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THE TOEPLITZNESS OF WEIGHTED COMPOSITION OPERATORS

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Dedicated to the memory of Takahiko Nakazi

Abstract. We will consider the asymptotic toeplitzness associated with weighted composition operators on the Hardy-Hilbert space H^2.

1. Introduction

Throughout let H^2 be the Hardy-Hilbert space of all analytic functions on the open unit disk $\mathbb{D}$ with square-summable Taylor coefficients. For $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ in H^2, identifying functions in H^2 with their boundary functions, the standard inner product is defined as

$$\begin{aligned}\langle f, g\rangle &= \sum_{n=0}^{\infty} a_n \overline{b_n} \\ &= \int_{\partial\mathbb{D}} f(e^{i\theta})\overline{g(e^{i\theta})}dm(\theta),\end{aligned}$$

where m is the normalized Lebesgue measure on the boundary $\partial\mathbb{D}$ of $\mathbb{D}$. Refer to [8, 15] for the basic properties of the classical Hardy spaces.

Let T be a bounded linear operator on H^2. Then T is a Toeplitz operator if and only if $S^*TS = T$, where S is the forward shift defined by $Sf(z) = zf(z)$ for $z \in \partial\mathbb{D}$ and $f \in H^2$ and S^* is the backward shift on H^2. In the natural way, for a bounded measurable function $u \in L^\infty(\partial\mathbb{D})$, a Toeplitz operator T_u on H^2 is defined as $T_u f = P(uf)$ for $f \in H^2$, where P is the orthogonal projection from $L^2(\partial\mathbb{D})$ to H^2. Recall that the compact Toeplitz operator on H^2 is only the zero operator. See [6, 13] for operator theory on H^2.

In [1], Barría and Halmos firstly called an operator T on H^2 asymptotically Toeplitz if the sequence of operators $\{S^{*^n} T S^n\}$ converges strongly on H^2. Then Feintuch [9] suggested the analogous conditions relative to either weak or norm

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operator convergence. So there are actually three different kinds of asymptotic toeplitzness.

Definition. Let T be a bounded linear operator on H^2.

(i) T is said to be *uniformly asymptotically Toeplitz* if there is a bounded linear operator A on H^2 such that $\|S^{*^n}TS^n - A\| \to 0$ as $n \to \infty$.

(ii) T is said to be *strongly asymptotically Toeplitz* if there is an operator A on H^2 such that $\|(S^{*^n}TS^n - A)f\| \to 0$ as $n \to \infty$ for any $f \in H^2$.

(iii) T is said to be *weakly asymptotically Toeplitz* if there is an operator A on H^2 such that $\langle (S^{*^n}TS^n - A)f, g\rangle \to 0$ as $n \to \infty$ for all $f, g \in H^2$.

Feintuch [9] showed the following result.

Theorem of Feintuch. *A bounded linear operator on H^2 is uniformly asymptotically Toeplitz if and only if it is the sum of a Toeplitz operator and a compact operator.*

The asymptotic toeplitzness of composition operators originally was considered by Shapiro. For an analytic self-map φ of $\mathbb{D}$, the composition operator C_φ is defined by $C_\varphi f = f \circ \varphi$. It has been known for a long time that such operators are bounded linear operators on H^2. See [3, 16, 19] for the study of composition operators. Nazarov and Shapiro [14] investigated properties of the asymptotic toeplitzness of composition operators and adjoints. Also, refer to [17, 18] for a survey of early results on the toeplitzness of composition operators. Recently the toeplitzness of products of composition operators and their adjoints is independently investigated in [4, 7].

The concept of composition operators has been generalized to weighted composition operators. Let u be a non-zero bounded analytic function on $\mathbb{D}$ and φ an analytic self-map of $\mathbb{D}$. We define the weighted composition operator M_uC_φ by

$$M_uC_\varphi f = u \cdot (f \circ \varphi)$$

for $f \in H^2$. Then M_uC_φ is a bounded linear operator on H^2.

In this article we would consider the asymptotic toeplitzness associated with weighted composition operators on H^2.

2. Toeplitzness of weighted composition operators

First we consider the condition for the weighted composition operator to be a Toeplitz operator.

