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THE DEGREE AND THE COPRIME-NESS FOR MATRIX-VALUED RATIONAL FUNCTIONS

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Abstract. In this note we give a relationship between the degree and coprime-ness of matrix-valued rational functions.

1. Introduction

The aim of this note is to provide a relationship between the degree and coprime-ness of matrix-valued rational functions. We first review a few essential facts for (block) Toeplitz operators and (block) Hankel operators. Let $L^2 \equiv L^2(\mathbb{T})$ be the set of square-integrable measurable functions on the unit circle $\mathbb{T} \equiv \partial\,\mathbb{D}$ in the complex plane and $H^2 \equiv H^2(\mathbb{T})$ be the corresponding Hardy space. Let $L^\infty \equiv L^\infty(\mathbb{T})$ be the set of bounded measurable functions on $\mathbb{T}$ and let $H^\infty \equiv H^\infty(\mathbb{T}) := L^\infty \cap H^2$. For a Hilbert space E, let $L^2_E \equiv L^2_E(\mathbb{T})$ be the Hilbert space of E-valued norm square-integrable measurable functions on $\mathbb{T}$ and $H^2_E \equiv H^2_E(\mathbb{T})$ be the corresponding Hardy space. We observe that $L^2_{\mathbb{C}^n} = L^2 \otimes \mathbb{C}^n$ and $H^2_{\mathbb{C}^n} = H^2 \otimes \mathbb{C}^n$. Let $M_{n\times m}$ denote the set of $n \times m$ complex matrices and write $M_n := M_{n\times n}$. If Φ is a matrix-valued function in $L^\infty_{M_n} \equiv L^\infty_{M_n}(\mathbb{T})$ $(= L^\infty(\mathbb{T}) \otimes M_n)$, then the block Toeplitz operator T_Φ and the block Hankel operator H_Φ on $H^2_{\mathbb{C}^n}$ are defined by

$$T_\Phi f = P(\Phi f) \quad \text{and} \quad H_\Phi f = JP^\perp(\Phi f) \quad (f \in H^2_{\mathbb{C}^n}),$$

where P and $P^\perp$ denote the orthogonal projections that map from $L^2_{\mathbb{C}^n}$ onto $H^2_{\mathbb{C}^n}$ and $\left(H^2_{\mathbb{C}^n}\right)^\perp$, respectively and J denotes the unitary operator from $L^2_{\mathbb{C}^n}$ to $L^2_{\mathbb{C}^n}$ given by $J(g)(z) = \overline{z} I_n g(\overline{z})$ for $g \in L^2_{\mathbb{C}^n}$ (I_n :=the $n \times n$ identity matrix). If $n = 1$, T_Φ and H_Φ are called the (scalar) Toeplitz operator and the (scalar) Hankel operator, respectively. For brevity we write I for the identity matrix and

$$I_\zeta := \zeta I \quad (\zeta \in L^\infty).$$

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For $\Phi \in L^\infty_{M_{n\times m}}$, write

$$\widetilde{\Phi}(z) := \Phi^*(\overline{z}). \tag{1.1}$$

A matrix function $\Theta \in H^\infty_{M_{n\times m}}$ is called *inner* if $\Theta^*(z)\Theta(z) = I_m$ for almost all $z \in \mathbb{T}$. The following facts are clear from the definition:

$$T^*_\Phi = T_{\Phi^*}, \quad H^*_\Phi = H_{\widetilde{\Phi}} \quad (\Phi \in L^\infty_{M_n}); \tag{1.2}$$

$$T_{\Phi\Psi} - T_\Phi T_\Psi = H^*_{\Phi^*} H_\Psi \quad (\Phi, \Psi \in L^\infty_{M_n}). \tag{1.3}$$

