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EFFECT OF INTEGER TRANSLATION ON RELATIVE ORDER AND RELATIVE TYPE OF ENTIRE AND MEROMORPHIC FUNCTIONS

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Abstract. In this paper some newly developed results based on the growth properties of relative order (relative lower order), relative type (relative lower type) and relative weak type of entire and meromorphic functions on the basis of integer translation applied upon them are investigated.

1. Introduction

Let $f(z)$ be an *entire function* defined in the finite complex plane $\mathbb{C}$. The *maximum modulus function* corresponding to *entire* $f(z)$ is defined as $M_f(r) = \max\{|f(z)| : |z| = r\}$. When $f(z)$ is *meromorphic*, one may define a different function $T_f(r)$ termed as *Nevanlinna's Characteristic function* of $f(z)$, playing same role as *maximum modulus* function in the following manner:

$$T_f(r) = N_f(r) + m_f(r),$$

where the function $N_f(r,a)$ $\left(\overline{N}_f(r,a)\right)$ known as *counting function* of a-points (distinct a-points) of *meromorphic* f is defined as

$$N_f(r,a) = \int_0^r \frac{n_f(t,a) - n_f(0,a)}{t} dt + n_f(0,a) \log r$$

$$\left(\overline{N}_f(r,a) = \int_0^r \frac{\overline{n}_f(t,a) - \overline{n}_f(0,a)}{t} dt + \overline{n}_f(0,a) \log r \right),$$

moreover we denote by $n_f(r,a)$ $(\overline{n}_f(r,a))$ the number of a-points (distinct a-points) of f in $|z| \leq r$ and an ∞ -point is a *pole* of $f(z)$. In many occasions $N_f(r,\infty)$ and $\overline{N}_f(r,\infty)$ are denoted by $N_f(r)$ and $\overline{N}_f(r)$ respectively.

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Further, the function $m_f(r,\infty)$ alternatively denoted by $m_f(r)$ known as the *proximity function* of $f(z)$ is defined as follows:

$$m_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(re^{i\theta}\right)\right| d\theta, \quad \text{where}$$
$$\log^+ x = \max(\log x, 0) \ \text{ for all } x \geqslant 0.$$

Also we may denote $m\left(r, \frac{1}{f-a}\right)$ by $m_f(r,a)$.

If $f(z)$ is an entire function, then the *Nevanlinna's Characteristic function* $T_f(r)$ of $f(z)$ is defined as

$$T_f(r) = m_f(r).$$

Further let $f(z)$ be a *meromorphic function* and $n \in \mathbb{N}$, then the *translation* of $f(z)$ be denoted by $f(z+n)$. For each $n \in \mathbb{N}$, one may obtain a function with some properties. Let us consider this family by $f_n(z)$ where

$$f_n(z) = \{f(z+n) : n \in \mathbb{N}\}.$$

We should recall that if α is a regular point of an analytic function $f(z)$ and if $f(\alpha) = 0$, then α is called a *zero* of $f(z)$. The point $z = \alpha$ is called a *zero* of $f(z)$ of order or multiplicity m (m being a positive integer) if in some neighbourhood of α, $f(z)$ can be expanded in a Taylor's series of the form $f(z) = \sum\limits_{n=m}^{\infty} a_n (z-\alpha)^n$ where $a_m \neq 0$.

It is clear that the number of *zeros* of $f(z)$ may be changed in a finite region after translation but it remains unaltered in the open complex plane $\mathbb{C}$, i.e.,

$$N_{f(z+n)}(r) = N_f(r) + e_n, \tag{1}$$

where e_n is a residue term such that $e_n \to 0$ as $r \to \infty$.

Also

$$m_{f(z+n)}(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(re^{i\theta}+n\right)\right| d\theta$$

$$\text{i.e., } m_{f(z+n)}(r) = m_f(r) + e_n^{'}, \tag{2}$$

where $e_n^{'}$ (may be distinct from e_n) be such that $e_n^{'} \to 0$ as $r \to \infty$.