Theorem 2.1. *Let u be a non-zero bounded analytic function on $\mathbb{D}$ and φ a non-constant analytic self-map of $\mathbb{D}$. Then M_uC_φ is Toeplitz if and only if φ is the identity.*

Proof. By the definition, M_uC_φ is Toeplitz if and only if $S^*M_uC_\varphi S = M_uC_\varphi$. Then, taking $f \equiv 1$, $S^*M_uC_\varphi S\ 1 = M_uC_\varphi\ 1$ and $S^*(u\varphi) = u$. So

$$\frac{u(z)\varphi(z) - u(0)\varphi(0)}{z} = u(z) \quad \text{and} \quad u(z)(\varphi(z) - z) = u(0)\varphi(0).$$

Next, taking $f(z) \equiv z$, $S^* M_u C_\varphi S\ z = M_u C_\varphi\ z$ and $S^*(u\varphi^2) = u\varphi$. Thus

$$u(z)\varphi(z)(\varphi(z) - z) = u(0)\varphi^2(0).$$

Consequently it holds that $\varphi(z)u(0)\varphi(0) = u(0)\varphi^2(0)$. If $u(0)\varphi(0) \neq 0$, $\varphi(z) = \varphi(0)$ =constant and this is a contradiction. If $u(0)\varphi(0) = 0$, then $u(z)(\varphi(z) - z) = 0$. By the analyticity, $\varphi(z) \equiv z$. □

In [12], Toeplitzness of weighted composition operators on H^2 is considered from another viewpoint.

Due to Feintuch's theorem, we can show the following.

Theorem 2.2. *Let u be a non-zero bounded analytic function on $\mathbb{D}$ and φ a non-constant analytic self-map of $\mathbb{D}$. Then $M_u C_\varphi$ is uniformly asymptotically Toeplitz if and only if $M_u C_\varphi$ is compact or φ is the identity.*

Proof. Suppose that $M_u C_\varphi$ is uniformly asymptotically Toeplitz. By Theorem of Feintuch, it holds that $M_u C_\varphi - T_f = K$, where $f \in L^\infty(\partial\mathbb{D})$ and K is a compact operator. Moreover, assume that $M_u C_\varphi$ is not compact and φ is not the identity. Let $K_\lambda(z) = 1/(1 - \overline{\lambda} z)$ for each $\lambda \in \mathbb{D}$ and $k_\lambda(z) = \sqrt{1 - |\lambda|^2} K_\lambda(z)$. By the compactness of K, $\langle (M_u C_\varphi - T_f)^* k_\lambda, k_\lambda \rangle \to 0$ as $|\lambda| \to 1$.

$$\begin{aligned}\langle (M_u C_\varphi - T_f)^* k_\lambda, k_\lambda \rangle &= (1 - |\lambda|^2)\langle \overline{u(\lambda)} K_{\varphi(\lambda)}, K_\lambda \rangle - \langle \overline{f} k_\lambda, k_\lambda \rangle \\ &= \frac{(1 - |\lambda|^2)\overline{u(\lambda)}}{1 - \overline{\varphi(\lambda)}\lambda} - P[\overline{f}](\lambda),\end{aligned}$$

where $P[\overline{f}]$ is the Poisson integral of $\overline{f}$.

As φ is not the identity, $\varphi(e^{i\theta}) \neq e^{i\theta}$ a.e. on $\partial\mathbb{D}$. Whenever λ tends to $e^{i\theta}$, then

$$\langle (M_u C_\varphi - T_f)^* k_\lambda, k_\lambda \rangle \to -\overline{f(e^{i\theta})} = 0 \quad \text{a.e. on} \quad \partial\mathbb{D}.$$

So $M_u C_\varphi = K$ and this contradicts that $M_u C_\varphi$ is not compact. □

The compactness of $M_u C_\varphi$ on H^2 is an interesting problem and was characterized in [2, 5] in terms of Carleson measures, but there would not be the function-theoretic characterization. Gunatillake [10] characterized some sufficient conditions for the compactness, assuming the continuity of u and φ. We here would present the condition independently of the continuity hypothesis, using the similar notion as in [11].

