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CERTAIN GENERALIZED AND MIXED TYPE GENERATING RELATIONS: AN OPERATIONAL APPROACH

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Abstract. In this paper, we discuss how the operational calculus can be exploited to the theory of generalized special functions of many variables and many indices. We obtained the generating relations for 3-index, 3-variable and 1-parameter Hermite polynomials. Some mixed type generating relations and bilateral generating relations of many indices and many variable like Lagurre-Hermite and Hermite-Sister Celine's polynomials are also obtained. Further we generalize some results on old symbolic notations using operational identities.

1. Introduction

Operational methods can be exploited to simplify the derivation of the properties associated with ordinary and generalized special functions and to define new families of functions. In the case of multi-variable generalized special functions, the use of operational techniques combined with the principle of monomiality provides new means of analysis for the solutions of a wide class of partial differential equations often encountered in physical problems.The idea of monomiality came from the concept of poweroid suggested by Steffensen [28]. The importance of the use of operational techniques in the study of special functions and their applications has been recognized by Dattoli and his co-workers, see for example [5, 7, 12, 13].

According to the principle of monomiality the polynomials $p_n(x)$ ($n \in \mathbb{N}$, $x \in \mathbb{C}$) are called quasi-monomials, if two operators $\hat{M}$ and $\hat{P}$, can be defined in such a way that

$$\begin{aligned} \hat{M}\{p_n(x)\} &= p_{n+1}(x), \\ \hat{P}\{p_n(x)\} &= np_{n-1}(x). \end{aligned} \tag{1.1}$$

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The operators $\hat{M}$ and $\hat{P}$ are called multiplicative and derivative operators and can be recognized as raising and lowering operators acting on the polynomials $p_n(x)$. Obviously $\hat{M}$ and $\hat{P}$ satisfy the commutative relation

$$[\hat{P},\ \hat{M}] = 1 \tag{1.2}$$

and thus display a Weyl group structure. Further consequence of (1.1) is the eigen property of $\hat{M}\hat{P}$

$$\hat{M}\hat{P}\{p_n(x)\} = np_n(x). \tag{1.3}$$

The polynomials $p_n(x)$ are obtained by taking the action of $\hat{M}$ on $p_0(x)$

$$p_n(x) = \hat{M}^n\, p_0(x), \tag{1.4}$$

(in the following we shall always set $p_0(x) = 1$) and consequently the exponential generating function of $p_n(x)$ is

$$G(x,t) = \sum_{n=0}^{\infty} p_n(x)\,\frac{t^n}{n!} = \exp(t\hat{M})\{1\}. \tag{1.5}$$

The principle of monomility for Hermite and Lageurre polynomials can be exploited in many useful and flexible ways. The reason of interest for family of Hermite polynomials is due to their mathematical importance and the fact that these polynomials give rise to the eigenstates of the quantum harmonic oscillator.

The 2-variable Hermite Kampd de Fdriet polynomials (2VHKdFP) $H_n(x,y)$ [2], defined by the generating function [5, p. 149, (1.10) and (1.14)]

$$\exp(xt + yt^2) = \sum_{n=0}^{\infty} H_n(x,y)\,\frac{t^n}{n!}, \tag{1.6}$$

have shown to be quasi-monomials under the action of the operators [5, p. 148 (1.9)],

$$\begin{aligned} \hat{M} &= x + 2y\frac{\partial}{\partial x}, \\ \hat{P} &= \frac{\partial}{\partial x}, \end{aligned} \tag{1.7}$$

and are characterized by the operational rule

$$H_n(x,y) = \exp\left(y\frac{\partial^2}{\partial x^2}\right)[x^n]. \tag{1.8}$$

Further, the 3-variable Hermite polynomials (3VHP) $H_n(x,y,z)$ are introduced [4, p. 114 (22)] have the generating function

$$\exp\left(xt + yt^2 + zt^3\right) = \sum_{n=0}^{\infty} H_n(x,y,z)\frac{t^n}{n!}, \tag{1.9}$$

which are quasi-monomials under the action of the operators

$$\begin{aligned}\hat{M} &= x + 2y\frac{\partial}{\partial x} + 3z\frac{\partial^2}{\partial x^2},\\ \hat{P} &= \frac{\partial}{\partial x},\end{aligned} \tag{1.10}$$

and satisfy the following operational rule:

$$H_n(x,y,z) = \exp\Big(y\frac{\partial^2}{\partial x^2} + z\frac{\partial^3}{\partial x^3}\Big)[x^n]. \tag{1.11}$$

Dattoli and Torre [14] introduced and discussed the theory of 2-variable Laguerre polynomials (2VLP) $L_n(x,y)$. The reason of interest for this family of Laguerre polynomials is due to their intrinsic mathematical importance and to the fact that these polynomials are shown to be natural solutions of a particular set of partial differential equations which often appear in the treatment of radiation physics problems such as the electromagnetic wave propagation.