Therefore from (1) and (2), one may obtain that

$$N_{f(z+n)}(r) + m_{f(z+n)}(r) = N_f(r) + e_n + m_f(r) + e_n^{'}$$
$$\text{i.e., } T_{f(z+n)}(r) = T_f(r) + e_n + e_n^{'}.$$

Now if n varies, then the Nevanlinna's Characteristic function for the family $f_n(z)$ is

$$T_{f_n}(r) = nT_f(r) + \sum_n \left(e_n + e_n^{'}\right). \tag{3}$$

However for any two *meromorphic functions* $f(z)$ and $g(z)$ the ratio $\frac{T_f(r)}{T_g(r)}$ as $r \to \infty$ is called the *growth* of $f(z)$ with respect to $g(z)$ in terms of their *Nevanlinna's Characteristic functions.*

The *order* of a meromorphic function f which is generally used in computational purpose is defined in terms of the growth of f with respect to the *exponential function* as

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log T_{\exp z}(r)} = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log\left(\frac{r}{\pi}\right)} = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log(r) + O(1)} .$$

Lahiri and Banerjee [4] introduced the *relative order* of a *meromorphic function* with respect to an *entire function* to avoid comparing growth just with $\exp z$. Extending the notion of *relative order* as cited in the reference, Datta and Biswas [1] gave the definition of *relative type* and *relative weak type* of a *meromorphic function* with respect to an *entire function.* In this paper we establish some newly developed results based on the growth properties of *relative order* (*relative lower order*), *relative type* (*relative lower type*) and *relative weak type* of entire and meromorphic functions on the basis of *integer translation* applied upon them.

2. Notation and preliminary remarks

We use the standard notations and definitions of the theory of entire and meromorphic functions which are available in [3] and [5]. Henceforth, we do not explain those in details. Now we just recall some definitions which will be needed in the sequel.

Definition 1. The *order* ρ_f and *lower order* λ_f of a meromorphic function $f(z)$ are defined as

$$\rho_f = \limsup_{r\to\infty} \frac{\log T_f(r)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log T_f(r)}{\log r}.$$

The notion of *type* (*lower type*) to determine the relative growth of two meromorphic functions having same non zero finite order is classical in complex analysis and is given by

Definition 2. The *type* σ_f and *lower type* $\overline{\sigma}_f$ of a meromorphic function $f(z)$ are defined as

$$\sigma_f = \limsup_{r\to\infty} \frac{T_f(r)}{r^{\rho_f}} \quad \text{and} \quad \overline{\sigma}_f = \liminf_{r\to\infty} \frac{T_f(r)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.$$

Analogously to determine the relative growth of two meromorphic functions having same non zero finite lower order, Datta and Jha [2] introduced the definition of *weak type* of a meromorphic function of finite positive lower order in the following way:

Definition 3 ([2])**.** The *weak type* τ_f of a meromorphic function $f(z)$ of finite positive lower order λ_f is defined by

$$\tau_f = \liminf_{r\to\infty} \frac{T_f(r)}{r^{\lambda_f}}.$$

Similarly, one can define the growth indicator $\overline{\tau}_f$ of a meromorphic function f of finite positive lower order λ_f as

$$\overline{\tau}_f = \limsup_{r\to\infty} \frac{T_f(r)}{r^{\lambda_f}}.$$

Given a non-constant entire function $f(z)$ defined in the open complex plane $\mathbb{C}$, its Nevanlinna's Characteristic function is strictly increasing and continuous. Hence there exists its inverse function $T_f^{-1} : (T_f(0), \infty) \to (0, \infty)$ with $\lim_{s\to\infty} T_f^{-1}(s) = \infty$.

