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HARMONIC AND BIHARMONIC MAPS ON DOUBLY TWISTED PRODUCT MANIFOLDS

Abdelhamid Boulal, Mustapha Djaa, and Seddik Ouakkas

Abstract. In this paper we investigate the geometry of doubly twisted product manifolds and we study the harmonicity and biharmonicity of maps between doubly twisted product Riemannian manifold. Also we characterize the conformal biharmonic maps and construct some new proper biharmonic maps.

1. Introduction

Let $\phi : (M^m, g) \to (N^n, h)$ be a smooth map between two Riemannian manifolds. The energy functional of ϕ is defined by

$$E(\phi) = \frac{1}{2}\int_K |d\phi|^2 dv_g, \tag{1.1}$$

where K is compact subset of M.

Definition 1. ϕ is called harmonic if it is a critical point of the energy functional $E(K)$ for all compact subsets $K \subset M$.

The Euler-Lagrange equation associated to (1.1) is given by the vanishing of the tension field

$$\tau(\phi) = Tr_g \nabla d\phi = 0. \tag{1.2}$$

For more detail see [1], [5], [6] and [8].

Definition 2. A map $\phi : (M, g) \to (N, h)$ between Riemannian manifolds is called *biharmonic* if it is a critical point of the *bienergy* functional:

$$E_2(\phi) = \frac{1}{2}\int_K |\tau(\phi)|^2 v_g \tag{1.3}$$

for all compact subset $K \subset M$.

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The Euler-Lagrange equation associated to (1.3) is given by the vanishing of the bitension field

$$\tau_2(\phi) = -J_\phi(\tau(\phi)) = -\big(\Delta^\phi \tau(\phi) + trace_g R^N(\tau(\phi), d\phi)d\phi\big), \tag{1.4}$$

where R^N is the curvature tensor field on N and J_ϕ is the Jacobi operator defined by

$$\begin{array}{rcl} J_\varphi : \Gamma(\phi^{-1}(TN)) & \to & \Gamma(\phi^{-1}(TN)) \\ V & \mapsto & \Delta^\phi V + trace_g R^N(V, d\phi)d\phi. \end{array} \tag{1.5}$$

One can refer to [2], [3], [4], [7] and [8] for background on biharmonic maps.

2. Some results on doubly twisted product manifolds

In this section, we give the definition and some geometric properties of doubly twisted product manifolds.

Definition 3. Let (M^m, g) and (N^n, h) be two Riemannian manifolds, and $f_1, f_2 : M \times N \to \mathbb{R}$ be smooth positive functions. The twisted metric on $M \times_{f_1, f_2} N$ is defined by

$$G = (f_1)^2 \pi^* g + (f_2)^2 \eta^* h, \tag{2.1}$$

where $\pi : (x, y) \in M \times N \to x \in M$ and $\eta : (x, y) \in M \times N \to y \in N$ are the canonical projections. For all $X, Y \in T(M \times N)$, we have

$$G(X, Y) = f_1^2 g(d\pi(X), d\pi(Y) + f_2^2 h(d\eta(X), d\eta(Y)).$$

Theorem 1. *Let (M^m, g) and (N^n, h) be two Riemannian manifolds. If $\overline{\nabla}$ denote the Levi-Civita connection on $(M \times_{f_1 f_2} N, G)$, then for all $X_1, Y_1 \in \mathcal{H}(M)$ and $X_2, Y_2 \in \mathcal{H}(N)$ we have:*

$$\begin{aligned} \overline{\nabla}_X Y = {} & \nabla_X Y + X(\ln f_1)(Y_1, 0) + X(\ln f_2)(0, Y_2) \\ & + Y(\ln f_1)(X_1, 0) + Y(\ln f_2)(0, X_2) \\ & - \frac{1}{2} g(X_1, Y_1)(\frac{1}{f_1^2} grad_M f_1^2, \frac{1}{f_2^2} grad_N f_1^2) \\ & - \frac{1}{2} h(X_2, Y_2)(\frac{1}{f_1^2} grad_M f_2^2, \frac{1}{f_2^2} grad_N f_2^2), \end{aligned}$$

where $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$ and $\nabla_X Y = (\nabla^M_{X_1} Y^1, \nabla^N_{X_2} Y^2)$.

Proof follows from the Kozul formula and the following lemma:

Lemma 1. *Let $X_1, Y_1, Z_1 \in \mathcal{H}(M)$ and $X_2, Y_2, Z_2 \in \mathcal{H}(N)$. Then*

$$\begin{aligned} X(f_1^2) \cdot g(Y_1, Z_1) &= 2X(\ln f_1) G((Y_1, 0), Z), \\ X(f_2^2) \cdot h(Y_2, Z_2) &= 2X(\ln f_2) G((0, Y_2), Z), \\ Y(f_1^2) \cdot g(X_1, Z_1) &= 2Y(\ln f_1) G((X_1, 0), Z), \\ Y(f_2^2) \cdot h(X_2, Z_2) &= 2Y(\ln f_2) G((0, X_2), Z), \end{aligned}$$

$$Z(f_1^2)\cdot g(X_1,Y_1) = g(X_1,Y_1)G((\frac{1}{f_1^2}grad_M f_1^2, \frac{1}{f_2^2}grad_N f_1^2), Z),$$
$$Z(f_2^2)\cdot h(X_2,Y_2) = h(X_2,Y_2)G((\frac{1}{f_1^2}grad_M f_2^2, \frac{1}{f_2^2}grad_N f_2^2), Z),$$

where $X=(X_1,X_2)$, $Y=(Y_1,Y_2)$ *and* $Z=(Z_1,Z_2)$.

From Theorem 1, we obtain:

Corollary 1. *For all* $X_1, Y_1 \in \mathcal{H}(M)$ *and* $X_2, Y_2 \in \mathcal{H}(N)$, *we have*

$$\begin{aligned}
\overline{\nabla}_{(X_1,0)}(Y_1,0) &= (\nabla^M_{X_1}Y_1,0) + X_1(\ln f_1)(Y_1,0) + Y_1(\ln f_1)(X_2,0)\\
&\quad - \frac{1}{2}g(X_1,Y_1)(\frac{1}{f_1^2}grad_M f_1^2, \frac{1}{f_2^2}grad_N f_1^2),\\
\overline{\nabla}_{(X_1,0)}(0,Y_2) &= X_1(\ln f_2)(0,Y_2) + Y_2(\ln f_1)(X_1,0),\\
\overline{\nabla}_{(0,X_2)}(Y_1,0) &= Y_1(\ln f_2)(0,X_2) + X_2(\ln f_1)(Y_1,0),\\
\overline{\nabla}_{(0,X_2)}(0,Y_2) &= (0,\nabla^N_{X_2}Y_2) + X_2(\ln f_2)(0,Y_2) + Y_2(\ln f_2)(0,X_2)\\
&\quad - \frac{1}{2}h(X_2,Y_2)(\frac{1}{f_1^2}grad_M f_2^2, \frac{1}{f_2^2}grad_N f_2^2).
\end{aligned}$$