For a non-constant analytic self-map φ of $\mathbb{D}$, denote $\Gamma(\varphi) = \{e^{i\theta} \in \partial\mathbb{D} : |\varphi(e^{i\theta})| = 1\}$, where we are identifying φ with its boundary function. For each $r, 0 < r < 1$, let

$$\{|\varphi| > r\} = \{e^{i\theta} \in \partial\mathbb{D} : |\varphi(e^{i\theta})| > r\}$$

and $\|f\|_\infty$ the essential supremum norm of a function f on $\partial\mathbb{D}$.

Theorem 2.3. *Let u be a non-zero bounded analytic function on $\mathbb{D}$ and φ a non-constant analytic self-map of $\mathbb{D}$. If M_uC_φ is compact on H^2, then $|\varphi| < 1$ a.e. on $\partial\mathbb{D}$.*

Moreover, if $\|u\chi_{\{|\varphi|>r\}}\|_\infty \to 0$ as $r \to 1$, then M_uC_φ is compact on H^2.

Proof. Let $\{z^n\}$ be an orthogonal basis in H^2. By the compactness,

$$\|(M_uC_\varphi)z^n\| \to 0 \text{ as } n \to \infty.$$

Suppose $m(\Gamma(\varphi)) > 0$. Then

$$\begin{aligned}\|(M_uC_\varphi)z^n\|^2 &= \int_{\partial\mathbb{D}} |u(e^{i\theta})|^2|\varphi(e^{i\theta})|^{2n}\, dm(\theta)\\ &\geq \int_{\Gamma(\varphi)} |u(e^{i\theta})|^2\, dm(\theta).\end{aligned}$$

So we obtain $u = 0$ on $\Gamma(\varphi)$. By the analyticity of u, $u \equiv 0$. This is a contradiction.

Moreover we assume that $\|u\chi_{\{|\varphi|>r\}}\|_\infty \to 0$ as $r \to 1$. For any $\varepsilon > 0$, there exists a constant δ, $0 < \delta < 1$ such that $\|u\chi_{\{|\varphi|>r\}}\|_\infty < \varepsilon$ for $\delta < r < 1$.

Let $\{f_n\}$ in H^2 with $\|f_n\| \leq 1$ such that f_n converges to 0 uniformly on compact subsets of $\mathbb{D}$. Then we have

$$\begin{aligned}&\|(M_uC_\varphi)f_n\|^2\\ &= \int_{\{|\varphi|>r\}} |u(e^{i\theta})f_n(\varphi(e^{i\theta}))|^2\, dm(\theta) + \int_{\{|\varphi|\leq r\}} |u(e^{i\theta})f_n(\varphi(e^{i\theta}))|^2\, dm(\theta)\\ &\leq \varepsilon^2\|C_\varphi f_n\|^2 + \sup_{|\varphi(e^{i\theta})|\leq r} |f_n(\varphi(e^{i\theta}))|^2\|u\|_\infty^2.\end{aligned}$$

Taking $n \to \infty$, $\|(M_uC_\varphi)f_n\|^2 \leq \varepsilon^2\|C_\varphi\|^2$. As ε is arbitrary, M_uC_φ is compact on H^2. □

For example, $u(z) = 1 - z$ and $\varphi(z) = (1 + z)/2$ satisfy this condition. Indeed, $|\varphi(e^{i\theta})| = \left|\cos\frac{\theta}{2}\right|$ and

$$\sup_{\{|\varphi|>r\}} |1 - e^{i\theta}| = \sup_{\{|\varphi|>r\}} 2\left|\sin\frac{\theta}{2}\right| \leq 2\sqrt{1 - r^2}.$$

As $r \to 1$, $\sup_{\{|\varphi|>r\}} |1 - e^{i\theta}| \to 0$. So M_uC_φ is compact on H^2.

Let $u(z) = \exp\left(\frac{z+1}{z-1}\right)$ be a singular inner function and $\varphi(z) = (1 + z)/2$ as in [10]. Then $\|u\chi_{\{|\varphi|>r\}}\|_\infty \not\to 0$ as $r \to 1$. In fact, M_uC_φ is not compact on H^2.