The 2VLP $L_n(x,y)$ are specified by the generating function [14]

$$\frac{1}{1-yt}\exp\Big(\frac{-xt}{1-yt}\Big) = \sum_{n=0}^{\infty} L_n(x,y)t^n, \tag{1.12}$$

are quasi-monomials under the action of the operators [8]

$$\hat{M} = y - D_x^{-1},\ \hat{P} = -\partial_x x \partial_x, \tag{1.13}$$

where D_x^{-1} denotes the inverse of the derivative operator.

2VLP $L_n(x,y)$ can be defined through the operational rule:

$$L_n(x,y) = \exp\Big(-y\Big(\frac{\partial}{\partial x}\ x\ \frac{\partial}{\partial x}\Big)\Big)\frac{(-x)^n}{n!}. \tag{1.14}$$

The use of operational methods in connection with the study of classical special functions, including the multi-dimensional and multi-index case has been recognized by Dattoli and his co-workers. They have shown that the operational methods can be used to simplify the derivations of many properties of ordinary and generalized special functions and also provide a unique tool of analysis to treat various polynomials. Motivated by the recent works [15, 16, 18, 21, 22] and importance of operational methods in introducing new families of polynomials, in this paper, we exploit the operational techniques to find the generating relation for Hermite polynomials of 3-index, 3-variable, 1-parameter $H_{l,m,n}(x,y,z;\tau)$.

2. Generating relation for 3-index, 3-variable, 1-parameter Hermite polynomials

In this section we find the generating relation for 3-index, 3-variable, 1-parameter Hermite polynomials:

The 3-index, 3-variable, 1-parameter Hermite polynomials are defined by the series

$$H_{l,m,n}(x,y,z;\tau) = l!m!n! \sum_{r=0}^{min(l,m,n)} \frac{\tau^r x^{l-r} y^{m-r} z^{n-r}}{r!(l-r)!(m-r)!(n-r)!}, \tag{2.1}$$

and Specified by the following generating function

$$\sum_{l,m,n=0}^{\infty} \frac{u^l v^m w^n}{l!m!n!} H_{l,m,n}(x,y,z;\tau) \ = \exp\left(xu + yv + zw + \tau uvw\right). \tag{2.2}$$

We observe that the polynomials $H_{l,m,n}(x,y,z;\tau)$ are quasi monomials under the action of the operators

$$\begin{aligned} \hat{M}_1 &= x + \tau \frac{\partial^2}{\partial y \partial z} \ \hat{P}_1 = \frac{\partial}{\partial x}, \\ \hat{M}_2 &= y + \tau \frac{\partial^2}{\partial z \partial x} \ \hat{P}_2 = \frac{\partial}{\partial y}, \\ \hat{M}_3 &= z + \tau \frac{\partial^2}{\partial x \partial y} \ \hat{P}_3 = \frac{\partial}{\partial z}. \end{aligned} \tag{2.3}$$

The operators $\hat{M}_1, \hat{P}_1, \hat{M}_2, \hat{P}_2, \hat{M}_3, \hat{P}_3$ satisfy the following identity

$$(\hat{M}_1 \hat{P}_1 + \hat{M}_2 \hat{P}_2 + \hat{M}_3 \hat{P}_3) \ H_{l,m,n}(x,y,z;\tau) = (l+m+n) H_{l,m,n}(x,y,z;\tau). \tag{2.4}$$

Further we see that

$$\frac{\partial}{\partial \tau} H_{l,m,n}(x,y,z;\tau) = \frac{\partial^3}{\partial x \partial y \partial z} H_{l,m,n}(x,y,z;\tau),$$

and

$$H_{l,m,n}(x,y,z;\tau) \mid_{\tau=0} = x^l y^m z^n. \tag{2.5}$$

and are defined by means of the operational rule

$$\exp\left(\tau \frac{\partial^3}{\partial x \partial y \partial z}\right) x^l y^m z^n = H_{l,m,n}(x,y,z;\tau). \tag{2.6}$$