For a matrix-valued function $\Phi \in H^2_{M_{n\times r}}$, we say that $\Delta \in H^2_{M_{n\times m}}$ is a *left inner divisor* of Φ if Δ is an inner matrix function such that $\Phi = \Delta A$ for some $A \in H^2_{M_{m\times r}}$ $(m \le n)$. We also say that two matrix functions $\Phi \in H^2_{M_{n\times r}}$ and $\Psi \in H^2_{M_{n\times m}}$ are *left coprime* if the only common left inner divisor of both Φ and Ψ is a unitary constant matrix and that $\Phi \in H^2_{M_{n\times r}}$ and $\Psi \in H^2_{M_{m\times r}}$ are *right coprime* if $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are left coprime. Two matrix functions Φ and Ψ in $H^2_{M_n}$ are said to be *coprime* if they are both left and right coprime. We would remark that if $\Phi \in H^2_{M_n}$ is such that $\det \Phi$ is not identically zero, then any left inner divisor Δ of Φ is square, i.e., $\Delta \in H^2_{M_n}$. If $\Phi \in H^2_{M_n}$ is such that $\det \Phi$ is not identically zero, then we say that $\Delta \in H^2_{M_n}$ is a *right inner divisor* of Φ if $\widetilde{\Delta}$ is a left inner divisor of $\widetilde{\Phi}$ (cf. [6]).

Let $\lambda \in \mathbb{D}$ and write $b_\lambda := \frac{z-\lambda}{1-\overline{\lambda}z}$, which is called a *Blaschke factor*. If M is a closed subspace of $\mathbb{C}^n$, then the matrix function of the form

$$e^{i\zeta} B_{\lambda,M} := e^{i\zeta}(B_\lambda P_M + P_{M^\perp})$$

($\zeta \in \mathbb{R}$, $B_\lambda := I_{b_\lambda}$ and $P_{\mathcal{X}}$:=the orthogonal projection of $\mathbb{C}^n$ onto $\mathcal{X}$) is called a *Blaschke-Potapov factor*. Also the function of the form

$$B := \nu \prod_{k=1}^{n} B_{\lambda_k, M_k} \quad (\nu \text{ is a unitary constant matrix})$$

is called a *finite Blaschke-Potapov product*. It is known [10] that $\Theta \in H^\infty_{M_n}$ is rational and inner if and only if it can be represented as a finite Blaschke-Potapov product. On the other hand, it is also known [2, Lemma 3.1] that if $F \in H^2_{M_n}$ and M is a non-zero closed subspace of $\mathbb{C}^n$, then

$$F \text{ has } B_{\lambda,M} \text{ as a right inner divisor } \iff M \subseteq \ker F(\lambda) \tag{1.4}$$

and that if $A, B \in H^2_{M_n}$ and B is a rational function such that $\det B$ is not identically zero, then

(1.5)

$$A \text{ and } B \text{ are right coprime } \iff \ker A(\alpha) \cap \ker B(\alpha) = \{0\} \text{ for any } \alpha \in \mathbb{D}.$$

For $\Phi \in L^\infty_{M_n}$, write

$$\Phi_+ := P_n \Phi \in H^2_{M_n} \quad \text{and} \quad \Phi_- := \left(P_n^\perp \Phi\right)^* \in H^2_{M_n}, \tag{1.6}$$

where P_n denotes the orthogonal projection from $L^2_{M_n}$ onto $H^2_{M_n}$. Thus we can write $\Phi = \Phi_-^* + \Phi_+$. Suppose $\Phi_+ = [\varphi_{ij}] \in H^2_{M_n}$ is such that Φ^* is of bounded type (in other words, each entry is a quotient of two functions in $H^\infty(\mathbb{T})$). Then it was ([1]) known that φ_{ij} can be written of the form $\varphi_{ij} = \theta_{ij}\overline{b_{ij}}$, where θ_{ij} is an inner function, $b_{ij} \in H^2$, and θ_{ij} and b_{ij} are coprime. Thus if θ is the least common multiple of θ_{ij}'s, then we can write

$$\Phi_+ = [\varphi_{ij}] = [\theta_{ij}\overline{b_{ij}}] = [\theta\overline{a_{ij}}] = \Theta A^* \quad (\Theta = I_\theta,\ A \in H^2_{M_n}). \tag{1.7}$$