Lahiri and Banerjee [4] introduced the definition of *relative order* of a meromorphic function $f(z)$ with respect to an entire function $g(z)$, denoted by $\rho_g(f)$ as follows:

$$\begin{aligned} \rho_g(f) &= \inf\{\mu > 0 : T_f(r) < T_g(r^{\mu}) \text{ for all sufficiently large } r\} \\ &= \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}. \end{aligned}$$

The definition coincides with the classical one [4] if $g(z) = \exp z$. Similarly, one can define the *relative lower order* of a meromorphic function $f(z)$ with respect to an entire $g(z)$ denoted by $\lambda_g(f)$ in the following manner:

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.$$

In the case of relative order, it therefore seems reasonable to define suitably the relative type and relative weak type of a meromorphic function with respect to an entire function to determine the relative growth of two meromorphic functions having same non zero finite relative order or relative lower order with respect to an entire function. Datta and Biswas [1] gave such definitions of relative type and relative weak type of a meromorphic function $f(z)$ with respect to an entire function $g(z)$ which are as follows:

Definition 4 ([1])**.** The relative type $\sigma_g(f)$ of a meromorphic function $f(z)$ with respect to an entire function $g(z)$ are defined as

$$\sigma_g(f) = \limsup_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\rho_g(f)}}, \quad \text{where } 0 < \rho_g(f) < \infty.$$

Likewise, one can define the lower relative type $\overline{\sigma}_g(f)$ in the following way:

$$\overline{\sigma}_g(f) = \liminf_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\rho_g(f)}}, \quad \text{where } 0 < \rho_g(f) < \infty.$$

Definition 5 ([1]). The relative weak type $\tau_g(f)$ of a meromorphic function $f(z)$ with respect to an entire function $g(z)$ with finite positive relative lower order $\lambda_g(f)$ is defined by

$$\tau_g(f) = \liminf_{r\to\infty} \frac{T_g^{-1}T_f(r)}{r^{\lambda_g(f)}}.$$

In a like manner, one can define the growth indicator $\overline{\tau}_g(f)$ of a meromorphic function f with respect to an entire function g with finite positive relative lower order $\lambda_g(f)$ as

$$\overline{\tau}_g(f) = \limsup_{r\to\infty} \frac{T_g^{-1}T_f(r)}{r^{\lambda_g(f)}}.$$

3. Main results

In this section we state the main results of the paper. First we recall two related lemmas which are needed in order to prove our results.

Lemma 1 ([2]). *If $f(z)$ is a meromorphic function of regular growth, i.e., $\rho_f = \lambda_f$, then*

$$\sigma_f = \overline{\sigma}_f = \tau_f = \overline{\tau}_f.$$

Lemma 2. *Let $f(z)$ be a meromorphic function. If $f_n(z) = f(z+n)$ for $n \in \mathbb{N}$, then*

$$\lim_{r\to\infty} \frac{T_{f_n}(r)}{T_f(r)} = n.$$

Proof. From (3) we get that

$$\frac{T_{f_n}(r)}{T_f(r)} = n + \frac{\sum_n \left(e_n + e_n^{'}\right)}{T_f(r)},$$

where $e_n \to 0$ and $e_n^{'} \to 0$ as $r \to \infty$. Since $T_f(r)$ is an increasing function of r, we get from above that

$$\lim_{r\to\infty} \frac{T_{f_n}(r)}{T_f(r)} = n.$$

Hence the lemma follows. □

In Lemma 2, we see that the growth rate of $T_{f_n}(r)$ with respect to $T_f(r)$ as $r \to \infty$. Now a question may arise about the limiting value of $\frac{T_{g_m}^{-1}T_{f_n}(r)}{T_g^{-1}T_f(r)}$ as $r \to \infty$ and for any entire function g with $g_m(z) = g(z+m)$ for $m \in \mathbb{N}$. The first theorem may provide the answer in this direction under some additional conditions.

Theorem 1. *Let $f(z)$ be a meromorphic function and $g(z)$ be an entire function with $0 < \tau_g \leq \overline{\tau}_g < \infty$ and $0 < \overline{\sigma}_g \leq \sigma_g < \infty$. If $f_n(z) = f(z+n)$ and $g_m(z) = g(z+m)$ for $m, n \in \mathbb{N}$, then*