Also by means of (2.6), we can transform a result involving such products as $a_{l,m,n} \ x^l y^m z^n$ into a corresponding result for (3I3V1PHP) $H_{l,m,n}(x,y,z;\tau)$ i.e

$$\exp\left(\tau \frac{\partial^3}{\partial x \partial y \partial z}\right) a_{l,m,n} \ x^l y^m z^n = a_{l,m,n} \ H_{l,m,n}(x,y,z;\tau). \tag{2.7}$$

Now we consider a product of three series of the relation [17]

$$\sum_{l=0}^{\infty} c^{k-l} {L_l}^{k-l}(-bc) x^l = e^{bx} \ (x+c)^k, \tag{2.8}$$

with $L_l{}^{\alpha}(x)$ being associated Laguerre polynomials [1],

$$L_n{}^{\alpha}(x) = \Gamma(\alpha+n+1)\sum_{r=0}^{\infty}\frac{(-1)^r x^r}{r!\Gamma(a+r+1)(n-r)!}, \tag{2.9}$$

accordingly we find from eq. (2.8) that

$$\begin{aligned}&\sum_{l,m,n=0}^{\infty} c_1{}^{k-l}c_2{}^{r-m}c_3{}^{s-n}L_l^{k-l}(-b_1c_1)L_m^{r-m}(-b_2c_2)L_n^{s-n}(-b_3c_3)x^l y^m z^n\\ &= \exp(b_1x+b_2y+b_3z)(x+c_1)^k(y+c_2)^r(z+c_3)^s.\end{aligned} \tag{2.10}$$

Now applying operational definition (2.6) in (2.10), we get

$$\begin{aligned}&\sum_{l,m,n=0}^{\infty} c_1{}^{k-l}c_2{}^{r-m}c_3{}^{s-n}L_l^{k-l}(-b_1c_1)L_m^{r-m}(-b_2c_2)L_n^{s-n}(-b_3c_3)\\ &\exp\left(\tau\frac{\partial^3}{\partial x\partial y\partial z}\right)x^l y^m z^n\\ &= \exp\left(\tau\frac{\partial^3}{\partial x\partial y\partial z}\right)\exp(b_1x+b_2y+b_3z)(x+c_1)^k(y+c_2)^r(z+c_3)^s.\end{aligned} \tag{2.11}$$

Which on using some identities of decoupling of exponentials, finally yields the generating relation

$$\begin{aligned}&\sum_{l,m,n=0}^{\infty} c_1{}^{k-l}c_2{}^{r-m}c_3{}^{s-n}L_l^{k-l}(-b_1c_1)L_m^{r-m}(-b_2c_2)\\ &L_n^{s-n}(-b_3c_3)H_{l,m,n}(x,y,z;\tau)\\ &= \exp(b_1x+b_2y+b_3z\ +\ b_1b_2b_3\tau)\\ &H_{k,r,s}(x+b_2\tau+b_3\tau+c_1, y+b_1\tau+b_3\tau+c_2, z+b_1\tau+b_2\tau+c_3;\tau),\end{aligned} \tag{2.12}$$

which is the generating relation for 3-index, 3-variable, 1-parameter Hermite polynomials $H_{l,m,n}(x,y,z;\tau)$.

3. Bilateral, unilateral and mixed type generating relations

In the previous section we have dealt with ordinary functions. Here we will refer to mixed generating functions. The theory of mixed generating functions has been pioneered by Carlitz [3] and Srivastava [9], who employed the Lagrange expansion as the essential tool to develop a unifying point of view on the problem and to derive families of mixed generating functions in a fairly direct way. In the previous papers [23, 25, 24, 26], we have derived the Hermite Tricomi function of 3-variable 2-parameter ${}_HC_n(x,y,z;\tau_1,\tau_2)$, Hermite-based Appell polynomials, Laguerre-based Appell polynomials and Bessels functions. In this section we will consider Hermite polynomials of two indices and two variables, $H_{m,n}(x,y;\tau)$,to obtain some interesting results. Hermite polynomials of two indices and two variables, $H_{m,n}(x,y;\tau)$ specified by [6]

$$H_{m,n}(x,y;\tau)=m!n!\sum_{s=0}^{\min(m,n)}\frac{\tau^s x^{m-s}y^{n-s}}{s!(m-s)!(n-s)!}, \tag{3.1}$$

and

$$\sum_{m,n=0}^{\infty}\frac{u^m v^n}{m!n!}H_{m,n}(x,y;\tau)=\exp\left(xu+yv+\tau uv\right), \tag{3.2}$$

and by the operational rule

$$H_{m,n}(x,y;\tau)=\exp\left(\tau\frac{\partial^2}{\partial x\partial y}\right)x^m y^n. \tag{3.3}$$