Let $\Phi \equiv \Phi_-^* + \Phi_+ \in L^\infty_{M_n}$ be such that Φ and Φ^* are of bounded type. Then in view of (1.7) we can write

$$\Phi_+ = \Theta_1 A^* \quad \text{and} \quad \Phi_- = \Theta_2 B^*, \tag{1.8}$$

where $\Theta_i = I_{\theta_i}$ with an inner function θ_i $(i = 1, 2)$, $A, B \in H^2_{M_n}$. If Ω is the greatest common left inner divisor of A and Θ in the representation (1.7):

$$\Phi = \Theta A^* = A^*\Theta \quad (\Theta \equiv I_\theta \text{ for an inner function } \theta),$$

then $\Theta = \Omega\Omega_l$ and $A = \Omega A_l$ for some inner matrix Ω_l and some $A_l \in H^2_{M_n}$. Therefore if $\Phi^* \in L^\infty_{M_n}$ is of bounded type, then we can write

$$\Phi = {A_l}^*\Omega_l, \quad \text{where } A_l \text{ and } \Omega_l \text{ are left coprime:} \tag{1.9}$$

in this case, $A_l^*\Omega_l$ is called the *left coprime factorization* of Φ and similarly, we can write

$$\Phi = \Omega_r A_r^*, \quad \text{where } A_r \text{ and } \Omega_r \text{ are right coprime:} \tag{1.10}$$

in this case, $\Omega_r A_r^*$ is called the *right coprime factorization* of Φ (cf. [3], [4]).

On the other hand, it was known [7] that for $\Phi \in L^\infty_{M_n}$, the following statements are equivalent:

(i) Φ is of bounded type;
(ii) $\ker H_\Phi = \Theta H^2_{\mathbb{C}^n}$ for some square inner matrix function Θ;
(iii) $\Phi = A\Theta^*$, where $A \in H^\infty_{M_n}$ and A and Θ are right coprime.

2. Main results

For an inner matrix function $\Theta \in H^2_{M_n}$, we write

$$\mathcal{H}(\Theta) := H^2_{\mathbb{C}^n} \ominus \Theta H^2_{\mathbb{C}^n}.$$

We begin with:

Definition 2.1. For $\Phi \in H^\infty_{M_n}$, define the (*analytic*) *degree* of Φ by

$$\deg(\Phi) := \operatorname{rank} H_{\Phi^*}.$$

For $\Phi \in L^\infty_{M_n}$, the *analytic degree* and *co-analytic degree* of Φ are defined by

$$\deg_+(\Phi) := \operatorname{rank} H_{\Phi^*} \quad \text{and} \quad \deg_-(\Phi) := \operatorname{rank} H_\Phi.$$

Even though the degree of matrix-valued functions is defined for square matrices, we may define the degree of any rectangular $n \times m$ matrix-valued function by defining the Hankel operators with $n \times m$ matrix-valued symbols, appropriately. However we concentrate on the square matrix cases for our purpose on the Toeplitz and the Hankel operator theory because frequently we want to deal with the commutators of two Hankel operators or the self-commutators of Hankel operators. On the other hand, it is well known that if $\Phi \in H^\infty_{M_n}$ is a matrix-valued rational function, then $\deg(\Phi)$ is equal to the *McMillan* degree of Φ (cf. [9, p. 81]).

Proposition 2.2. *Suppose $\Phi \in H^\infty_{M_n}$ is such that Φ^* is of bounded type, so that we may write*

$$\Phi = \Theta_1 A^* = B^* \Theta_2 \quad (A, B \in H^\infty_{M_n};\ \text{the } \Theta_i \text{ are inner}),$$

where Θ_1 and A are right coprime and Θ_2 and B are left coprime. Then

$$\deg(\Phi) = \deg(\det \Theta_1) = \deg(\det \Theta_2).$$

Proof. We first observe that if Θ is a square inner matrix function, then

$$\dim \mathcal{H}(\Theta) = \deg(\det \Theta). \tag{2.1}$$

Indeed,

$$\begin{aligned}\dim \mathcal{H}(\Theta) &= \dim \ker T_{\Theta^*} = -\text{index}\, T_\Theta \\ &= -\text{index}\, T_{\det \Theta} = \dim \ker T_{\overline{\det \Theta}} \\ &= \dim \mathcal{H}(\det \Theta) = \deg(\det \Theta),\end{aligned}$$