$$\max\left\{\left(\frac{n}{m}\right)^{\frac{1}{\lambda_g}}\cdot\left(\frac{\tau_g}{\overline{\tau}_g}\right)^{\frac{1}{\lambda_g}},\left(\frac{n}{m}\right)^{\frac{1}{\rho_g}}\cdot\left(\frac{\overline{\sigma}_g}{\sigma_g}\right)^{\frac{1}{\rho_g}}\right\}\leq\liminf_{r\to\infty}\frac{T_{g_m}^{-1}T_{f_n}(r)}{T_g^{-1}T_f(r)}$$
$$\leq\limsup_{r\to\infty}\frac{T_{g_m}^{-1}T_{f_n}(r)}{T_g^{-1}T_f(r)}\leq\min\left\{\left(\frac{n}{m}\right)^{\frac{1}{\lambda_g}}\cdot\left(\frac{\overline{\tau}_g}{\tau_g}\right)^{\frac{1}{\lambda_g}},\left(\frac{n}{m}\right)^{\frac{1}{\rho_g}}\cdot\left(\frac{\sigma_g}{\overline{\sigma}_g}\right)^{\frac{1}{\rho_g}}\right\}.$$

Proof. For any $\varepsilon(>0)$, we get from Lemma 2 for all sufficiently large values of r that

$$T_{f_n}(r)\leq(n+\varepsilon)\,T_f(r) \tag{4}$$

and

$$T_{f_n}(r)\geq(n-\varepsilon)\,T_f(r)\,. \tag{5}$$

Also from Lemma 2, we get for all sufficiently large values of r that

$$T_{g_m}(r)\geq(m-\varepsilon)\,T_g(r)$$
$$i.e.,\ r\geq T_{g_m}^{-1}\left[(m-\varepsilon)\,T_g(r)\right]$$
$$i.e.,\ T_g^{-1}\left(\frac{r}{m-\varepsilon}\right)\geq T_{g_m}^{-1}(r) \tag{6}$$

and

$$T_{g_m}(r)\leq(m+\varepsilon)\,T_g(r)$$
$$i.e.,\ r\leq T_{g_m}^{-1}\left[(m+\varepsilon)\,T_g(r)\right]$$
$$i.e.,\ T_g^{-1}\left(\frac{r}{m+\varepsilon}\right)\leq T_{g_m}^{-1}(r)\,. \tag{7}$$

Now from (4) and (6) it follows for all sufficiently large values of r that

$$T_{g_m}^{-1}T_{f_n}(r)\leq T_{g_m}^{-1}\left[(n+\varepsilon)\,T_f(r)\right]$$
$$i.e.,\ T_{g_m}^{-1}T_{f_n}(r)\leq T_g^{-1}\left[\left(\frac{n+\varepsilon}{m-\varepsilon}\right)T_f(r)\right]. \tag{8}$$

Again from (5) and (7), it follows for all sufficiently large values of r that

$$T_{g_m}^{-1}T_{f_n}(r)\geq T_{g_m}^{-1}\left[(n-\varepsilon)\,T_f(r)\right]$$
$$i.e.,\ T_{g_m}^{-1}T_{f_n}(r)\geq T_g^{-1}\left[\left(\frac{n-\varepsilon}{m+\varepsilon}\right)T_f(r)\right]. \tag{9}$$

Now for the definition of type and lower type, we get for all sufficiently large values of r that

$$T_g\left(\left\{\frac{T_f(r)}{(\sigma_g+\varepsilon)}\right\}^{\frac{1}{\rho_g}}\right)\leq T_f(r)$$
$$i.e.,\quad T_g^{-1}T_f(r)\geq\left\{\frac{T_f(r)}{(\sigma_g+\varepsilon)}\right\}^{\frac{1}{\rho_g}} \tag{10}$$

and

$$T_g\left(\left\{\left(\frac{n+\varepsilon}{(m-\varepsilon)\left(\overline{\sigma}_g-\varepsilon\right)}\right)T_f(r)\right\}^{\frac{1}{\rho_g}}\right)\geq\left[\left(\frac{n+\varepsilon}{m-\varepsilon}\right)T_f(r)\right]$$

$$\text{(11)}\qquad i.e.,\ \left[\left(\frac{n+\varepsilon}{(m-\varepsilon)\left(\overline{\sigma}_g-\varepsilon\right)}\right)T_f(r)\right]^{\frac{1}{\rho_g}}\geq T_g^{-1}\left[\left(\frac{n+\varepsilon}{m-\varepsilon}\right)T_f(r)\right].$$