(I). A partly bilateral and partly unilateral generating function for $L_n^{\alpha}(x)$ due to Exton in the following modified form is given by [19, 27]

$$\exp\left(y+z-\frac{xz}{y}\right)=\sum_{m\ =-\infty}^{\infty}\sum_{n\ =-m^*}^{\infty}\left(\frac{L_n^m(x)}{(m+n)!}\right)y^m z^n, \quad m^*=\max\ (0,-m),\ m\in z:=[o,\pm1,\ldots]. \tag{3.4}$$

By operating $\exp\left(\tau\frac{\partial^2}{\partial y\partial z}\right)$ on both sides of (3.4) and using (3.3),we obtain

$$\exp\left(\tau\frac{\partial^2}{\partial y\partial z}\right)\exp\left(y+z-\frac{xz}{y}\right)=\sum_{m\ =-\infty}^{\infty}\sum_{n\ =-m^*}^{\infty}\left(\frac{L_n^m(x)}{(m+n)!}\right)H_{m,n}(y,z;\tau). \tag{3.5}$$

Now by decoupling the right hand side of (3.5), we obtain an interesting partly bilateral and partly unilateral generating function

$$\exp\left(y+z-\frac{xz}{y}-\frac{\tau x}{2y^2}\right)=\sum_{m\ =-\infty}^{\infty}\sum_{n\ =-m^*}^{\infty}\left(\frac{L_n^m(x)}{(m+n)!}\right)H_{m,n}(y,z;\tau). \tag{3.6}$$

(II). We consider the following product:

$$L_n^{\alpha}(x)L_m^{\beta}(y)=\sum_{k=0}^{n}\sum_{j=0}^{m}\binom{n+\alpha}{n-k}\binom{m+\beta}{m-j}\frac{(-x)^k(-y)^j}{k!j!}, \tag{3.7}$$

Now operating $\exp\left(\tau\frac{\partial^2}{\partial x\partial y}\right)$ on both sides of (3.7) and using (3.3) and the identity [11]

$$\exp\left(\tau\frac{\partial^2}{\partial x\partial y}\right)L_n^{\alpha}(x)L_m^{\beta}(y)=L_{m,n}^{\alpha,\beta}(x,y;\tau) \tag{3.8}$$

we obtain the identity relating Hermite and Lagurre polynomials:

$$L_{m,n}^{\alpha,\beta}(x,y;\tau)=\sum_{k=0}^{n}\sum_{j=0}^{m}\binom{n+\alpha}{n-k}\binom{m+\beta}{m-j}\frac{(-1)^{k+j}}{k!j!}H_{k,j}(y,z;\tau). \tag{3.9}$$

(III). Further we consider the identity

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} L_l^{n-l}(x) L_m^{n-m}(y) = e^z \frac{(-1)^{l+m}}{l!m!} h_{l,m}(x-z, y-z/z). \tag{3.10}$$

Now operating $\exp\left(\tau \frac{\partial^2}{\partial x \partial y}\right)$ on both sides of (3.10), using (3.3) and exploiting the identity(3.8), we obtain Hermite-Hermite polynomials of two indices:

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} L_{l,m}^{(n-l)(n-m)}(x,y;\tau) = e^z \frac{(-1)^{l+m}}{l!m!} \ {}_H h_{l,m}(x-z, y-z/z). \tag{3.11}$$

(IV). We consider the following bilateral generating function [1]

$$\sum_{m,n} \frac{u^m v^n}{m!n!} H_{m,n}(x,y,z,w,k) \ = \exp\left(xu + yv + zu^2 + wv^2 + kuv\right), \tag{3.12}$$

where the polynomials $H_{m,n}$ are explicitly provided by the series

$$H_{m,n}(x,y,z,w,k) = m!n! \sum_{l=0}^{min(m,n)} \frac{k^l H_{m-l}(x,z) H_{n-l}(y,w)}{l!(m-l)!(n-l)!}. \tag{3.13}$$

We can define a new polynomials accordingly

$$ {}_H f_{m,n}(x,y,z,w,k) = \sum_{k=0}^{m} \sum_{s=0}^{n} (-1)^{k+s} (c)_{m+k} (c)_{n+s} \rho_{m-k} \rho_{n-s} \frac{H_{k,s}(x,y,z,w,k)}{k!s!(m-k)!(n-s)!}, \tag{3.14}$$

provided by the generating function

$$\begin{aligned} \sum_{m,n} u^m v^n \ {}_H f_{m,n}(x,y,z,w,k) \ &= \frac{1}{(1-u)(1-v)} \\ \exp\left(\frac{-4xu}{(1-u)^2} - \frac{-4yv}{(1-v)^2} + \frac{zu^2}{(1-u)^4} + \frac{wv^2}{(1-v)^4} + \frac{kuv}{(1-u)(1-v)}\right)&, \end{aligned} \tag{3.15}$$

where ${}_H f_{m,n}(x,y,z,w,k)$ is two index five variable Hermite Sister Celine's polynomials,which is new in literature.

