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# CERTAIN RESULTS ON ALMOST KENMOTSU MANIFOLDS WITH CONFORMAL REEB FOLIATION

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Abstract. The object of the present paper is to study some curvature properties of almost Kenmotsu manifolds with conformal Reeb foliation. Among others it is proved that an almost Kenmotsu manifold with conformal Reeb foliation is Ricci semisymmetric if and only if it is an Einstein manifold. Finally, we study Yamabe soliton in this manifold.

## 1. Introduction

Geometry of Kenmotsu manifolds was originated by Kenmotsu [13] and became an interesting research area in differential geometry. As a generalization of Kenmotsu manifolds, the notion of almost Kenmotsu manifolds was first introduced by Janssens and Vanhecke [12]. In recent years, for some results regarding such manifolds we refer the reader to ([6–10, 14, 15], [23–26, 28, 29]).

A Riemannian manifold $M^{2n+1}$ is called locally symmetric if its curvature tensor $R$ is parallel, that is, $\nabla R = 0$, where $\nabla$ is the Levi-Civita connection. It was introduced by Shirokov in [18]. The notion of semisymmetric manifolds, a proper generalization of locally symmetric manifolds worked out by Cartan in 1927, is defined by $R(X,Y) \cdot R = 0$, where $R(X,Y)$ acts on $R$ as a derivation. A complete intrinsic classification of these manifolds was given by Szabo [19].

A Riemannian manifold $(M, g), n \geq 3$, is said to be Ricci-semisymmetric if the Ricci tensor satisfies the curvature condition

$$R.S = 0,$$

where $S$ is the Ricci tensor.

The class of Ricci-semisymmetric manifolds includes the set of Ricci-symmetric manifolds ($\nabla S = 0$) as a proper subset. Ricci-semisymmetric manifolds were investigated by several authors.

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We define the subsets $U_R$, $U_S$ of a Riemannian manifold $M$ by $U_R = \{x \in M : R - \frac{\kappa}{(n-1)n} G \neq 0 \text{ at } x\}$ and $U_S = \{x \in M : S - \frac{\kappa}{n} g \neq 0 \text{ at } x\}$ respectively, where $G(X,Y)Z = g(Y,Z)X - g(X,Z)Y$. Evidently we have $U_S \subset U_R$. A Riemannian manifold is said to be pseudo-symmetric [20] if at every point of $M$ the tensor $R.R$ and $Q(g,R)$ are linearly dependent. This is equivalent to

$$R.R = f_R Q(g,R)$$

on $U_R$, where $f_R$ is some function on $U_R$. Clearly, every semi-symmetric manifold is pseudo-symmetric but the converse is not true, in general [20].

A Riemannian manifold $M$ is said to Ricci pseudo-symmetric if $R.S$ and $Q(g,S)$ on $M$ are linearly dependent. This is equivalent to

$$R.S = f_S Q(g,S)$$

on $U_S$, where $f_S$ is a function defined on $U_S$.

A Riemannian manifold $(M,g), n \geq 3$, is said to be Ricci generalized pseudosymmetric [20] if and only if the relation

$$R.R = L_R Q(S,R), \tag{1}$$

where $L_R$ is a function on the set $U = \{x \in M : Q(S,R) \neq 0 \text{ at } x\}$. A very important subclass of this class of manifolds realizing the condition

$$R.R = Q(S,R), \tag{2}$$

where

$$Q(S,R)(U,V,W;X,Y) = ((X \wedge_S Y) \cdot R)(U,V)W \tag{3}$$

for all smooth vector fields $X, Y, U, V, W$ on $M$, and the endomorphism $X \wedge_S Y$ defined by

$$(X \wedge_S Y)Z = S(Y,Z)X - S(X,Z)Y. \tag{4}$$

On the other hand, it is well known that a Riemannian metric $g$ of an $n$-dimensional complete Riemannian manifold $(M^n, g)$ is said to be a Yamabe soliton if it satisfies

$$\pounds_V g = (\lambda - r)g \tag{5}$$

for a constant $\lambda \in \mathbb{R}$ and a smooth vector field $V$ on $M^n$, where $r$ is the scalar curvature of $g$ and $\pounds$ denotes the Lie-derivative operator. A Yamabe soliton is said to be shrinking, steady or expanding according to $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$ respectively and $\lambda$ is said to be the soliton constant.