Theorem 2. *Let* (M^m,g) *and* (N^n,h) *be two Riemannian manifolds. If* $\overline{\nabla}$ *denote the Levi-Civita connection on* $(M\times_{f_1 f_2} N, G)$ *and* $\widehat{R}$ *is the curvature tensor associate to* $\overline{\nabla}$, *then for all* $X_1, Y_1, Z_1 \in \mathcal{H}(M)$ *and* $X_2, Y_2, Z_2 \in \mathcal{H}(N)$ *we have*

$$\begin{aligned}
&\widehat{R}((X_1,0),(Y_1,0))(Z_1,0)\\
&= (R^M(X_1,Y_1)Z_1,0) - g(Y_1,Z_1)(\nabla^M_{X_1}grad_M \ln f_1,0)\\
&\quad + g(X_1,Z_1)(\nabla^M_{Y_1}grad_M \ln f_1,0)\\
&\quad - g(\nabla^M_{Y_1}grad_M \ln f_1 - Y_1(\ln f_1)grad_M \ln f_1, Z_1)(X_1,0)\\
&\quad + g(\nabla^M_{X_1}grad_M \ln f_1 - X_1(\ln f_1)grad_M \ln f_1, Z_1)(Y_1,0)\\
&\quad - |grad_M \ln f_1|^2\Big(g(Y_1,Z_1)(X_1,0) - g(X_1,Z_1)(Y_1,0)\Big)\\
&\quad + (\frac{f_1}{f_2})^2|grad_N \ln f_1|^2\Big(g(Y_1,Z_1)(X_1,0) - g(X_1,Z_1)(Y_1,0)\Big)\\
&\quad + g(X_1(\ln f_1)Y_1 - Y_1(\ln f_1)X_1, Z_1)(grad_M \ln f_1, 0)\\
&\quad + (\frac{f_1}{f_2})^2 g\Big(X_1(\ln(f_1^2.f_2)) - Y_1(\ln(f_1^2.f_2)), Z_1\Big)(0, grad_N \ln f_1),\\
&\widehat{R}((X_1,0),(Y_1,0))(0,Z_2)\\
&= X_1(Z_2(\ln f_1))(Y_1,0) - Y_1(Z_2(\ln f_1))(X_1,0)\\
&\quad + h(Z_2, grad_N \ln f_1)\Big(Y_1(\ln f_2)X_1 - X_1(\ln f_2)Y_1, 0\Big),\\
&\widehat{R}((X_1,0),(0,Y_2))(Z_1,0)
\end{aligned}$$

$$
\begin{aligned}
&= g\Big(\nabla^M_{X_1} grad_M \ln f_2 + X_1(\ln \frac{f_2}{f_1}) grad_M \ln f_2 - X_1(\ln f_2) grad_M \ln f_1 \\
&\quad + \big(grad_M \ln f_1(\ln f_2) + (\frac{f_1}{f_2})^2 grad_N \ln f_1(\ln f_2)\big) X_1, Z_1\Big)(0, Y_2) \\
&\quad + \Big[Z_1(\ln f_2) Y_2(\ln f_1) - Y_2(Z_1(\ln f_1))\Big](X_1, 0) \\
&\quad - g(Y_2(\ln f_1) X_1, Z_1)(grad_M \ln f_2, 0) \\
&\quad + (\frac{f_1}{f_2})^2 g(X_1, Z_1)\Big(0, \nabla^N_{Y_2} grad_N \ln f_1 + Y_2(\ln \frac{f_1}{f_2}) grad_N \ln f_1 \\
&\quad - Y_2(\ln f_1) grad_N \ln f_2\Big), \\
&\quad \widehat{R}((X_1, 0), (0, Y_2))(0, Z_2) \\
&= \ - h\Big(\nabla^N_{Y_2} grad_N \ln f_1 + Y_2(\ln \frac{f_1}{f_2}) grad_N \ln f_1 - Y_2(\ln f_1) grad_N \ln f_2 \\
&\quad + \big(grad_N \ln f_2(\ln f_1) + (\frac{f_2}{f_1})^2 grad_M \ln f_2(\ln f_1)\big) Y_2, Z_2\Big)(X_1, 0) \\
&\quad + \Big[X_1(Z_2(\ln f_2)) - Z_2(\ln f_1) X_1(\ln f_2)\Big](0, Y_2) \\
&\quad + h(X_1(\ln f_2) Y_2, Z_2)(0, grad_N \ln f_1) \\
&\quad - (\frac{f_2}{f_1})^2 h(Y_2, Z_2)\Big(\nabla^M_{X_1} grad_M \ln f_2 + X_1(\ln \frac{f_2}{f_1}) grad_M \ln f_2 \\
&\quad - X_1(\ln f_2) grad_M \ln f_1, 0\Big), \\
&\quad \widehat{R}((0, X_2), (0, Y_2))(0, Z_2) \\
&= (0, R^N(X_2, Y_2) Z_2) - h(Y_2, Z_2)(0, \nabla^N_{X_2} grad_N \ln f_2) \\
&\quad + h(X_2, Z_2)(0, \nabla^N_{Y_2} grad_N \ln f_2) \\
&\quad - h(\nabla^N_{Y_2} grad_N \ln f_2 - Y_2(\ln f_2) grad_N \ln f_2, Z_2)(0, X_2) \\
&\quad + h(\nabla^N_{X_2} grad_N \ln f_2 - X_2(\ln f_2) grad_N \ln f_2, Z_2)(0, Y_2) \\
&\quad - \Big(|grad_N \ln f_2|^2 - (\frac{f_2}{f_1})^2 |grad_M \ln f_2|^2\Big) h(Y_2, Z_2)(0, X_2) \\
&\quad + \Big(|grad_N \ln f_2|^2 - (\frac{f_2}{f_1})^2 |grad_M \ln f_2|^2\Big) h(X_2, Z_2)(0, Y_2) \\
&\quad + h(X_2(\ln f_2) Y_2 - Y_2(\ln f_2) X_2, Z_2)(0, grad_N \ln f_2) \\
&\quad + (\frac{f_2}{f_1})^2 g\Big(X_2(\ln(f_2^2 . f_1)) - Y_2(\ln(f_2^2 \cdot f_1)), Z_2\Big)(grad_M \ln f_2, 0), \\
&\quad \widehat{R}((0, X_2), (0, Y_2))(Z_1, 0) \\
&= X_2(Z_1(\ln f_2))(0, Y_2) - Y_2(Z_1(\ln f_2))(0, X_2) \\
&\quad + g(Z_1, grad_M \ln f_2)\Big(0, Y_2(\ln f_1) X_2 - X_2(\ln f_1) Y_2\Big),
\end{aligned}
$$

$$
\begin{aligned}
&\widehat{R}((0,X_2),(Y_1,0))(Z_1,0)\\
=& -g\Big(\nabla^M_{Y_1} grad_M \ln f_2 + Y_1(\ln\frac{f_2}{f_1}) grad_M \ln f_2 - Y_1(\ln f_2) grad_M \ln f_1\\
&+ \big(grad_M \ln f_1(\ln f_2) + (\frac{f_1}{f_2})^2 grad_N \ln f_1(\ln f_2)\big)Y_1, Z_1\Big)(0,X_2)\\
&- \Big[Z_1(\ln f_2)X_2(\ln f_1) - X_2(Z_1(\ln f_1))\Big](Y_1,0)\\
&+ g(X_2(\ln f_1)Y_1, Z_1)(grad_M \ln f_2, 0)\\
&- (\frac{f_1}{f_2})^2 g(Y_1,Z_1)\Big(0, \nabla^N_{X_2} grad_N \ln f_1 + X_2(\ln\frac{f_1}{f_2}) grad_N \ln f_1\\
&- X_2(\ln f_1) grad_N \ln f_2\Big),\\
&\widehat{R}((0,X_2),(Y_1,0))(0,Z_2)\\
=& h\Big(\nabla^N_{X_2} grad_N \ln f_1 + X_2(\ln\frac{f_1}{f_2}) grad_N \ln f_1 - X_2(\ln f_1) grad_N \ln f_2\\
&+ \big(grad_N \ln f_2(\ln f_1) + (\frac{f_2}{f_1})^2 grad_M \ln f_2(\ln f_1)\big)X_2, Z_2\Big)(Y_1,0)\\
&- \Big[Y_1(Z_2(\ln f_2)) - Z_2(\ln f_1)Y_1(\ln f_2)\Big](0,X_2)\\
&- h(Y_1(\ln f_2)X_2, Z_2)(0, grad_N \ln f_1)\\
&+ (\frac{f_2}{f_1})^2 h(X_2,Z_2)\Big(\nabla^M_{Y_1} grad_M \ln f_2 + Y_1(\ln\frac{f_2}{f_1}) grad_M \ln f_2\\
&- Y_1(\ln f_2) grad_M \ln f_1, 0\Big).
\end{aligned}
$$

3. Harmonic maps on doubly twisted product manifolds

Let $(M^m,g),(N^n,h)$ and (P^p,k) be Riemannian manifolds of dimensions m,n and p respectively. Let $f_1,f_2 : M\times N \to \mathbb{R}$ be smooth positive functions, and $(M\times_{f_1f_2} N, G)$ be the doubly twisted product manifolds.