4. Symbolic notation and operational techniques

Many relations involving finite series of polynomials can be put into particularly neat form by use of an old symbolic notation. Whenever $\doteqdot$ is used to replace $=$, it is to be understood that exponents will be lowered to subscripts on any symbol which is undefined here except with subscripts, For example simple Laguerre polynomial $L_n(x)$ satisfies the relation

$$\frac{x^n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k n! L_k(x)}{k!(n-k)!}. \tag{4.1}$$

In symbolic notation (4.1) can be written as

$$\frac{x^n}{n!} \doteqdot [1 - L(x)]^n. \tag{4.2}$$

In this section we consider symbolic notation in view of operational techniques and will prove and generalized some of the results. For example, we can prove result (4.2) by replacing L(x) by $(1 - \hat{D}_x^{-1})$ in r.h.s.

(I). Let us consider [20]

$$L_n(\alpha x) \doteqdot [1 - \alpha + \alpha L(x)]^n, \tag{4.3}$$

replacing L(x) by $(1 - \hat{D}_x^{-1})$ in r.h.s, we obtain l.h.s which can be generalized by replacing L(x) by $(y - \hat{D}_x^{-1})$ in r.h.s of eq. (4.3) as follows

$$L_n(\alpha x, y) \doteqdot [y - \alpha y + \alpha L(x, y)]^n. \tag{4.4}$$

(II). Next consider [20]

$$H_n(x + \alpha) \doteqdot [H(x) + 2\alpha]^n, \tag{4.5}$$

replacing H(x) by $(2x + \frac{\partial}{\partial x})$ in r.h.s, we obtain l.h.s which can be generalized by replacing H(x) by $(2x + 2y\frac{\partial}{\partial x})$ in r.h.s of eq. (4.5) as follows

$$H_n(x + \alpha, y) \doteqdot [H(x, y) + 2\alpha]^n. \tag{4.6}$$

(III). Further consider [20]

$$H_n(\alpha x) \doteqdot [H(x) + 2x(\alpha - 1)]^n, \tag{4.7}$$

replacing H(x) by $(2x + \frac{\partial}{\partial x})$ in r.h.s, we obtain l.h.s which can be generalized by replacing $H(x)$ by $(2x + 2y\frac{\partial}{\partial x})$ in r.h.s of eq. (4.7) as follows

$$H_n(\alpha x, y) \doteqdot [H(x, y) + 2x(\alpha - 1)]^n. \tag{4.8}$$

The above results can further be generalized by replacing $H(x)$ by $(x + 2y\frac{\partial}{\partial x} + 3z\frac{\partial^2}{\partial x^2})$, for Hermite polynomials of three variables.

Here we consider some cases of mixed generating function of symbolic notation using operational techniques.

(IV). Let us obtain the expression for the polynomial $\phi_n(x, y)$ given by [20]

$$\phi(x, y) \doteqdot H_n(xL(y)), \tag{4.9}$$

replacing L(y) by $(1 - \hat{D}_y^{-1})$ in r.h.s, we obtain

$$\begin{aligned}
\phi(x, y) &\doteqdot H_n(x(1 - \hat{D}_y^{-1})), \\
\sum_{n=0}^{\infty} \frac{t^n}{n!}\phi_n(x, y) &\doteqdot \exp[2x(1 - \hat{D}_y^{-1})t - t^2], \\
\sum_{n=0}^{\infty} \frac{t^n}{n!}\phi_n(x, y) &\doteqdot \exp(2xt - t^2)\exp(2xyt), \\
\sum_{n=0}^{\infty} \frac{t^n}{n!}\phi_n(x, y) &\doteqdot \exp(2xt - t^2)\,{}_0F_1[-; 1; -2xyt],
\end{aligned} \tag{4.10}$$

which can be generalized by replacing L(y) by $(z - \hat{D}_y^{-1})$ in r.h.s of eq. (4.9) as follows