Given a smooth Riemannian manifold $(M^n, g_0)$, the evolution of the metric $g_0$ in time $t$ to $g = g(t)$ through the following equation

$$\frac{\partial}{\partial t} g_t = -rg, g(0) = g_0, \tag{6}$$

is known as the Yamabe flow (which was introduced by Hamilton [11]). A Yamabe soliton is a special soliton of the Yamabe flow that moves by one parameter family of diffeomorphisms $\phi_t$ generated by a fixed vector field V on $M^n$ (for more details see [5]).

The significance of Yamabe flow lies in the fact that it is a natural geometric deformation to metrics of constant scalar curvature. One notes that Yamabe flow corresponds to the fast diffusion case of the porous medium equation (the plasma equation) in mathematical physics. Just as a Ricci soliton is a special solution of the Ricci flow, a Yamabe soliton is a special solution of the Yamabe flow that moves by one parameter family of diffeomorphisms $\phi_t$ generated by a fixed vector field V on M, and homotheties, i.e., $g(.,t)) = \sigma(t)\phi_*(t)g_0$.

Given a Yamabe soliton, if $V = Df$ holds for a smooth function $f$ on $M^n$, equation (5) becomes

$$Hessf = \frac{1}{2}(\lambda - r)g, \tag{7}$$

where $Hessf$ denotes the Hessian of $f$ and $D$ denotes the gradient operator of $g$ on $M^n$. In this case $f$ is called the potential function of the Yamabe soliton and $g$ is said to be a gradient Yamabe soliton. A Yamabe soliton (respectively, gradient Yamabe soliton) is said to be trivial when $V$ is Killing (respectively, $f$ is constant).

Wang [21] studied Yamabe solitons on a three-dimensional Kenmotsu manifolds. Bejan et al. studied Ricci soliton in 3-dimensional Paracontact geometry [1]. Again, Wang et al. studied Ricci soliton on an almost Kenmotsu manifold ([22,27]). Moreover in [9] De and Pathak studied Ricci generalized pseudosymmetric Kenmotsu manifolds.

**Definition** ([16]). A vector field $X$ on a contact manifold $M$ is said to be infinitesimal contact transformation if there exists a smooth function $\sigma$ on $M$ such that

$$(\pounds_X\eta)Y = \sigma\eta(Y)$$

for every smooth vector fields $X$ and $Y$. If $\sigma \equiv 0$, then $X$ is called a strict contact infinitesimal transformation.

Motivated by the above studies, in this paper we study almost Kenmotsu manifolds with conformal Reeb foliation.

The paper is organized as follows: In Section 2, we give a brief account on almost Kenmotsu manifolds with conformal Reeb foliation. Section 3 deals with Ricci semisymmetric almost Kenmotsu manifolds with conformal Reeb foliation, while Section 4 is devoted to study Ricci generalized pseudosymmetric almost Kenmotsu manifolds with conformal Reeb foliation. Next in Section 5, we study some transformation on such a manifold. Finally, we study Yamabe soliton on almost Kenmotsu manifolds with conformal Reeb foliation.

## 2. Almost Kenmotsu manifolds

A differentiable $(2n+1)$-dimensional manifold $M$ is said to have a $(\phi, \xi, \eta)$-structure or an almost contact structure, if it admits a $(1,1)$ tensor field $\phi$, a characteristic vector field $\xi$ and a 1-form $\eta$ satisfying ([2,3]),

$$\phi^2 = -I + \eta \otimes \xi,\ \eta(\xi) = 1, \tag{8}$$

where $I$ denote the identity endomorphism. Here also $\phi\xi = 0$ and $\eta \circ \phi = 0$; both can be derived from (8) easily.