Therefore from (8) and (11), it follows for all sufficiently large values of r that

$$\text{(12)}\qquad T_{g_m}^{-1}T_{f_n}(r)\leq\left[\left(\frac{n+\varepsilon}{(m-\varepsilon)\left(\overline{\sigma}_g-\varepsilon\right)}\right)T_f(r)\right]^{\frac{1}{\rho_g}}.$$

Therefore from (10) and (12), it follows for all sufficiently large values of r that

$$\frac{T_{g_m}^{-1}T_{f_n}(r)}{T_g^{-1}T_f(r)}\leq\frac{\left[\left(\frac{n+\varepsilon}{(m-\varepsilon)(\overline{\sigma}_g-\varepsilon)}\right)T_f(r)\right]^{\frac{1}{\rho_g}}}{\left\{\frac{T_f(r)}{(\sigma_g+\varepsilon)}\right\}^{\frac{1}{\rho_g}}}$$

$$i.e.,\ \frac{T_{g_m}^{-1}T_{f_n}(r)}{T_g^{-1}T_f(r)}\leq\left(\frac{(n+\varepsilon)\left(\sigma_g+\varepsilon\right)}{(m-\varepsilon)\left(\overline{\sigma}_g-\varepsilon\right)}\right)^{\frac{1}{\rho_g}}$$

$$\text{(13)}\qquad i.e.,\ \limsup_{r\to\infty}\frac{T_{g_m}^{-1}T_{f_n}(r)}{T_g^{-1}T_f(r)}\leq\left(\frac{n}{m}\right)^{\frac{1}{\rho_g}}\cdot\left(\frac{\sigma_g}{\overline{\sigma}_g}\right)^{\frac{1}{\rho_g}}.$$

Similarly from (9), it can be shown for all sufficiently large values of r that

$$\text{(14)}\qquad \liminf_{r\to\infty}\frac{T_{g_m}^{-1}T_{f_n}(r)}{T_g^{-1}T_f(r)}\geq\left(\frac{n}{m}\right)^{\frac{1}{\rho_g}}\cdot\left(\frac{\overline{\sigma}_g}{\sigma_g}\right)^{\frac{1}{\rho_g}}.$$

Therefore from (13) and (14), we obtain that

$$\left(\frac{n}{m}\right)^{\frac{1}{\rho_g}}\cdot\left(\frac{\overline{\sigma}_g}{\sigma_g}\right)^{\frac{1}{\rho_g}}\leq\liminf_{r\to\infty}\frac{T_{g_m}^{-1}T_{f_n}(r)}{T_g^{-1}T_f(r)}\leq\limsup_{r\to\infty}\frac{T_{g_m}^{-1}T_{f_n}(r)}{T_g^{-1}T_f(r)}$$

$$\text{(15)}\qquad \leq\left(\frac{n}{m}\right)^{\frac{1}{\rho_g}}\cdot\left(\frac{\sigma_g}{\overline{\sigma}_g}\right)^{\frac{1}{\rho_g}}.$$

Similarly, using the weak type one can easily verify that

$$\left(\frac{n}{m}\right)^{\frac{1}{\lambda_g}}\cdot\left(\frac{\tau_g}{\overline{\tau}_g}\right)^{\frac{1}{\lambda_g}}\leq\liminf_{r\to\infty}\frac{T_{g_m}^{-1}T_{f_n}(r)}{T_g^{-1}T_f(r)}\leq\limsup_{r\to\infty}\frac{T_{g_m}^{-1}T_{f_n}(r)}{T_g^{-1}T_f(r)}$$

$$\text{(16)}\qquad \leq\left(\frac{n}{m}\right)^{\frac{1}{\lambda_g}}\cdot\left(\frac{\overline{\tau}_g}{\tau_g}\right)^{\frac{1}{\lambda_g}}.$$

Thus the theorem follows from (15) and (16). □

Corollary 1. *Under the same conditions of Theorem 1, if $g(z)$ is of regular growth, then by Lemma 1 one can easily obtain that*

$$\lim_{r\to\infty}\frac{T_{g_m}^{-1}T_{f_n}(r)}{T_g^{-1}T_f(r)}=\left(\frac{n}{m}\right)^{\frac{1}{\rho_g}}.$$