where the third equality follows from the Fredholm theory of block Toeplitz operators (cf. [5]). We thus have

$$\begin{aligned}\deg(\Phi) = \text{rank}\, H_{\Phi^*} &= \dim (\ker H^*_{\Phi^*})^\perp \\ &= \dim \left(\ker H_{\widetilde{B}\widetilde{\Theta}_2^*}\right)^\perp \\ &= \dim \left(\widetilde{\Theta}_2 H^2_{\mathbb{C}^n}\right)^\perp \quad (\text{since } \widetilde{B} \text{ and } \widetilde{\Theta}_2 \text{ are right coprime}) \\ &= \dim \mathcal{H}(\widetilde{\Theta}_2) = \deg(\det \widetilde{\Theta}_2) \quad (\text{by (2.1)}).\end{aligned}$$

If $\Psi = [\psi_{ij}] \in H^\infty_{M_n}$, then $\widetilde{\Psi} = [\widetilde{\psi_{ji}}] = [\widetilde{\psi_{ij}}]^t$, so that $\det \widetilde{\Psi} = \det [\widetilde{\psi_{ij}}] = \widetilde{\det \Psi}$. Therefore $\deg(\Phi) = \deg(\det \widetilde{\Theta}_2) = \deg(\det \Theta_2)$ and similarly, $\deg(\widetilde{\Phi}) = \deg(\det \Theta_1)$. Since $\deg(\Phi) = \text{rank}\, H_{\Phi^*} = \text{rank}\, H^*_{\Phi^*} = \text{rank}\, H_{\widetilde{\Phi}^*} = \deg(\widetilde{\Phi})$, the result follows at once. □

We here take a chance to compute the degree of a function $\Phi := \left(\begin{smallmatrix} z & -b_\alpha z \\ 0 & z^2 \end{smallmatrix}\right)$. First of all we make a right coprime factorization of Φ:

$$\Phi \equiv \begin{pmatrix} z & -b_\alpha z \\ 0 & z^2 \end{pmatrix} = \begin{pmatrix} b_\alpha z & 0 \\ 0 & z^2 \end{pmatrix} \begin{pmatrix} b_\alpha & 0 \\ -1 & 1 \end{pmatrix}^*.$$

Indeed by (1.5) we can see that

$$\Theta \equiv \begin{pmatrix} b_\alpha z & 0 \\ 0 & z^2 \end{pmatrix} \text{ and } A \equiv \begin{pmatrix} b_\alpha & 0 \\ -1 & 1 \end{pmatrix} \text{ are right coprime.}$$

Thus by Proposition 2.2, $\deg \Phi = \deg(\det \Theta) = \deg(b_\alpha z^3) = 4$.

Theorem 2.3. *If $\Theta \in H^\infty_{M_n}$ is an inner matrix function, then $\deg(\det \Theta) < \infty$ if and only if Θ is a finite Blaschke-Potapov product.*

Proof. If Θ is a finite Blaschke-Potapov product of the form $\Theta = \nu \prod_{j=1}^m B_{\alpha_j, M_j}$, then $\det \Theta = \prod_{j=1}^m (b_{\alpha_j})^{\dim M_j}$, so that $\deg(\det\Theta) = \sum_{j=1}^m \dim M_j < \infty$. Conversely, if $\deg(\det \Theta) = \dim \mathcal{H}(\Theta) < \infty$, put $\Theta := [\theta_{ij}]_{ij=1}^n$. Since $\text{rank} H^*_{\overline{\theta_{ij}}} \le \text{rank} H^*_{\Theta^*} = \dim \mathcal{H}(\widetilde{\Theta}) = \dim \mathcal{H}(\Theta) < \infty$, it follows from the Kronecker's lemma [8, p. 183] that θ_{ij}'s are rational functions. Thus Θ is a rational inner matrix function and hence a finite Blaschke-Potapov product. □