3.1. Harmonicity of $\phi : (P,k) \longrightarrow (M\times_{f_1f_2} N, G)$

Theorem 3. *If $\varphi : P \to M$ and $\psi : P \to N$ are regular maps, then the tension field of*

$$
\begin{aligned}
\phi : (P^p,\ell) &\longrightarrow (M\times_{f_1f_2} N, G)\\
x &\longmapsto (\varphi(x),\psi(x))
\end{aligned}
$$

is given by the following relation:

$$
\begin{aligned}
\tau(\phi) =& \Big(\tau(\varphi),\tau(\psi)\Big) + 2\Big(d\varphi(grad_P(\ln f_1\circ\phi)), d\psi(grad_P(\ln f_2\circ\phi))\Big)\\
&- e(\varphi)\Big(\frac{1}{f_1^2} grad_M f_1^2, \frac{1}{f_2^2} grad_N f_1^2\Big)\circ\phi
\end{aligned}
\tag{3.1}
$$

$$- e(\psi)\Big(\frac{1}{f_1^2}grad_M f_2^2, \frac{1}{f_2^2}grad_N f_2^2\Big)\circ\phi.$$

Proof. Choose a local orthonormal frame $(e_i)_i$ with respect to ℓ on M. Then by definition of tension field, we have

$$\begin{aligned}
\tau(\phi) &= tr_k \nabla d\phi \\
&= \nabla_{e_i} d\phi(e_i) - d\phi(\nabla^P_{e_i} e_i) \\
&= \widehat{\nabla}_{(d\varphi(e_i), d\psi(e_i))}(d\varphi(e_i), d\psi(e_i)) - \Big(d\varphi(\nabla^P_{e_i} e_i), d\psi(\nabla^P_{e_i} e_i)\Big) \\
&= \nabla_{(d\varphi(e_i), d\psi(e_i))}(d\varphi(e_i), d\psi(e_i)) + 2(d\varphi(e_i), d\psi(e_i))(\ln f_1)(d\varphi(e_i), 0) \\
&\quad + 2(d\varphi(e_i), d\psi(e_i))(\ln f_2)(0, d\psi(e_i)) \\
&\quad - e(\varphi)\Big(\frac{1}{f_1^2}grad_M f_1^2, \frac{1}{f_2^2}grad_N f_1^2\Big)\circ\phi \\
&\quad - e(\psi)\Big(\frac{1}{f_1^2}grad_M f_2^2, \frac{1}{f_2^2}grad_N f_2^2\Big)\circ\phi - \Big(d\varphi(\nabla^P_{e_i} e_i), d\psi(\nabla^P_{e_i} e_i)\Big) \\
&= \Big(\tau(\varphi), \tau(\psi)\Big) + 2\Big(d\varphi(grad_P(\ln f_1\circ\phi)), d\psi(grad_P(\ln f_2\circ\phi))\Big) \\
&\quad - e(\varphi)\Big(\frac{1}{f_1^2}grad_M f_1^2, \frac{1}{f_2^2}grad_N f_1^2\Big)\circ\phi \\
&\quad - e(\psi)\Big(\frac{1}{f_1^2}grad_M f_2^2, \frac{1}{f_2^2}grad_N f_2^2\Big)\circ\phi.
\end{aligned}$$

□

From Theorem 3, we deduce:

Corollary 2. *The tension field of*

$$\begin{aligned}
\phi_1 : (M, g) &\longrightarrow (M \times_{f_1 f_2} N, G) \\
x &\longmapsto (\varphi(x), y_0)
\end{aligned}$$

is given by

$$\begin{aligned}
\tau(\phi_1) = (\tau(\varphi), 0) &+ 2(d\varphi(grad_M(\ln f_1\circ\phi)), 0) \\
&- e(\varphi)(\frac{1}{f_1^2}grad_M f_1^2, \frac{1}{f_2^2}grad_N f_1^2)\circ\phi.
\end{aligned}$$

Corollary 3. *The tension field of*

$$\begin{aligned}
\phi_2 : (N, h) &\longrightarrow (M \times_{f_1 f_2} N, G) \\
y &\longmapsto (x_0, \psi(y))
\end{aligned}$$

is given by

$$\begin{aligned}
\tau(\phi_2) = (0, \tau(\psi)) &+ 2(0, d\psi(grad_N(\ln f_2\circ\phi))) \\
&- e(\psi)(\frac{1}{f_1^2}grad_M f_2^2, \frac{1}{f_2^2}grad_N f_2^2)\circ\phi.
\end{aligned}$$

Corollary 4. *If $\varphi = Id_M$ and $\psi = Id_N$, then*

$$\tau(\phi_1) = (2-m)(grad_M \ln f_1, 0) - \frac{m}{2f_2^2}(0, grad_N f_1^2),$$
$$\tau(\phi_2) = (2-n)(0, grad_N \ln f_2) - \frac{n}{2f_1^2}(grad_M f_2^2, 0).$$

From definition of conformal map and Theorem 3, we obtain:

Theorem 4. *Let $\varphi : M \to M$ be conformal map with dilatation λ, then the tension field of*

$$\begin{array}{rcl} \phi : (M,g) & \longrightarrow & (M \times_{f_1 f_2} M, G) \\ x & \longmapsto & (\varphi(x), \varphi(x)) \end{array}$$

is given by

$$\begin{aligned} \tau(\phi) = {} & (2-m)\Big(d\varphi(grad \ln \lambda), d\varphi(grad \ln \lambda)\Big) \\ & + 2(d\varphi(grad \ln f_1 \circ \phi), d\varphi(grad \ln f_2 \circ \phi)) \\ & - \frac{m}{2}\lambda^2\Big(\frac{1}{f_1^2}(gradf_1^2 + gradf_2^2), \frac{1}{f_2^2}(gradf_1^2 + gradf_2^2)\Big) \circ \phi. \end{aligned}$$

3.2. Harmonicity of $\phi : (M \times_{f_1 f_2} N, G) \longrightarrow (P, k)$

Let $\phi : (x,y) \in (M \times_{f_1 f_2} N, G) \longrightarrow \phi(x,y) \in (P,k)$ be smooth map. If we denote by

$$\begin{array}{rcl} \phi_N = \phi_N^x : (N,h) & \longrightarrow & (P,k) \\ y & \longmapsto & \phi_N^x(y) = \phi(x,y) \end{array}$$

and

$$\begin{array}{rcl} \phi_M = \phi_M^y : (M,g) & \longrightarrow & (P,k) \\ x & \longmapsto & \phi_M^y(x) = \phi(x,y), \end{array}$$

then for all $X \in \mathcal{H}(M)$, $Y \in \mathcal{H}(N)$ and $(x,y) \in M \times N$, we have:

$$\begin{cases} d_{(x,y)}\phi(X,0) = d_x\phi_M^y(X) = d_x\phi_M(X), \\ d_{(x,y)}\phi(0,Y) = d_y\phi_N^x(Y) = d_y\phi_N(Y). \end{cases}$$

Theorem 5. *The tension field of $\phi : (M \times_{f_1 f_2} N, G) \longrightarrow (P,k)$ is given by:*

$$\begin{aligned} \tau(\phi) = {} & \frac{1}{f_1^2}\Big(\tau(\phi_M) + (m-2)d\phi_M(grad_M \ln f_1) + nd\phi_M(grad_M \ln f_2)\Big) \\ & + \frac{1}{f_2^2}\Big(\tau(\phi_N) + (n-2)d\phi_N(grad_N \ln f_2) + md\phi_N(grad_N \ln f_1)\Big). \end{aligned} \tag{3.2}$$

Proof. Any local orthonormal frame $\{E_i, i = \overline{1,m}\}$ and $\{F_j, j = \overline{1,n}\}$ on (M^m, g) and (N^n, h) respectively, induces a local orthonormal frame on the doubly twisted product manifolds $(M \times_{f_1 f_2} N, G)$ by

$$\{U_i = (E_i, 0),\ U_{m+j} = (0, \frac{1}{f}F_j) : i = \overline{1,m},\ j = \overline{1,n}\}.$$