If a manifold $M$ with a $(\phi, \xi, \eta)$-structure admits a Reimannian metric $g$ such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

for any vector fields $X$, $Y$ of $T_pM^{2n+1}$, then $M$ is said to have an almost contact structure $(\phi, \xi, \eta, g)$. The fundamental 2-form $\Phi$ on an almost contact metric manifold is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any $X$, $Y$ of $T_pM^{2n+1}$. The condition for an almost contact metric manifold being normal is equivalent to vanishing of the $(1,2)$-type torsion tensor $N_\phi$, defined by $N_\phi = [\phi, \phi] + 2d\eta \otimes \xi$, where $[\phi, \phi]$ is the Nijenhuis torsion of $\phi$. Recently in ([10,15]), almost contact metric manifold such that $\eta$ is closed and $d\Phi = 2\eta \wedge \phi$ are studied and they are called almost Kenmotsu manifolds. Obviously, a normal almost Kenmotsu manifold is a Kenmotsu manifold. Also Kenmotsu manifolds can be characterized by $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$, for any vector fields $X$,$Y$. It is well known [13] that a Kenmotsu manifold $M^{2n+1}$ is locally a warped product $I \times_f N^{2n}$ where $N^{2n}$ is a Kähler manifold, $I$ is an open interval with coordinate $t$ and the warping function $f$, defined by $f = ce^t$ for some positive constant c. Let us denote the distribution orthogonal to $\xi$ by $\mathcal{D}$ and defined by $\mathcal{D} = Ker(\eta) = Im(\phi)$. In an almost Kenmotsu manifold, since $\eta$ is closed, $\mathcal{D}$ is an intregrable distribution. Let $M^{2n+1}$ be an almost Kenmotsu manifold. We denote by $h = \frac{1}{2}\pounds_\xi \phi$ and $l = R(\cdot, \xi)\xi$ on $M^{2n+1}$. The tensor fields $l$ and $h$ are symmetric operators and satisfy the following relations [4]:

$$h\xi = 0,\ l\xi = 0,\ tr(h) = 0,\ tr(h\phi) = 0,\ h\phi + \phi h = 0, \tag{9}$$

$$\nabla_X \xi = -\phi^2 X - \phi h X (\Rightarrow \nabla_\xi \xi = 0), \tag{10}$$

$$\phi l \phi - l = 2(h^2 - \phi^2), \tag{11}$$

$$R(X, Y)\xi = \eta(X)(Y - \phi h Y) - \eta(Y)(X - \phi h X) + (\nabla_Y \phi h)X - (\nabla_X \phi h)Y \tag{12}$$

for any vector fields $X, Y$. The $(1,1)$-type symmetric tensor field $h' = h \circ \phi$ is anticommuting with $\phi$ and $h'\xi = 0$. Also it is clear that ([4,15,29])

$$h = 0 \Leftrightarrow h' = 0,\ \ h'^2 = (k+1)\phi^2 (\Leftrightarrow h^2 = (k+1)\phi^2). \tag{13}$$

In a recent paper [15] Pastore and Saltarelli studied almost Kenmotsu manifolds with conformal Reeb foliation. In this paper they proved that an almost Kenmotsu manifold satisfying $R(X, \xi) \cdot R = 0$, for any smooth vector field $X$, is a Kenmotsu manifold of constant curvature $-1$. It is well known that in the contact manifold the vanishing of the curvature tensor $h = \frac{1}{2}\pounds_\xi \phi$ means that the Reeb vector field is killing. According to Pastore and Saltarelli [15] for an almost Kenmotsu manifolds $h = 0$ means that the Reeb foliation is conformal. Here we recall a proposition:

**Proposition 2.1** ([15]). *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with conformal Reeb foliation. Then, for any vector fields $X$ and $Y$, one has*

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{14}$$

$$R(X,\xi)\xi = \phi^2 X, \tag{15}$$

$$R(\xi, X)Y = -g(X,Y)\xi + \eta(Y)X, \tag{16}$$

$$S(X,\xi) = -2n\eta(X). \tag{17}$$

## 3. Ricci semisymmetric almost Kenmotsu manifolds with conformal Reeb foliation

In this section we characterize Ricci semisymmetric almost Kenmotsu manifolds with conformal Reeb foliation. Suppose the manifold $M^{2n+1}$ is Ricci semisymmetric almost Kenmotsu manifolds with conformal Reeb foliation. Then

$$(R(X,Y) \cdot S)(U,V) = 0,$$

which implies

$$S(R(X,Y)U,V) + S(U,R(X,Y)V) = 0 \tag{18}$$

for all smooth vector fields $X, Y, U, V$.