Corollary 2.4. *Every left (right) inner divisor of $B_\lambda := I_{b_\lambda} \in H^\infty_{M_n}$ is a Blaschke-Potapov factor of the form $e^{i\zeta} B_{\lambda, M}$ with $\dim M \le n$.*

Proof. Let Δ_1 be a left inner divisor of B_λ. Then we can write $B_\lambda = \Delta_1 \Delta_2$ for some inner Δ_2. Thus $b_\lambda^n = \det(B_\lambda) = \det(\Delta_1)\det(\Delta_2)$, and hence $\det(\Delta_1) = e^{i\zeta} b_\lambda^m$ $(\zeta \in \mathbb{R},\ m \le n)$. Thus by Theorem 2.3, Δ_1 is a finite Blaschke Potapov product and therefore $\Delta_1 = e^{i\zeta} B_{\lambda, M}$ $(\dim M = m)$, which gives the result. □

Theorem 2.5. *Let $\Phi \in L^\infty_{M_n}$ be a matrix-valued rational function, so that we may write*

$$\begin{aligned} \Phi &= \Theta_1^* A \quad (\textit{left coprime factorization}) \\ &= C\Theta_2^* \quad (\textit{right coprime factorization}), \end{aligned}$$

If $\Theta_1 = \nu \prod_{j=1}^m B_{\alpha_j, M_j}$ (ν is a constant unitary matrix), then $\Theta_2 = \prod_{j=1}^m B_{\alpha_j, N_j}$ (up to right unitary constant matrix), where $\dim M_j = \dim N_j$ for all $j = 1, 2, \ldots, m$. In particular,

$$\det \Theta_1 = \det \Theta_2 \quad \textit{and} \quad \det A = \det C.$$

Proof. Observe that $\Phi^* = A^* \Theta_1 = A^* \nu B_{\alpha_1, M_1} \prod_{j=2}^m B_{\alpha_j, M_j}$. Write $\Psi := (\nu^* A)^* B_{\alpha_1, M_1}$. Then $\Psi B_{\alpha_1, M_1^\perp} = (\nu^* A)^* I_{b_{\alpha_1}} = I_{b_{\alpha_1}} (\nu^* A)^*$, so that $\Psi = I_{b_{\alpha_1}} (\nu^* A)^* B^*_{\alpha_1, M_1^\perp}$. Thus if $\Psi = \Delta_1 C_1^*$ (Δ_1 and C_1 are right coprime), then Δ_1 is a left inner divisor of $I_{b_{\alpha_1}}$ and

$$\dim \mathcal{H}(\Delta_1) = \deg_-(\widetilde{\Psi}^*) = \deg_-(\Psi^*) = \deg\left(\det B_{\overline{\alpha_1}, M_1}\right) = \dim M_1.$$

It thus follows from Corollary 2.4 that $\Delta_1 = e^{i\zeta} B_{\alpha_1, N_1}$, where $\dim N_1 = \dim M_1$. An induction gives that $\dim N_j = \dim M_j$ for $j = 1, \ldots, m$, so that

$$\det \Theta_1 = \prod_{j=1}^m (b_{\alpha_j})^{\dim M_j} = \prod_{j=1}^m (b_{\alpha_j})^{\dim N_j} = \det \Theta_2.$$

Moreover, $\det A = \det B$. This completes the proof. □

Corollary 2.6. *If* $\Phi,\ \Psi \in H^\infty_{M_n}$, *then* $\deg(\Phi\Psi) \le \deg(\Phi) + \deg(\Psi)$.