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\, \phi_n(x,y,z,w) \;\doteqdot\; \exp(2xzt - wt^2)_0F_1[-;1;-2xyt], \tag{4.11}$$

(V). Next we obtain the expression for the polynomial $\sigma_n(x)$

$$\sigma_n(x) \;\doteqdot\; L_n(H(x)), \tag{4.12}$$

replacing $H(x)$ by $(2x + \frac{\partial}{\partial x})$ in r.h.s and using already quoted decoupling formula we obtain

$$\sigma_n(x) \;\doteqdot\; \frac{1}{1-t}\exp\left(\frac{-2xt}{1-t} - \frac{t^2}{(1-t)^2}\right), \tag{4.13}$$

which can be viewed as Hermite -Laguerre polynomials ${}_HL_n(x)$ which can further be generalized by replacing $H(x)$ by $(2x + y\frac{\partial}{\partial x})$ in r.h.s of eq. (4.12)

$$\sigma_n(x,y) \;\doteqdot\; \frac{1}{1-t}\exp\left(\frac{-2xt}{1-t} - \frac{yt^2}{(1-t)^2}\right), \tag{4.14}$$

which can be further generalized in the following form

$$\begin{aligned} \sigma_n(x,y,z) &\doteqdot\; L_n(H(x,y),z), \\ \sigma_n(x,y,z) &\doteqdot\; L_n((2x + y\frac{\partial}{\partial x}),z), \\ \sigma_n(x,y,z) &\doteqdot\; \frac{1}{1-zt}\exp\left(\frac{-2xt}{1-zt} - \frac{yt^2}{(1-zt)^2}\right), \end{aligned} \tag{4.15}$$

which can be viewed as Hermite-Laguerre polynomials ${}_HL_n(x,y;z)$.
(VI). We consider a well known result of Sister Celine [20]

$$f_n(x^2) \;\doteqdot\; L_n(2xH(x)), \tag{4.16}$$

replacing $H(x)$ by $(2x + \frac{\partial}{\partial x})$ in r.h.s of eq. (4.16) we obtain

$$f_n(x^2) \;\doteqdot\; \frac{1}{1-t}\exp\left(\frac{-4x^2t}{(1-t)^2}\right), \tag{4.17}$$

which can be generalized, in the following form

$$f_n(x^2,y) \;\doteqdot\; \frac{1}{1-yt}\exp\left(\frac{-4x^2t}{(1-yt)^2}\right), \tag{4.18}$$

replacing x^2 by x in eq.(4.18),we get

$$f_n(x,y) \;\doteqdot\; \frac{1}{1-yt}\exp\left(\frac{-4xt}{(1-yt)^2}\right), \tag{4.19}$$

which is Sister Celine's polynomial of two variables.

5. Conclusion

It is concluded that many results can be proved and generalized by exploiting operational techniques. An example is provided by obtaining expression for $H_n(H(x))$.

$$\sum_{n=0}^{\infty}\frac{H_n(H(x))t^n}{n!} \doteqdot \sum_{n=0}^{\infty}H_n\left(2x+\frac{\partial}{\partial x}\right)\frac{t^n}{n!},$$

$$\sum_{n=0}^{\infty}\frac{H_n(H(x))t^n}{n!} \doteqdot \exp\left(2\left(2x+\frac{\partial}{\partial x}\right)t-t^2\right) \doteqdot \exp(4xt-5t^2),$$

$$\sum_{n=0}^{\infty}\frac{H_n(H(x))t^n}{n!} \doteqdot \exp\left(\frac{4xt\sqrt{5}}{\sqrt{5}}-(t\sqrt{5})^2\right) \doteqdot \sum_{n=0}^{\infty}H_n\left(\frac{2x}{\sqrt{5}}\right)\frac{(\sqrt{5}t)^n}{n!},$$

$$\sum_{n=0}^{\infty}\frac{H_n(H(x))t^n}{n!} \doteqdot \sum_{n=0}^{\infty}H_n\left(\frac{2x}{\sqrt{5}}\right)5^{n/2}\frac{(t)^n}{n!},$$

$$H_n(H(x)) \doteqdot H_n\left(\frac{2x}{\sqrt{5}}\right)5^{n/2}.$$

which is a result of Rainville [20], which can further be generalized for 2 and 3 variables Hermite polynomials by replacing $H(x)$ by $(2x+y\frac{\partial}{\partial x})$ and $(x+2y\frac{\partial}{\partial x}+3z\frac{\partial^2}{\partial x^2})$.

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