Substituting $X = \xi$ in (18) yields

$$S(R(\xi,Y)U,V) + S(U,R(\xi,Y)V) = 0. \tag{19}$$

Using (16) and (17) in (19) implies

$$\eta(U)S(Y,V) + \eta(V)S(U,Y) + 2ng(Y,U)\eta(V) + 2ng(Y,V)\eta(U) = 0. \tag{20}$$

Replacing $U$ by $\xi$ in (20) yields

$$S(Y,V) = -2ng(Y,V).$$

Conversely, let the manifold be an Einstein manifold. Then obviously $R \cdot S = 0$.

This leads to the following:

**Theorem 3.1.** *Let $M^{2n+1}(\phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with conformal Reeb foliation. Then the manifold is Ricci semisymmetric if and only if the manifold is an Einstein one.*

*Remark* 3.2. The above theorem is the generalization of Theorem 5.2 of [15].

## 4. Ricci generalized Pseudosymmetric almost Kenmotsu manifolds with conformal Reeb foliation

Let us consider Ricci generalized pseudosymmetric almost Kenmotsu manifolds with conformal Reeb foliation. Then

$$(R(X,Y)\cdot R)(U,V)W = L_R[((X\wedge_S Y)\cdot R)(U,V)W],$$

which implies

$$\begin{aligned}
&R(X,Y)R(U,V)W - R(R(X,Y)U,V)W - R(U,R(X,Y)V)W\\
&-R(U,V)R(X,Y)W = L_R[(X\wedge_S Y)R(U,V)W - R((X\wedge_S Y)U,V)W\\
&-R(U,(X\wedge_S Y)V)W - R(U,V)(X\wedge_S Y)W]
\end{aligned}\tag{21}$$

for all smooth vector fields $X,Y,U,V,W$.

Using (4) in (21) we obtain

$$\begin{aligned}
&R(X,Y)R(U,V)W - R(R(X,Y)U,V)W - R(U,R(X,Y)V)W\\
&-R(U,V)R(X,Y)W = L_R R[S(Y,R(U,V)W)X - S(X,R(U,V)W)Y\\
&-R(S(Y,U)X,V)W + R(S(X,U)Y,V)W - R(U,S(Y,V)X)W\\
&+R(U,S(X,V)Y)W - S(Y,W)R(U,V)X + S(X,W)R(U,V)Y].
\end{aligned}\tag{22}$$

Substituting $Y=\xi$ in (22) yields

$$\begin{aligned}
&R(X,\xi)R(U,V)W - R(R(X,\xi)U,V)W - R(U,R(X,\xi)V)W\\
&-R(U,V)R(X,\xi)W = L_R[S(\xi,R(U,V)W)X - S(X,R(U,V)W)\xi\\
&-R(S(\xi,U)X,V)W + R(S(X,U)\xi,V)W - R(U,S(\xi,V)X)W\\
&+R(U,S(X,V)\xi)W - S(\xi,W)R(U,V)X + S(X,W)R(U,V)\xi].
\end{aligned}\tag{23}$$

Using (16) and (17) in (23) we obtain

$$\begin{aligned}
&\bar{R}(U,V,W,X)\xi - \eta(R(U,V)W)X + \eta(U)R(X,V)W\\
&-g(X,U)R(\xi,V)W + \eta(V)R(U,X)W - g(X,V)R(U,\xi)W\\
&+\eta(W)R(U,V)X - g(X,W)R(U,V)\xi = -2nL_R[\eta(R(U,V)W)X\\
&+\eta(U)R(X,V)W + \eta(V)R(U,X)W + \eta(W)R(U,V)X]\\
&-L[S(X,R(U,V)W)\xi - S(X,U)R(\xi,V)W - S(X,V)R(U,\xi)W\\
&-S(X,W)R(U,V)\xi],
\end{aligned}\tag{24}$$

where $\bar{R}(U,V,W,X) = g(R(U,V)W,X)$.