Using formula of tension field, we have:

$$\begin{aligned}\tau(\phi) &= tr_G \nabla^P d\phi \\ &= \sum_{k=1}^{m+n} \nabla^P d\phi(U_k, U_k) \\ &= + \sum_{i=1}^{m} \Big\{ \frac{1}{f_1} \nabla^P_{d\phi(E_i,0)} \frac{1}{f_1} d\phi(E_i, 0) - d\phi(\frac{1}{f_1} \widehat{\nabla}_{(E_i,0)} \frac{1}{f_1}(E_i, 0)) \Big\} \\ &\quad + \sum_{j=1}^{n} \Big\{ \frac{1}{f_2} \nabla^P_{d\phi(0,F_j)} \frac{1}{f_2} d\phi(0, F_j) - d\phi(\frac{1}{f_2} \widehat{\nabla}_{(0,F_j)} \frac{1}{f_2}(0, F_j)) \Big\},\end{aligned}$$

$$\begin{aligned}\tau(\phi) = & \sum_{i=1}^{m} \frac{1}{f_1} \Big\{ E_i(\frac{1}{f_1}) d\phi(E_i, 0) + \frac{1}{f_1} \nabla_{(E_i,0)} d\phi(E_i, 0) - d\phi(E_i(\frac{1}{f_1})(E_i, 0) \\ & + \frac{1}{f_1}((\nabla^M_{E_i} E_i, 0) + 2E_i(\ln f_1)(E_i, 0) \\ & + \frac{m}{2}(\frac{1}{f_1^2} grad_M f_1^2, \frac{1}{f_2^2} grad_N f_1^2)))) \Big\} \\ & + \sum_{j=1}^{n} \frac{1}{f_2} \Big\{ F_j(\frac{1}{f_2}) d\phi(0, F_j) + \frac{1}{f_2} \nabla_{(0,F_i)} d\phi(0, F_j) - d\phi(F_j(\frac{1}{f_2})(0, F_j) \\ & + \frac{1}{f_2}((0, \nabla^N_{F_j} F_j) + 2F_j(\ln f_2)(0, F_j) \\ & + \frac{n}{2}(\frac{1}{f_1^2} grad_M f_2^2, \frac{1}{f_2^2} grad_N f_2^2)))) \Big\},\end{aligned}$$

$$\begin{aligned}\tau(\phi) = & \frac{1}{f_1^2} \Big\{ - d\phi_M(grad_M \ln f_1) + \nabla_{d\phi_M(E_i)} d\phi_M(E_i) - d\phi_M(\nabla^M_{E_i} E_i) \\ & - (1-m) d\phi_M(grad_M \ln f_1) + \frac{m}{2f_2^2} d\phi_N(grad_N f_1^2) \Big\} \\ & + \frac{1}{f_2^2} \Big\{ - d\phi_N(grad_N \ln f_2) + \nabla_{d\phi_N(Fj)} d\phi_N(F_j) - d\phi_N(\nabla^N_{F_j} F_j) \\ & - (1-n) d\phi_N(grad_N \ln f_2) + \frac{n}{2f_1^2} d\phi_M(grad_M f_2^2) \Big\}\end{aligned}$$

hence

$$\begin{aligned}\tau(\phi) = & \frac{1}{f_1^2} \Big(\tau(\phi_M) + (m-2) d\phi_M(grad_M \ln f_1) + n d\phi_M(grad_M \ln f_2) \Big) \\ & + \frac{1}{f_2^2} \Big(\tau(\phi_N) + (n-2) d\phi_N(grad_N \ln f_2) + + m d\phi_N(grad_N \ln f_1) \Big). \end{aligned} \tag{3.3}$$

□

3.3. Harmonicity of $\phi : (M \times_{f_1 f_2} N, G) \longrightarrow (P \times_{\alpha_1,\alpha_2} B, G')$

Let $(M^m, g), (N^n, h)$, (P^p, ℓ) and (Q^q, ρ) be Riemannian manifolds of dimensions m, n, p and q respectively, $f_1, f_2 : M \times N \to \mathbb{R}$ (resp $\alpha_1, \alpha_2 : P \times Q \to \mathbb{R}$) be smooth positive functions, and $(M \times_{f_1 f_2} N, G = G_{f_1 f_2})$ (resp $(P \times_\alpha Q, G' = G_{\alpha_1,\alpha_2})$) be the doubly twisted product manifolds of (M^m, g) and (N^n, h) (respectively (P^p, ℓ) and (Q^q, ρ)).

Theorem 6. *Let $\varphi : (M, g) \longrightarrow (P, \ell)$ and $\psi : (N, h) \longrightarrow (Q, \rho)$ be a smooth maps. The tension field of*

$$\begin{array}{rcl} \phi : (M \times_{f_1 f_2} N, G) & \longrightarrow & (P \times_{\alpha_1,\alpha_2} Q, G') \\ (x, y) & \longmapsto & (\varphi(x), \psi(y)) \end{array}$$

is given by:

$$\begin{aligned} \tau(\phi) = \frac{1}{f_1^2}\Big[&(\tau(\varphi), 0) + (m-2)(d\varphi(grad_M \ln f_1), 0) + n(d\varphi(grad_M \ln f_2), 0) \\ &+ 2(d\varphi(grad_M(\ln\alpha_1 \circ \phi)), 0) - e(\varphi)(\frac{1}{\alpha_1^2} grad_P \alpha_1^2, \frac{1}{\alpha_2^2} grad_Q \alpha_1^2)\Big] \\ &+ \frac{1}{f_2^2}\Big[(0, \tau(\psi)) + (n-2)(0, d\psi(grad_N \ln f_2)) + m(0, d\psi(grad_N \ln f_1)) \\ &+ 2(0, d\psi(grad_N(\ln\alpha_2 \circ \phi))) - e(\psi)(\frac{1}{\alpha_1^2} grad_P \alpha_2^2, \frac{1}{\alpha_2^2} grad_Q \alpha_2^2)\Big]. \end{aligned} \tag{3.4}$$

Proof. Let $(E_i)_i$ (resp$(F_j)_j$) be an orthonormal frame on (M, g) (resp(N, h)). If $\overline{\nabla}$(resp$\widetilde{\nabla}$) denote the Levi-Civita connection on the doubly twisted product manifolds $(M \times_f N, G_f)$ and $(P \times_\alpha Q, G_\alpha)$ respectively, then we have

$$\begin{aligned} \tau(\phi) &= tr_G \widetilde{\nabla} d\phi \\ &= \frac{1}{f_1}\widetilde{\nabla}_{(d\varphi(E_i),0)} \frac{1}{f_1}(d\varphi(E_i), 0) - d\phi(\frac{1}{f_1}\overline{\nabla}_{(E_i,0)} \frac{1}{f_1}(E_i, 0)) \\ &\quad + \frac{1}{f_2}\widetilde{\nabla}_{(0,d\psi(F_i))} \frac{1}{f_2}(0, d\psi(F_i) - d\phi(\frac{1}{f_2}\overline{\nabla}_{(0,F_i)} \frac{1}{f_2}(0, F_i)) \\ &= \frac{1}{f_1^2}\Big[-(d\varphi(grad_M \ln f_1), 0) + (\nabla^P_{d\varphi(E_i)} d\varphi(E_i), 0) \\ &\quad + 2(d\varphi(grad_M(\ln\alpha_1 \circ \varphi)), 0) - (d\varphi(\nabla^M_{E_i} E_i), 0) \\ &\quad - (1-m)(d\varphi(grad_M \ln f_1), 0) - e(\varphi)(\frac{1}{\alpha_1^2} grad_P \alpha_1^2, \frac{1}{\alpha_2^2} grad_Q \alpha_1^2)\Big] \\ &\quad + \frac{m}{f_2^2}(0, d\psi(grad_N \ln f_1)) + \frac{1}{f_2^2}\Big[-(0, d\psi(grad_M \ln f_2)) \\ &\quad + (0, \nabla^B_{d\psi(F_i)} d\psi(F_i)) + 2(0, d\psi(grad_N(\ln\alpha_2 \circ \psi))) \\ &\quad - (0, d\psi(\nabla^N_{F_i} F_i)) - (1-n)(0, d\psi(grad_N \ln f_2)) \end{aligned}$$

$$
\begin{aligned}
&- e(\psi)(\frac{1}{\alpha_1^2} grad_P \alpha_2^2, \frac{1}{\alpha_2^2} grad_Q \alpha_2^2)\Big] + \frac{n}{f_1^2}(d\varphi(grad_M \ln f_2), 0) \\
= &\frac{1}{f_1^2}\Big[(\tau(\varphi), 0) + (m-2)(d\varphi(grad_M \ln f_1), 0) + n(d\varphi(grad_M \ln f_2), 0) \\
&+ 2(d\varphi(grad_M(\ln \alpha_1 \circ \varphi)), 0) - e(\varphi)(\frac{1}{\alpha_1^2} grad_P \alpha_1^2, \frac{1}{\alpha_2^2} grad_Q \alpha_1^2)\Big] \\
&+ \frac{1}{f_2^2}\Big[(0, \tau(\psi)) + (n-2)(0, d\psi(grad_N \ln f_2)) + m(0, d\psi(grad_M \ln f_1)) \\
&+ 2(0, d\psi(grad_N(\ln \alpha_2 \circ \psi))) - e(\psi)(\frac{1}{\alpha_1^2} grad_P \alpha_2^2, \frac{1}{\alpha_2^2} grad_Q \alpha_2^2)\Big].
\end{aligned}
$$

