)| = |S \cap V(T_t)| - |I(T)\cap V(T_t)| \geq 0$. Since $|S\cap V(T_t)| = |S|$ and $|I(T)\cap V(T_t)| = |I(T)|$, it follows that $|S| \geq |I(T)|$. Suppose that $t \notin S$. By Theorem 2.5, it follows that $|A(T)| \geq h(t) + g(t)$. Since vertex t is labeled by Algorithm 1, it follows that $h(t)+g(t) \geq 0$. So $|A(T)| = |S\cap V(T_t)| - |I(T)\cap V(T_t)| \geq 0$. That is $|S| \geq |I(T)|$.

Therefore, for any cases, we have $|S| \geq |I(T)|$. It is obvious that $|S| \leq |I(T)|$. Hence $\gamma_{[1,2]}(T) = |S| = |I(T)| = n - \ell$. □

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