Next we consider the strongly asymptotically Toeplitzness. If M_uC_φ is compact, then M_uC_φ is uniformly asymptotically Toeplitz and so strongly (weakly) asymptotically Toeplitz.

Theorem 2.4. *Let u be a non-zero bounded analytic function on $\mathbb{D}$ and φ a non-constant analytic self-map of $\mathbb{D}$ such that M_uC_φ is not compact. If $|\varphi| < 1$ a.e. on $\partial\mathbb{D}$, then M_uC_φ is strongly (and so weakly) asymptotically Toeplitz with asymptotic symbol zero.*

Proof. For $f \in H^2$,

$$\begin{aligned}\|S^{*^n}(M_uC_\varphi)S^n f\|^2 &\leq \|u\varphi^n f\circ\varphi\|^2 \\ &= \int_{\partial\mathbb{D}} |u(e^{i\theta})\varphi^n(e^{i\theta})f(\varphi(e^{i\theta}))|^2 \; dm(\theta) \\ &= \int_{\{|\varphi|<1\}} |u(e^{i\theta})\varphi^n(e^{i\theta})f(\varphi(e^{i\theta}))|^2 \; dm(\theta).\end{aligned}$$

By the Lebesgue Dominated Convergence Theorem, $\|S^{*^n}(M_uC_\varphi)S^n f\| \to 0$. Thus M_uC_φ is strongly asymptotically Toeplitz with asymptotic symbol zero. □

We could obtain the converse of the theorem above under the hypothesis.

Theorem 2.5. *Let u be a non-zero bounded analytic function on $\mathbb{D}$ and φ a non-constant analytic self-map of $\mathbb{D}$ with $\varphi(z) \not\equiv z$. Suppose that $\varphi(0) = 0$. If M_uC_φ is strongly asymptotically Toeplitz with asymptotic symbol zero, then $|\varphi| < 1$ a.e. on $\partial\mathbb{D}$.*

Proof. As $\varphi(0) = 0$, $\varphi(z) = z\psi(z)$ where ψ is analytic on $\mathbb{D}$. Then

$$S^{*^n}(M_uC_\varphi)S^n 1 = S^{*^n}(u\varphi^n) = u\psi^n.$$

Suppose $m(\Gamma(\varphi)) > 0$. We have

$$\begin{aligned}\|S^{*^n}(M_uC_\varphi)S^n 1\|^2 &= \int_{\partial\mathbb{D}} |u(e^{i\theta})\psi^n(e^{i\theta})|^2 \; dm(\theta) \\ &\geq \int_{\Gamma(\varphi)} |u(e^{i\theta})|^2 \; dm(\theta).\end{aligned}$$

Thus, since $\|S^{*^n}(M_uC_\varphi)S^n 1\| \to 0$ as $n \to \infty$, $u = 0$ on $\Gamma(\varphi)$. By the analyticity of u, $u \equiv 0$. This is a contradiction. □

Finally we obtain the criterion for M_uC_φ to be weakly asymptotically Toeplitz.

Theorem 2.6. *Let u be a non-zero bounded analytic function on $\mathbb{D}$ and φ a non-constant analytic self-map of $\mathbb{D}$. If M_uC_φ is weakly asymptotically Toeplitz with asymptotic symbol zero, then φ is not a nontrivial rotation. Furthermore, if φ is not a rotation with $\varphi(0) = 0$, M_uC_φ is weakly asymptotically Toeplitz with asymptotic symbol zero.*

The proof is done by the same way as in [14]. In this case the behavior of the weight u does not cause the weakly asymptotic Toeplitzness.

3. Adjoint asymptotic toeplitzness

In this section we consider the adjoint of M_uC_φ. But it is easily checked that the Toeplitzness, uniformly asymptotic Toeplitzness and weakly asymptotic Toeplitzness of $(M_uC_\varphi)^*$ are ones of M_uC_φ.

We could show the following by the same method as in [14].

Theorem 3.1. *Let u be a non-zero bounded analytic function on $\mathbb{D}$ and φ a non-constant analytic self-map of $\mathbb{D}$. Suppose that $\varphi(0) = 0$ and φ is not a rotation. Then $(M_uC_\varphi)^*$ is strongly asymptotically Toeplitz.*

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