□

From Theorem 6, follows:

Corollary 5. *The tension field of*

$$
\begin{array}{rcl}
\phi = Id_{M\times N} : (M \times_{f_1 f_2} N, G) & \longrightarrow & (M \times_{\alpha_1 \alpha_2} N, G') \\
(x, y) & \longmapsto & (x, y)
\end{array}
$$

is given by

$$
\begin{aligned}
\tau(\phi) = &\frac{1}{f_1^2}\Big(grad_M \ln(f_2^n(\frac{f_1}{\alpha_1})^{m-2}) - \frac{n}{2(f_2\alpha_1)^2} grad_M \alpha_2^2, 0\Big) \\
&\frac{1}{f_2^2}\Big(0, grad_N \ln(f_1^m(\frac{f_2}{\alpha_2})^{n-2}) - \frac{m}{2(f_1\alpha_2)^2} grad_N \alpha_1^2\Big).
\end{aligned}
$$

Corollary 6. *If $\varphi : M \to M$ and $\psi : N \to N$ are harmonic maps, then the tension fields of*

$$
\begin{array}{rcl}
\phi_1 : (M \times_{f_1 f_2} N, G) & \longrightarrow & (M \times N, g \oplus h) \\
(x, y) & \longmapsto & (\varphi(x), \psi(y))
\end{array}
$$

and

$$
\begin{array}{rcl}
\phi_2 : (M \times N, g \oplus h) & \longrightarrow & (M \times_{\alpha_1 \alpha_2} N, G') \\
(x, y) & \longmapsto & (\varphi(x), \psi(y))
\end{array}
$$

are given by the following formulas:

$$
\begin{aligned}
\tau(\phi_1) = &\frac{1}{f_1^2}\Big[(m-2)(d\varphi(grad_M \ln f_1), 0) + n(d\varphi(grad_M \ln f_2), 0)\Big] \\
&+ \frac{1}{f_2^2}\Big[(n-2)(0, d\psi(grad_N \ln f_2)) + m(0, d\psi(grad_M \ln f_1))\Big], \\
\tau(\phi_2) = &\Big[2(d\varphi(grad_M(\ln \alpha_1 \circ \phi_1)), 0) - e(\varphi)(\frac{1}{\alpha_1^2} grad_P \alpha_1^2, \frac{1}{\alpha_2^2} grad_Q \alpha_1^2)\Big] \\
&+ \Big[2(0, d\psi(grad_N(\ln \alpha_2 \circ \phi_2))) - e(\psi)(\frac{1}{\alpha_1^2} grad_P \alpha_2^2, \frac{1}{\alpha_2^2} grad_Q \alpha_2^2)\Big].
\end{aligned}
$$

Corollary 7. *The tension fields of*

$$\begin{array}{rcl} Id_1 : (M \times_{f_1 f_2} N, G) & \longrightarrow & (M \times N, g \oplus h) \\ (x,y) & \longmapsto & (x,y) \end{array}$$

and

$$\begin{array}{rcl} Id_2 : (M \times N, g \oplus h) & \longrightarrow & (M \times_{\alpha_1 \alpha_2} N, G') \\ (x,y) & \longmapsto & (x,y) \end{array}$$

are given by the following formulas:

$$\begin{aligned} \tau(Id_1) &= \frac{1}{f_1^2}\Big[(m-2)(grad_M \ln f_1, 0) + n(grad_M \ln f_2, 0)\Big] \\ &\quad + \frac{1}{f_2^2}\Big[(n-2)(0, grad_N \ln f_2) + m(0, grad_N \ln f_1)\Big], \\ \tau(Id_2) &= \Big[\frac{1}{\alpha_1^2}(grad_M(\alpha_1^2), 0) - \frac{m}{2}(\frac{1}{\alpha_1^2} grad_M \alpha_1^2, \frac{1}{\alpha_2^2} grad_N \alpha_1^2)\Big] \\ &\quad + \Big[\frac{1}{\alpha_2^2}(0, grad_N(\alpha_2^2)) - \frac{n}{2}(\frac{1}{\alpha_1^2} grad_M \alpha_2^2, \frac{1}{\alpha_2^2} grad_N \alpha_2^2)\Big]. \end{aligned}$$

Therefore,

(1) Id_1 *is harmonic if and only if*

$$f_1^{(m-2)} f_2^n = C_1(y) \quad \textit{and} \quad f_1^m f_2^{(n-2)} = C_2(x),$$

where $C_2 \in C^\infty(M)$ *and* $C_1 \in C^\infty(N)$

(2) Id_2 *is harmonic if and only if*

$$(2-m)\alpha_1^2 = n\alpha_2^2 + C_1(y) \quad \textit{and} \quad (2-n)\alpha_2^2 = m\alpha_1^2 + C_2(x),$$

where $C_2 \in C^\infty(M)$ *and* $C_1 \in C^\infty(N)$.

Theorem 7. *Let* $\varphi : (M^m, g) \to (P^m, \ell)$ *and* $\psi : (N^n, h) \to (Q^n, \rho)$ *are conformal maps with dilation* λ *and* μ *respectively. Then the tension field of*

$$\begin{array}{rcl} \phi : (M \times_{f_1 f_2} N, G) & \longrightarrow & (P \times_{\alpha_1 \alpha_2} Q, G') \\ (x,y) & \longmapsto & (\varphi(x), \psi(y)) \end{array}$$

is given by

$$\begin{aligned} \tau(\phi) &= \frac{1}{f_1^2}\Big[((m-2)(d\varphi(grad_M \ln(\frac{f_1}{\lambda})), 0) + n(d\varphi(grad_M \ln f_2), 0) \\ &\quad + 2(d\varphi(grad_M(\ln \alpha_1 \circ \phi)), 0) - \frac{m}{2}\lambda^2(\frac{1}{\alpha_1^2} grad_P \alpha_1^2, \frac{1}{\alpha_2^2} grad_Q \alpha_1^2)\Big] \\ &\quad + \frac{1}{f_2^2}\Big[(n-2)(0, d\psi(grad_N \ln(\frac{f_2}{\mu}))) + m(0, d\psi(grad_N \ln f_1)) \\ &\quad + 2(0, d\psi(grad_N(\ln \alpha_2 \circ \phi))) - \frac{n}{2}\mu^2(\frac{1}{\alpha_1^2} grad_P \alpha_2^2, \frac{1}{\alpha_2^2} grad_Q \alpha_2^2)\Big]. \end{aligned}$$

Proof. Since φ and ψ are conform, then we have

$$\tau(\varphi) = (2-m)d\varphi(grad_M ln\lambda) \quad \text{and} \quad e(\varphi) = \frac{m}{2}\lambda^2 \tag{3.5}$$

and

$$\tau(\psi) = (2-m)d\varphi(grad_M ln\mu) \quad \text{and} \quad e(\psi) = \frac{m}{2}\mu^2 \tag{3.6}$$

(see [1]). Substituting (3.5) and (3.6) in (3.4), Theorem 7 follows. □

From Proposition 7, we obtain:

Theorem 8. *Let $\varphi : (M^m, g) \to (P^m, \ell)$ and $\psi : (N^n, h) \to (Q^n, \rho)$ are conformal maps with dilation λ and μ respectively. Then the tension field of*

$$\begin{aligned} \phi : (M \times_{f_1 f_2} N, G) &\longrightarrow (P \times Q, \ell \oplus \rho) \\ (x, y) &\longmapsto (\varphi(x), \psi(y)) \end{aligned}$$

is given by

$$\begin{aligned} \tau(\phi) &= \frac{1}{f_1^2}\Big[((m-2)(d\varphi(grad_M \ln(\frac{f_1}{\lambda})), 0) + n(d\varphi(grad_M \ln f_2), 0) \\ &\quad + \frac{1}{f_2^2}\Big[(n-2)(0, d\psi(grad_N \ln(\frac{f_2}{\mu}))) + m(0, d\psi(grad_N \ln f_1)) \\ &= (\frac{1}{f_1^2} d\varphi(grad_M \ln f_2^n (\frac{f_1}{\lambda})^{(m-2)}), 0) + (0, \frac{1}{f_2^2} d\psi(grad_N \ln f_1^m (\frac{f_2}{\mu})^{(n-2)})) \end{aligned}$$

and ϕ is harmonic if and only if

$$\begin{aligned} f_2^n \cdot f_1^{(m-2)} &= C_1(y)\lambda^{(m-2)}(x), \\ f_1^m \cdot f_2^{(n-2)} &= C_2(x)\mu^{(n-2)}(y). \end{aligned}$$

Remark 1. From Corollary 7 and Theorem 8, we can construct an infinite examples of harmonic maps.

4. Biharmonicity of $\phi : (M \times_{f_1, f_2} N, G) \longrightarrow (P^p, k)$

Lemma 2. *Let $\lambda \in C^\infty(M \times N)$ be a mooth function and $\sigma \in \Gamma(\phi^{-1}TP)$. Then*

$$\begin{aligned} J_\phi(\lambda\sigma) &= \frac{1}{f_1^2}\Big[\lambda J_{\phi_M}(\sigma) + \Delta_M(\lambda)\sigma + 2\nabla^{\phi_M}_{grad-M\lambda}\sigma + (m-2)(grad_M \ln f_1)(\lambda)\sigma \\ &\quad + (m-2)\lambda\nabla^{\phi_M}_{grad_M \ln f_1}\sigma + n\lambda\nabla^{\phi_M}_{grad_M \ln f_2}\sigma + n(grad_M \ln f_2)(\lambda)\sigma\Big] \\ &\quad + \frac{1}{f_2^2}\Big[\lambda J_{\phi_N}(\sigma) + \Delta_N(\lambda)\sigma + 2\nabla^{\phi_N}_{grad-N\lambda}\sigma + (n-2)(grad_N \ln f_2)(\lambda)\sigma \\ &\quad + (n-2)\lambda\nabla^{\phi_N}_{grad_N \ln f_2}\sigma + m\lambda\nabla^{\phi_N}_{grad_N \ln f_1}\sigma + m(grad_N \ln f_1)(\lambda)\sigma\Big]. \end{aligned} \tag{4.1}$$

Proof. Let $(E_i)_{i=1}^m$ and $(F_j)_{j=1}^n$ be a local orthonormal frame on M and N respectively. from the expression of Jacobi operatorDe (formula (1.5)), we have

$$J_\phi(\lambda\sigma) = trace_G(\nabla^\phi)^2(\lambda\sigma) + trace_G R^p(\lambda\sigma, d\phi)d\phi. \tag{4.2}$$

By calculating each term, we obtain:

$$\begin{aligned} trace_G(\nabla^\phi)^2(\lambda\sigma) = & \sum_{i=1}^m \Big[\frac{1}{f_1}\nabla^\phi_{(E_i,0)}\frac{1}{f_1}\nabla^\phi_{(E_i,0)}\lambda\sigma - \nabla^\phi_{\widehat{\nabla}_{\frac{1}{f}(E_i,0)}\frac{1}{f_1}(E_i,0)}\lambda\sigma\Big] \\ & + \sum_{j=1}^n \Big[\frac{1}{f_2}\nabla^\phi_{(0,F_j)}\frac{1}{f_2}\nabla^\phi_{(0,F_j)}\lambda\sigma - \nabla^\phi_{\widehat{\nabla}_{\frac{1}{f_2}(0,F_j)}\frac{1}{f_2}(0,F_j)}\lambda\sigma\Big], \end{aligned} \tag{4.3}$$

$$\begin{aligned} \sum_{i=1}^m \frac{1}{f_1}\nabla^\phi_{(E_i,0)}\frac{1}{f_1}\nabla^\phi_{(E_i,0)}\lambda\sigma = & \frac{1}{f_1^2}\Big[- (grad_M \ln f_1)(\lambda)\sigma - \lambda\nabla^{\phi_M}_{grad_M \ln f_1}\sigma \\ & + \Delta_M(\lambda)\sigma + 2\nabla^{\phi_M}_{grad_M \lambda}\sigma + \lambda\nabla^{\phi_M}_{E_i}\nabla^{\phi_M}_{E_i}\sigma\Big], \end{aligned} \tag{4.4}$$

$$\begin{aligned} \sum_{i=1}^m \nabla^\phi_{\widehat{\nabla}_{\frac{1}{f_1}(E_i,0)}\frac{1}{f_1}(E_i,0)}\lambda\sigma = & \frac{1-m}{f_1^2}\Big[(grad_M \ln f_1)(\lambda)\sigma + \lambda\nabla^{\phi_M}_{grad_M \ln f_1}\sigma\Big] \\ & - \frac{m}{f_2^2}\Big[(grad_N \ln f_1)(\lambda)\sigma + \lambda\nabla^{\phi_N}_{grad_N \ln f_1}\sigma\Big], \end{aligned} \tag{4.5}$$

$$\begin{aligned} \sum_{j=1}^n \frac{1}{f_2}\nabla^\phi_{(0,F_j)}\frac{1}{f_2}\nabla^\phi_{(0,F_j)}\lambda\sigma = & \frac{1}{f_2^2}\Big[- (grad_N \ln f_2)(\lambda)\sigma - \lambda\nabla^{\phi_N}_{grad_N \ln f_2}\sigma \\ & + \Delta_N(\lambda)\sigma + 2\nabla^{\phi_N}_{grad_N \lambda}\sigma + \lambda\nabla^{\phi_N}_{F_j}\nabla^{\phi_N}_{F_j}\sigma\Big], \end{aligned} \tag{4.6}$$

$$\begin{aligned} \sum_{j=1}^n \nabla^\phi_{\widehat{\nabla}_{\frac{1}{f_2}(0,F_j)}\frac{1}{f_2}(0,F_j)}\lambda\sigma = & \frac{1-n}{f_2^2}\Big[(grad_N \ln f_2)(\lambda)\sigma + \lambda\nabla^{\phi_N}_{grad_N \ln f_2}\sigma\Big] \\ & - \frac{n}{f_1^2}\Big[(grad_M \ln f_2)(\lambda)\sigma + \lambda\nabla^{\phi_M}_{grad_M \ln f_2}\sigma\Big], \end{aligned} \tag{4.7}$$

$$\begin{aligned} trace_G R^p(\lambda\sigma, d\phi)d\phi = & \frac{\lambda}{f_1^2} trace_g R^p(\sigma, d\phi_M)d\phi_M \\ & + \frac{\lambda}{f_2^2} trace_h R^p(\sigma, d\phi_N)d\phi_N. \end{aligned} \tag{4.8}$$

Subsisting (4.5), (4.6) and (4.7) in (4.4) and summing with (4.8) we obtain (4.1). □