Theorem 2. *Let $f(z)$ be a meromorphic function and $g(z)$ be an entire function with $0<\lambda_g\le\rho_g<\infty$. If $f_n(z)=f(z+n)$ and $g_m(z)=g(z+m)$ for $m,n\in\mathbb{N}$, then*

$$\frac{\lambda_g}{\rho_g}\le\liminf_{r\to\infty}\frac{\log T_{g_m}^{-1}T_{f_n}(r)}{\log T_g^{-1}T_f(r)}\le\limsup_{r\to\infty}\frac{\log T_{g_m}^{-1}T_{f_n}(r)}{\log T_g^{-1}T_f(r)}\le\frac{\rho_g}{\lambda_g}.$$

Proof. From (8) and (9), we get for all sufficiently large values of r that

$$\log T_{g_m}^{-1}T_{f_n}(r)\le\log T_g^{-1}\left[\left(\frac{n+\varepsilon}{m-\varepsilon}\right)T_f(r)\right]\tag{17}$$

and

$$\log T_{g_m}^{-1}T_{f_n}(r)\ge\log T_g^{-1}\left[\left(\frac{n-\varepsilon}{m+\varepsilon}\right)T_f(r)\right].\tag{18}$$

Now for the definition of order and lower order, we get for all sufficiently large values of r that

$$\begin{aligned}T_g\left(\{T_f(r)\}^{\frac{1}{\rho_g+\varepsilon}}\right)&\le T_f(r)\\ \textit{i.e.,}\quad \log T_g^{-1}T_f(r)&\ge\frac{1}{(\rho_g+\varepsilon)}\log T_f(r)\end{aligned}\tag{19}$$

and

$$T_g\left[\left\{\left(\frac{n+\varepsilon}{(m-\varepsilon)}\right)T_f(r)\right\}^{\frac{1}{\lambda_g-\varepsilon}}\right]\ge\left[\left(\frac{n+\varepsilon}{m-\varepsilon}\right)T_f(r)\right]$$

$$\textit{i.e.,}\quad\left[\left(\frac{n+\varepsilon}{(m-\varepsilon)}\right)T_f(r)\right]^{\frac{1}{\lambda_g-\varepsilon}}\ge T_g^{-1}\left[\left(\frac{n+\varepsilon}{m-\varepsilon}\right)T_f(r)\right]$$

$$\textit{i.e.,}\quad\frac{1}{(\lambda_g-\varepsilon)}\log T_f(r)+O(1)\ge\log T_g^{-1}\left[\left(\frac{n+\varepsilon}{m-\varepsilon}\right)T_f(r)\right].\tag{20}$$

Therefore from (17) and (20), it follows for all sufficiently large values of r that

$$\log T_{g_m}^{-1}T_{f_n}(r)\le\frac{1}{(\lambda_g-\varepsilon)}\log T_f(r)+O(1).\tag{21}$$

Therefore from (19) and (21), it follows for all sufficiently large values of r that

$$\frac{\log T_{g_m}^{-1}T_{f_n}(r)}{\log T_g^{-1}T_f(r)}\le\left(\frac{\rho_g+\varepsilon}{\lambda_g-\varepsilon}\right)\cdot\frac{\log T_f(r)+O(1)}{\log T_f(r)}$$

$$\text{(22)} \qquad i.e., \ \limsup_{r\to\infty} \frac{\log T_{g_m}^{-1} T_{f_n}(r)}{\log T_g^{-1} T_f(r)} \leq \frac{\rho_g}{\lambda_g}.$$

Similarly, from (18) it can be shown for all sufficiently large values of r that

$$\liminf_{r\to\infty} \frac{\log T_{g_m}^{-1} T_{f_n}(r)}{\log T_g^{-1} T_f(r)} \geq \frac{\lambda_g}{\rho_g}. \tag{23}$$