Proof. If M_Φ denotes the multiplication operator with symbol Φ, then a straightforward calculation shows that $JM_\Phi J = M^*_{\widetilde{\Phi}}$ and $H_\Phi = PJM_\Phi|_{H^2_{\mathbb{C}^n}}$. Using theses equalities we can show that $H_{\Phi\Psi} = T^*_{\widetilde{\Phi}} H_\Psi + H_\Phi T_\Psi$. We thus have $\deg(\Phi\Psi) = \operatorname{rank} H_{\Psi^*\Phi^*} = \operatorname{rank}(T_{\widetilde{\Psi}} H_{\Phi^*} + H_{\Psi^*} T_{\Psi^*}) \le \operatorname{rank} H_{\Phi^*} + \operatorname{rank} H_{\Psi^*} = \deg(\Phi) + \deg(\Psi)$. □

We need not expect that $\deg(\Phi) = \deg(\det\Phi)$ for $\Phi \in H^\infty_{M_n}$. To see this, let

$$\Phi := \begin{pmatrix} z & -b_\alpha z \\ 0 & 1 \end{pmatrix}.$$

Then $\det\Phi = z$, and hence $\deg(\det\Phi) = 1$. On the other hand, by a straightforward calculation, we can see that Φ has the right coprime factorization such as

$$\Phi = \begin{pmatrix} b_\alpha z & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} b_\alpha & 0 \\ -1 & 1 \end{pmatrix}^* \quad \text{(right coprime factorization)}.$$

Thus by Proposition 2.2, $\deg(\Phi) = \deg\left(\det\left(\begin{smallmatrix} b_\alpha z & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) = \deg(b_\alpha z) = 2$.

Theorem 2.7. *Suppose* $\Theta, A \in H^\infty_{M_n}$ *with* Θ *a finite Blaschke-Potapov product. Then the following statements are equivalent*:

(i) $\det\Theta$ *and* $\det A$ *are coprime*;
(ii) $A\nu$ *and* Θ *are right coprime for each unitary constant matrix* ν;
(iii) τA *and* Θ *are left coprime for each unitary constant matrix* τ.

Proof. (i)⇒(ii) and (i)⇒(iii): We first claim that if $A, B \in H^\infty_{M_n}$, then

$$\det A \text{ and } \det B \text{ are coprime} \implies A \text{ and } B \text{ are coprime.} \tag{2.2}$$

For (2.2), we suppose A and B are not right coprime. Then $A = A_1\Delta$ and $B = B_1\Delta$ for some inner matrix function Δ which is not a unitary constant matrix. Thus $\det A = \det A_1 \det\Delta$ and $\det B = \det B_1 \det\Delta$. But since Δ is not a unitary constant matrix it follows that $\deg(\det\Delta) = \dim\mathcal{H}(\Delta) \neq 0$, which implies that $\det\Delta$ is not constant. Thus $\det A$ and $\det B$ are not coprime. We thus have

$$\det A \text{ and } \det B \text{ are coprime} \implies A \text{ and } B \text{ are right coprime.} \tag{2.3}$$

It then follows from (2.3) that if $\det A$ and $\det B$ are coprime, and hence $\det\widetilde{A}$ and $\det\widetilde{B}$ are coprime, then $\widetilde{A}$ and $\widetilde{B}$ are right coprime, so that A and B are left coprime. This together with (2.3) proves (2.2). Now since $\det A\nu = e^{i\zeta}\det A$ and $\det\tau A = e^{i\xi}\det A$ for some $\zeta, \xi \in \mathbb{R}$, the implications follow at once from (2.2).

(ii)⇒(i): Let

$$\Theta := \nu \prod_{j=1}^m B_{\alpha_j, M_j} \quad \text{and} \quad m_j := \dim M_j.$$

Then $\det\Theta = e^{i\zeta}\prod_{j=1}^m (b_{\alpha_j})^{m_j}$. If $\det\Theta$ and $\det A$ are not coprime, then $A(\alpha_{j_0})$ is not invertible for some j_0, $1\le j_0\le m$. Thus there exists a non-zero vector $\mathbf{x}\in\ker A(\alpha_{j_0})$. Since $\det\Theta(\alpha_{j_0})=0$, it follows that $\Theta(\alpha_{j_0})$ is not invertible, so that there exists a non-zero vector $\mathbf{y}\in\ker\Theta(\alpha_{j_0})$. If we choose a unitary constant matrix ν_0 such that $\nu_0\mathbf{y}=\mathbf{x}$, then $\mathbf{y}\in\ker(A\nu_0)(\alpha_{j_0})\cap\ker\Theta(\alpha_{j_0})$, which by (1.6), implies that $A\nu_0$ and Θ are not right coprime.