Theorem 9. *Let (M^m, g), (N^n, h), (P^p, k) be Riemannian manifolds and $f_1, f_2 : M \times N \to \mathbb{R}$ be smooth positive functions. Then the bitension fields of $\phi : (M^m \times_{f_1,f_2} N^n, G) \longrightarrow (P^p, k)$ is given by*

$$\begin{aligned}
\tau_2(\phi) = {} & \frac{1}{f_1^4}\Big[\tau_2(\phi_M) + (m-2)J_{\phi_M}(d\phi_M(grad_M \ln f_1)) \\
& + nJ_{\phi_M}(d\phi_M(grad_M \ln f_2)) + 2\Delta_M(\ln f_1)V \\
& + 2(m-4)|grad_M \ln f_1|^2 V + (6-m)\nabla^{\phi_M}_{grad_M \ln f_1} V \\
& - n\nabla^{\phi_M}_{grad_M \ln f_2} V + 2n(grad_M \ln f_2)(\ln f_1)V\Big] \\
& + \frac{1}{f_2^4}\Big[\tau_2(\phi_N) + (n-2)J_{\phi_N}(d\phi_N(grad_N \ln f_2)) \\
& + mJ_{\phi_N}(d\phi_N(grad_N \ln f_1)) + 2\Delta_N(\ln f_2)W \\
& + 2(n-4)|grad_N \ln f_2|^2 W + (6-n)\nabla^{\phi_N}_{grad_N \ln f_2} W - m\nabla^{\phi_N}_{grad_N \ln f_1} W \\
& + 2m(grad_N \ln f_1)(\ln f_2)W\Big] \\
& + \frac{1}{(f_1 f_2)^2}\Big[J_{\phi_M}(W) + J_{\phi_N}(V) + 2\Delta_M(\ln f_2)W + +2\Delta_N(\ln f_1)V \\
& + (2-m)\nabla^{\phi_M}_{grad_M \ln f_1} W + (2-n)\nabla^{\phi_N}_{grad_N \ln f_2} V \\
& + 2(n-2)|grad_M \ln f_2|^2 W + 2(m-2)|grad_N \ln f_1|^2 V \\
& + (4-n)\nabla^{\phi_M}_{grad_M \ln f_2} W + (4-m)\nabla^{\phi_N}_{grad_N \ln f_1} V \\
& + 2(m-2)(grad_M \ln f_1)(\ln f_2)W + 2(n-2)(grad_N \ln f_2)(\ln f_1)V\Big],
\end{aligned} \tag{4.9}$$

where

$$V = \tau(\phi_M) + (m-2)d\phi_M(grad_M \ln f_1) + nd\phi_M(grad_M \ln f_2),$$

and

$$W = \tau(\phi_N) + (n-2)d\phi_N(grad_N \ln f_2) + md\phi_N(grad_N \ln f_1).$$

Proof. From Lemma 2, we obtain

$$\begin{aligned}
J_\phi(\frac{1}{f_1^2}V) = {} & \frac{1}{f_1^4}\Big[\tau_2(\phi_M) + (m-2)J_{\phi_M}(d\phi_M(grad_M \ln f_1)) \\
& + nJ_{\phi_M}(d\phi_M(grad_M \ln f_2)) + 2\Delta_M(\ln f_1)V \\
& + 2(m-4)|grad_M \ln f_1|^2 V + (6-m)\nabla^{\phi_M}_{grad_M \ln f_1} V \\
& - n\nabla^{\phi_M}_{grad_M \ln f_2} V + 2n(grad_M \ln f_2)(\ln f_1)V\Big] \\
& + \frac{1}{(f_1 f_2)^2}\Big[J_{\phi_N}(V) + 2\Delta_N(\ln f_1)V + (2-n)\nabla^{\phi_N}_{grad_N \ln f_2} V \\
& + 2(m-2)|grad_N \ln f_1|^2 V + (4-m)\nabla^{\phi_N}_{grad_N \ln f_1} V
\end{aligned} \tag{4.10}$$

$$+ 2(n-2)(grad_N \ln f_2)(\ln f_1)V\Big],$$

$$\begin{aligned} J_\phi(\frac{1}{f_2^2}W) = \ & + \frac{1}{f_2^4}\Big[\tau_2(\phi_N) + (n-2)J_{\phi_N}(d\phi_N(grad_N \ln f_2)) \\ & + mJ_{\phi_N}(d\phi_N(grad_N \ln f_1)) + 2\Delta_N(\ln f_2)W \\ & + 2(n-4)|grad_N \ln f_2|^2 W + (6-n)\nabla^{\phi_N}_{grad_N \ln f_2} W \\ & - m\nabla^{\phi_N}_{grad_N \ln f_1} W + 2m(grad_N \ln f_1)(\ln f_2)W\Big] \\ & + \frac{1}{(f_1 f_2)^2}\Big[J_{\phi_M}(W) + 2\Delta_M(\ln f_2)W + (2-m)\nabla^{\phi_M}_{grad_M \ln f_1} W \\ & + 2(n-2)|grad_M \ln f_2|^2 W + (4-n)\nabla^{\phi_M}_{grad_M \ln f_2} W \\ & + 2(m-2)(grad_M \ln f_1)(\ln f_2)W\Big]. \end{aligned} \tag{4.11}$$

By formulae (4.10), (4.11) and (1.4) we obtain Theorem 9. □

Example 1. If $a, b \in \mathbb{R}_+^*$ such $a \neq b$, then

$$\begin{aligned} \phi : \mathbb{R} \times_{a,b} \mathbb{R} \ & \longrightarrow \ \mathbb{R} \\ (x, y) \ & \longmapsto \ x^2 - y^2 \end{aligned}$$

is a proper biharmonic map (i.e., biharmonic no harmonic map). From Theorem 5 we have

$$\tau(\phi) = \frac{2}{a^2} - \frac{2}{b^2}$$

and from Theorem (9), we have

$$\tau_2(\phi) = 0.$$

Theorem 10. *Let $f \in C^\infty(M)$ be a smooth function and $\varphi : (M^m, g) \longrightarrow (P^m, k)$ $(m \geq 3)$ be a conformal map with dilation λ. Then the bitension*

$$\phi : (x, y) \in (M \times_{f,f} N, G) \longrightarrow \phi(x, y) = \varphi(x) \in (P, k)$$

is given by

$$\begin{aligned} \tau_2(\phi) = \ & \frac{1}{f^4}\Big[J_\varphi(d\varphi(grad_M \ln \mu)) + (6-m-n)\nabla^\varphi_{grad_M \ln f} d\varphi(grad_M \ln \mu) \\ & + \big(2\Delta_M(\ln f) + 2\Delta_N(\ln f) + 2(m+n-4)|grad_M \ln f|^2 \\ & + 2(m+n-4)|grad_N \ln f|^2\big)d\varphi(grad_M \ln \mu)\Big], \end{aligned} \tag{4.12}$$

where $\mu = f^{m+n-2}\lambda^{2-m}$.

Remark 2. If $(P^m, k) = (M^m, g)$ and $\varphi(x) = x$, then

$$\tau(\phi) = \frac{m+n-2}{f^2} grad_M \ln f,$$

$$\tau_2(\phi) = \frac{m+n-2}{f^2}\Big[grad_M(\Delta(\ln f)) + 2Rici^M(grad_M \ln f)$$

$$+\frac{1}{2}(6-m-n)grad_M(|grad_M \ln f|^2) + \big(2\Delta_M(\ln f) + 2\Delta_N(\ln f)$$
$$+ 2(m+n-4)(|grad_M \ln f|^2 + |grad_N \ln f|^2)\big)grad_M \ln f\Big].$$

Corollary 8. *Let $f(x) = e^x$. Then*

$$\begin{array}{rcl}\phi : \mathbb{R} \times_{f,f} N^n & \longrightarrow & \mathbb{R} \\ (x,y) & \longmapsto & x\end{array}$$

is proper biharmonic map (i.e., biharmonic no harmonic map) if and only if $\dim N = 3$.

Proof. From Theorem 10 and Remark 2, we obtain

$$\tau(\phi) = (n-1)e^{-2x}\frac{d}{dx}$$
$$\tau_2(\phi) = 2(n-1)(n-3)e^{-2x}\frac{d}{dx}.$$
□

Using Theorem 9 we obtain:

Theorem 11. *Let $f \in C^\infty(M \times N)$ be a smooth function and $\varphi : (M^m, g) \longrightarrow (P^m, k)$ $(m \geq 3)$ be a conformal submersion with dilation λ. Then $\phi : (x,y) \in (M \times_{f,f} N, G) \longrightarrow \phi(x,y) = \varphi(x) \in (P,k)$ is biharmonic maps if and only if the following equation is verified*

$$\begin{aligned}0 = {}& grad_M(\Delta \ln \mu) + 2Ricci^M(grad_M \ln \mu) + 4(2-m)\nabla^M_{grad_M \ln f} grad_M \ln \lambda \\ & + (4-m)(2-m)grad_M(|grad_M \ln \lambda|^2) \\ & + 4(m+n-2)\nabla^M_{grad_M \ln \lambda} grad_M \ln f \\ & + \frac{1}{2}(6-m-n)(m+n-2)grad_M(|grad_M \ln f|^2) \\ & - \Big[2\Delta(\ln \mu) + (m+n-2)(6-m-n)|grad_M \ln f|^2\Big] grad_M \ln \lambda \\ & + (6-m-n)\Big[(2-m)|grad_M \ln \lambda|^2 \\ & + 2(m+n-2)d\ln f(grad_M \ln \lambda)\Big] grad_M \ln f \\ & + \Big[2\Delta_M(\ln f) + 2\Delta_N(\ln f) + (2-m)|grad_M \ln \lambda|^2 \\ & + 2(m+n-4)(|grad_M \ln f|^2 + |grad_N \ln f|^2)\Big] grad_M \ln \mu,\end{aligned} \tag{4.13}$$

where $\mu = \lambda^{2-m} f^{m+n-2}$.