Therefore from (22) and (23) , we obtain that

$$\frac{\lambda_g}{\rho_g} \leq \liminf_{r\to\infty} \frac{\log T_{g_m}^{-1} T_{f_n}(r)}{\log T_g^{-1} T_f(r)} \leq \limsup_{r\to\infty} \frac{\log T_{g_m}^{-1} T_{f_n}(r)}{\log T_g^{-1} T_f(r)} \leq \frac{\rho_g}{\lambda_g}.$$

Thus the theorem follows from above. □

Corollary 2. *Under the same conditions of Theorem 2 if $g(z)$ is of regular growth, then one may get that*

$$\lim_{r\to\infty} \frac{\log T_{g_m}^{-1} T_{f_n}(r)}{\log T_g^{-1} T_f(r)} = 1.$$

As an application of Corollary 2, we prove the following theorems.

Theorem 3. *Let $f(z)$ be a meromorphic function and $g(z)$ be an entire function with regular growth. If $f_n(z) = f(z+n)$ and $g_m(z) = g(z+m)$ for $m, n \in \mathbb{N}$, then the relative order and relative lower order of $f_n(z)$ with respect to $g_m(z)$ are same as those of $f(z)$ with respect to $g(z)$.*

Proof. In view of Corollary 2, we obtain that

$$\begin{aligned}\rho_{g_m}(f_n) &= \limsup_{r\to\infty} \frac{\log T_{g_m}^{-1} T_{f_n}(r)}{\log r} \\ &= \limsup_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r} \cdot \lim_{r\to\infty} \frac{\log T_{g_m}^{-1} T_{f_n}(r)}{\log T_g^{-1} T_f(r)} \\ &= \rho_g(f) \cdot 1 = \rho_g(f).\end{aligned}$$

In a similar manner, $\lambda_{g_m}(f_n) = \lambda_g(f)$.

Thus the theorem follows. □

Theorem 4. *Let $f(z)$ be a meromorphic function and $g(z)$ be an entire function with regular growth. If $f_n(z) = f(z+n)$ and $g_m(z) = g(z+m)$ for $m, n \in \mathbb{N}$, then the relative type and relative lower type of $f_n(z)$ with respect to $g_m(z)$ are $\left(\frac{n}{m}\right)^{\frac{1}{\rho_g}}$ times that of $f(z)$ with respect to $g(z)$ if $\rho_g(f)$ is positive finite.*

Proof. From Corollary 1 and Theorem 3, we get that

$$\begin{aligned}\sigma_{g_m}(f_n) &= \limsup_{r\to\infty} \frac{T_{g_m}^{-1} T_{f_n}(r)}{r^{\rho_{g_m}(f_n)}} \\ &= \lim_{r\to\infty} \frac{T_{g_m}^{-1} T_{f_n}(r)}{T_g^{-1} T_f(r)} \cdot \limsup_{r\to\infty} \frac{T_g^{-1} T_f(r)}{r^{\rho_g(f)}}\end{aligned}$$

$$= \left(\frac{n}{m}\right)^{\frac{1}{\rho_g}} \cdot \sigma_g(f).$$

Similarly, $\overline{\sigma}_{g_m}(f_n) = \left(\frac{n}{m}\right)^{\frac{1}{\rho_g}} \cdot \overline{\sigma}_g(f)$.
This proves the theorem. □

Theorem 5. *Let $f(z)$ be a meromorphic function and $g(z)$ be an entire function with regular growth. If $f_n(z) = f(z+n)$ and $g_m(z) = g(z+m)$ for $m, n \in \mathbb{N}$, then $\tau_{g_m}(f_n)$ and $\overline{\tau}_{g_m}(f_n)$ are $\left(\frac{n}{m}\right)^{\frac{1}{\rho_g}}$ times that of $f(z)$ with respect to $g(z)$, i.e.,*

$$\tau_{g_m}(f_n) = \left(\frac{n}{m}\right)^{\frac{1}{\rho_g}} \cdot \tau_g(f) \text{ and } \overline{\tau}_{g_m}(f_n) = \left(\frac{n}{m}\right)^{\frac{1}{\rho_g}} \cdot \overline{\tau}_g(f)$$

when $\lambda_g(f)$ is positive finite.

We omit the proof of Theorem 5 because it can be carried out in the line of Theorem 4.

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