(iii)⇒(ii): If τA and Θ are left coprime for each unitary constant matrix τ, then $\widetilde{A}\widetilde{\tau}$ and $\widetilde{\Theta}$ are right coprime, so that by the equivalence of (i) and (ii), $\widetilde{\det A}$ and $\widetilde{\det\Theta}$ are coprime, and hence $\det A$ and $\det\Theta$ are coprime, which implies that $A\nu$ and Θ are right coprime. □

The converse of (2.2) is not true in general. For example, let

$$A := \frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}b_\alpha z & 0\\ 0 & 1\end{pmatrix}\quad\text{and}\quad B := \begin{pmatrix}b_\alpha & 0\\ -1 & 1\end{pmatrix}.$$

Then by (1.5), A and B are right coprime because $\ker A(\alpha)\cap\ker B(\alpha)=\{0\}$ for all $\alpha\in\mathbb{D}$. Observe that

$$\widetilde{A} := \frac{1}{\sqrt{2}}\begin{pmatrix}b_{\overline{\alpha}} z & b_{\overline{\alpha}} z\\ -1 & 1\end{pmatrix}\quad\text{and}\quad \widetilde{B} := \begin{pmatrix}b_{\overline{\alpha}} & -1\\ 0 & 1\end{pmatrix}.$$

A similar argument shows that $\widetilde{A}$ and $\widetilde{B}$ are right coprime and hence A and B are left coprime. Therefore A and B are coprime. But evidently, $\det A\equiv b_\alpha z$ and $\det B\equiv b_\alpha$ are not coprime.

Corollary 2.8. *Let $\Phi\in H^\infty_{M_n}$ be a matrix-valued rational function, so that we may write*

$$\Phi=\Theta_1 A^* = B^*\Theta_2\ \ (A,B\in H^\infty_{M_n};\ \textit{the } \Theta_i \textit{ are finite Blaschke-Potapov product}).$$

Then we have:

(i) *If* $\det\Theta_1$ *and* $\det A$ *are coprime, then* $\deg(\Phi)=\deg(\det\Theta_1)$;
(ii) *If* $\det\Theta_2$ *and* $\det B$ *are coprime, then* $\deg(\Phi)=\deg(\det\Theta_2)$.

Proof. This follows from Proposition 2.2 and Theorem 2.7. □

The following corollary was well-known. Here we give a direct and simple proof by using the coprime factorization.

Corollary 2.9. *If $\varphi\in H^\infty$ is a rational function of the reduced form $\varphi=\frac{q}{p}$, (p and q are polynomials), then* $\operatorname{rank} H_{\overline{\varphi}}=\max\{\deg q,\deg p\}$.

Proof. Suppose $n=\deg p\ge\deg q=m$. Without loss of generality, we may write $p(z)=\prod_{i=1}^n(1-\overline{\alpha_i}z)$ ($\alpha_i\ne 0$). Since $\varphi=\frac{q}{p}\in H^\infty$, it follows that $0<|\alpha_i|<1$ for all i. Write

$$\varphi(z)=\frac{q(z)}{\prod_{i=1}^n(1-\overline{\alpha_i}z)}=\Big(\prod_{i=1}^n b_{\alpha_i}\Big)\overline{a}\qquad (b_{\alpha_i}:=\frac{z-\alpha_i}{1-\overline{\alpha_i}z}),$$

where $a(z) = \prod_{i=1}^{n} \frac{z^n \overline{q(z)}}{1-\overline{\alpha_i} z}$. We want to show that