Corollary 9. *Let $f \in C^\infty(M \times N)$ be a smooth function and $\varphi : (M^m, g) \longrightarrow (P^m, k)$ $(m \geq 3)$ be a be conformal submersion with dilation λ. If φ is a proper biharmonic maps, then $\phi : (x,y) \in (M \times_{f,f} N, G) \longrightarrow \phi(x,y) = \varphi(x) \in (P,k)$ is a biharmonic maps if and only if the following equation is verified*

$$0 = grad_M(\Delta \ln f) + 2Ricci^M(grad_M \ln f)$$

$$
\begin{aligned}
&+\frac{4(2-m)}{m+n-2}\Big[\nabla^M_{grad_M \ln f} grad_M \ln\lambda + |grad_M \ln\lambda|^2 grad_M \ln f\Big]\\
&+2\Big[(6-m-n)d\ln f(grad_M \ln\lambda) + \Delta_M(\ln f)\\
&+(m+n-4)|grad_M \ln f|^2\Big] grad_M \ln f + 4\nabla^M_{grad_M \ln\lambda} grad_M \ln f\\
&+\frac{1}{2}(6-m-n)grad_M(|grad_M \ln f|^2)\\
&-\Big[2\Delta(\ln f)(6-m-n)|grad_M \ln f|^2\Big] grad_M \ln\lambda.
\end{aligned}
$$

Example 2. Let $\varphi : x \in \mathbb{R}^m - \{0\} \longrightarrow \varphi(x) = \frac{x}{|x|^2} \in \mathbb{R}^m - \{0\}$. Then φ is a conformal maps with dilation

$$\lambda(x) = \frac{1}{\mid x \mid^2} = \frac{1}{r^2}$$

and φ is a proper biharmonic maps if and only $m = 4$.

Example 3. Soit

$$
\begin{aligned}
\phi : (\mathbb{R}^4 - \{0\}) \times_f N^2 &\longrightarrow (\mathbb{R}^4 - \{0\})\\
(x, y) &\longmapsto \frac{x}{\mid x \mid^2}
\end{aligned}
$$

Let $\alpha \in C^\infty([0, +\infty[, \mathbb{R})$ and $f = e^{\alpha(r)}$ where $r = |x|$. Then we have

$$
\begin{aligned}
grad \ln f &= \alpha' \frac{\partial}{\partial r},\\
\mid grad \ln f \mid^2 &= (\alpha')^2,\\
\Delta \ln f &= \alpha'' + \frac{3}{r}\alpha',\\
grad(\Delta \ln f) &= (\alpha''' + \frac{3}{r}\alpha'' - \frac{3}{r^2}\alpha')\frac{\partial}{\partial r}
\end{aligned}
$$

and ϕ is biharmonic maps if and only if α is solution of the differential equation

$$\alpha''' + 2\alpha'\alpha'' - \frac{1}{r}\alpha'' - \frac{3}{r^2}\alpha' - \frac{6}{r}(\alpha')^2 + 4(\alpha')^3 = 0. \tag{4.14}$$

Example 4. Let $f \in C^\infty(\mathbb{R})$ be a smooth function. Then

$$
\begin{aligned}
\phi : (M^m, g) &\longrightarrow (M \times_{f,f} \mathbb{R}, G)\\
x &\longmapsto (x, y_0)
\end{aligned}
$$

ϕ is a proper biharmonic maps if and only if

$$
\begin{cases}
grad_{\mathbb{R}} \ln f \neq 0,\\
\frac{m}{2} grad_{\mathbb{R}}(|grad_{\mathbb{R}} \ln f|^2) + |grad_{\mathbb{R}} \ln f|^2 grad_{\mathbb{R}} \ln f = 0.
\end{cases}
$$

Let $\gamma \in C^\infty([0, +\infty[, \mathbb{R})$ be a smooth function. If we put $f(t) = e^\gamma(t)$, then we have

$$
\begin{cases}
grad_{\mathbb{R}} \ln f = \gamma' \frac{d}{dt},\\
grad_{\mathbb{R}}(|grad_{\mathbb{R}} \ln f|^2) = 2\gamma'\gamma'' \frac{d}{dt}.
\end{cases}
$$

Hence

$$\tau(\phi) = -m\gamma'(0, \frac{d}{dt}),$$
$$\tau_2(\phi) = -m(m\gamma'' + (\gamma')^2)(0, \frac{d}{dt})$$

and ϕ is biharmonic no harmonic maps if and only $f(t) = (\frac{1}{m}t + a)^{\frac{1}{m}}$ avec $a \in \mathbb{R}$.

Example 5. Let $\alpha \in C^\infty([0, +\infty[, \mathbb{R})$ be a smooth function. If $f(x) = e^{\alpha(x)}$ and

$$\begin{array}{rcl} \psi : \mathbb{R} & \longrightarrow & (\mathbb{R} \times_{f,f} N^n, G) \\ x & \longmapsto & (x, y_0). \end{array}$$

Then we have

$$\tau(\psi) = \alpha'(\frac{d}{dx}, 0),$$
$$\tau_2(\psi) = \Big(\alpha''' - 5\alpha'\alpha'' - 2(\alpha')^3\Big)(\frac{d}{dx}, 0).$$

So ψ is proper biharmonic maps if and only if

$$\alpha' \neq 0 \quad \text{and} \quad \alpha''' - 5\alpha'\alpha'' - 2(\alpha')^3 = 0. \tag{4.15}$$

The solutions of equation (4.15) under the form $\beta(x) = \alpha'(x) = \frac{a}{x}$ are given by $a = \frac{5 \pm \sqrt{41}}{4}$ (i.e., $f(x) = x^a$).

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References

[1] P. Baird, *Harmonic Maps between Riemannain Manifolds*, Clarendon Press Oxford 2003.
[2] A. Balmus, S. Montaldo, and C. Oniciuc, *Biharmonic maps between warped product manifolds*, J. Geom. Phys. **57** (2007), no. 2, 499–466.
[3] M. Djaa, A. M. Cherif, K. Zegga, and S. Ouakkas, *On the generalized of harmonic and bi-harmonic maps*, Int. Electron. J. Geom. **5** (2012), no. 1, 1–11.
[4] A. M. Cherif and M. Djaa, *Geometry of energy and bienergy variations between Riemannian manifolds*, Kyungpook Math. J. **55** (2015), no. 3, 715–730.
[5] J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Maths. **86** (1964), 109–160.
[6] J. Eells et L. Lemaire, *Another report on harmonic maps*, Bull. London Math. Soc. **20** (1988), no. 5, 385–524.
[7] S. Ouakkas, *Biharmonic maps, conformal deformations and the Hopf maps*, Differential Geom. Appl. **26** (2008), no. 5, 495–502.
[8] S. Ouakkas, R. Nasri, and M. Djaa, *On the f-harmonic and f-biharmonic maps*, JP J. Geom. Topol. **10** (2010), no. 1, 11–27.

Abdelhamid Boulal
Department of Mathematics
University of Tiaret
140000, Tiaret, Algeria
E-mail address: Abdelhamidboulal14@gmail.com

Mustapha Djaa
Department of Mathematics
University of Relizane
48000, Relizane, Algeria
E-mail address: Djaamustapha@Live.com

Seddik Ouakkas
Department of Mathematics
University of Saida
20000, Saida, Algeria
E-mail address: S20.ouakkas@gmail.com