$$\prod_{i=1}^{n} b_{\alpha_i} \text{ and } a \text{ are coprime.}$$

Note that $\prod_{i=1}^{n} b_{\alpha_i}$ and a are coprime if and only if $(z^n\overline{q})(\alpha_i) \neq 0$ for all $i = 1, 2, \ldots, n$. If $q(z)$ is a constant, it is trivial. If instead $q(z)$ is not constant, then we can write $q(z) = c\prod_{i=1}^{m}(z-\beta_i)$. Then $z^n\overline{q(z)} = z^{n-m} \cdot z^m\overline{q(z)} = cz^{n-m}\prod_{i=1}^{m}(1-\overline{\beta_i}z)$ $(1 \leq m \leq n)$. Thus if $(z^n\overline{q})(\alpha_i) = 0$, then $\beta_i = \frac{1}{\overline{\alpha_j}}$ for some i, j, which implies that $p(z)$ and $q(z)$ have a common zero, a contradiction. This proves that $\prod_{i=1}^{n} b_{\alpha_i}$ and a are coprime. Therefore $\operatorname{rank} H_{\overline{\varphi}} = \dim \mathcal{H}\left(\prod_{i=1}^{n} b_{\alpha_i}\right) = n = \deg p$.

If $\deg p < \deg q$, then we can write $q = ph + r$, where h is a polynomial of degree $n_0 := m - n > 0$. Then $\deg r \leq \deg q$. Thus we have $\varphi = h + \frac{r}{p}$. Observe that $h(z) = z^{n_0}\overline{d(z)}$ and $\frac{r(z)}{p(z)} = \prod_{i=1}^{n} b_{\alpha_i}\overline{a}$, where $\alpha_i \neq 0$, $d(0) \neq 0$, $a(z) = \prod_{i=1}^{n} \frac{z^n\overline{r(z)}}{1-\overline{\alpha_i}z}$, and $a(\alpha_i) \neq 0$ for all $i = 1, 2, \ldots, n$. Hence,

$$\varphi(z) = z^{n_0}\overline{d(z)} + \prod_{i=1}^{n} b_{\alpha_i}\overline{a} = z^{n_0}\prod_{i=1}^{n} b_{\alpha_i}\overline{\Big(d(z)\prod_{i=1}^{n} b_{\alpha_i} - z^{n_0}a(z)\Big)},$$

where $z^{n_0}\prod_{i=1}^{n} b_{\alpha_i}$ and $d(z)\prod_{i=1}^{n} b_{\alpha_i} - z^{n_0}a(z)$ are coprime. We thus have that $\deg \varphi = n_0 + n = m = \deg q$. □

Corollary 2.10. *Given a complex (possibly finite) sequence $\{\alpha_n\}$ having no limit point and a sequence $\{n_i\}$ of natural numbers satisfying $\sum n_i \leq r$, there exists a function $\varphi \in H^\infty$ such that*

(i) *φ has a zero of order n_i at each α_n;*
(ii) $\operatorname{rank} H_{\overline{\varphi}} = r$.

Proof. If $\{\alpha_n\}$ is an infinite sequence, let φ be the entire function appeared in the Weierstrass Product Theorem: i.e., if we arrange $0 < |\alpha_1| \leq |\alpha_2| \leq \cdots$, then $\varphi(z) := \prod_{n=1}^{\infty}(1 - \frac{z}{\alpha_n})e^{p_n(z)}$ with $p_n(z) := \frac{z}{\alpha_n} + \frac{1}{2}(\frac{z}{\alpha_n})^2 + \cdots + \frac{1}{n}(\frac{z}{\alpha_n})^n$. Then $\operatorname{rank} H_{\overline{\varphi}} = \deg \varphi = \infty$. If $\{\alpha_n\}$ is a finite sequence and $r = \infty$, let s be a singular inner function and $f(z) = \prod_{n=1}^{N}(1 - \frac{z}{\alpha_n})^{n_i}$. Putting $\varphi = sf$ gives the required function. If instead $\{\alpha_n\}$ is a finite sequence and $r < \infty$, choose $\alpha > \max\{1, |\alpha_n|\}$. Put $p(z) = (z-\alpha)^r$ and $q(z) = \prod_{n=1}^{N}(1 - \frac{z}{\alpha_n})^{n_i}$. Also putting $\varphi = \frac{q}{p}$ gives the required function. □

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