Taking inner product with $\xi$ in (24) implies

$$\begin{aligned}
&\bar{R}(U,V,W,X) - \eta(R(U,V)W)\eta(X) + \eta(U)\eta(R(X,V)W)\\
&-g(X,U)\eta(R(\xi,V)W) + \eta(V)\eta(R(U,X)W) - g(X,V)\eta(R(U,\xi)W)\\
&+\eta(W)\eta(R(U,V)X) - g(X,W)\eta(R(U,V)\xi) = -2nL_R[\eta(R(U,V)W)\eta(X)\\
&+\eta(U)\eta(R(X,V)W) + \eta(V)\eta(R(U,X)W) + \eta(W)\eta(R(U,V)X)]\\
&-L[S(X,R(U,V)W) - S(X,U)\eta(R(\xi,V)W) - S(X,V)\eta(R(U,\xi)W)\\
&-S(X,W)\eta(R(U,V)\xi)].
\end{aligned}\tag{25}$$

Putting $W = \xi$ in (25) implies

$$L_R[2n\eta(R(U,V)X) - S(X,R(U,V)\xi)] = 0.$$

Hence either $L_R = 0$, or,

$$2n\eta(R(U,V)X) - S(X,R(U,V)\xi = 0.$$

Case 1: Suppose $L_R = 0$. Then from (21) it follows that

$$R \cdot R = 0.$$

Case 2: Suppose

$$2n\eta(R(U,V)X) - S(X,R(U,V)\xi = 0. \tag{26}$$

Substituting $U = \xi$ in (26) and making use of (16), (17) we get

$$S(X,V) = -2ng(X,V).$$

Therefore the manifold becomes an Einstein manifold.

It is known that the Reeb foliation of a $(k,\mu)$-almost Kenmotsu manifold is conformal. Therefore, in view of ([18, Theorem 1.2]), we observe that a Riemannian semisymmetric almost Kenmotsu manifold with conformal Reeb foliation is of constant sectional curvature $-1$.

Thus we can state the following:

**Theorem 4.1.** *A Ricci-generalized pseudosymmetric almost Kenmotsu manifold $M^{2n+1}(\phi,\xi,\eta,g)$ with conformal Reeb foliation is of constant sectional curvature $-1$ or Einstein.*

## 5. Some transformation in an almost Kenmotsu manifolds with conformal Reeb foliation

We now consider a transformation $\mu$ which transform an almost Kenmotsu structure $(\phi,\xi,\eta,g)$ with conformal Reeb foliation into another almost Kenmotsu structure $(\bar{\phi},\bar{\xi},\bar{\eta},\bar{g})$. We denote by the notation 'bar' the geometric object which are transformed by the transformation $\mu$ [17].

We first suppose that in an almost Kenmotsu manifold with conformal Reeb foliation the Riemannian curvature tensor remains invariant under the transformation $\mu$.

Thus we have

$$\bar{R}(X,Y)Z = R(X,Y)Z \tag{27}$$

for all smooth vector fields $X, Y, Z$.

From (27) it follows that

$$\eta(\bar{R}(X,Y)Z) = \eta(R(X,Y)Z) \tag{28}$$

for all smooth vector fields $X, Y, Z$.

Using (14) in (28) yields

$$g(X,Z)\eta(Y) - g(Y,Z)\eta(X) = \eta(\bar{R}(X,Y)Z). \tag{29}$$

Putting $Y = \bar{\xi}$ in (29) we obtain

$$\eta(\bar{\xi})g(X,Z) - \eta(X)g(\bar{\xi},Z) = \eta(\bar{\xi})\bar{g}(X,Z) - \bar{\eta}(Z)\eta(X). \tag{30}$$

Interchanging $X$ and $Z$ in (30) implies

$$\eta(\bar{\xi})g(X,Z) - \eta(Z)g(\bar{\xi},X) = \eta(\bar{\xi})\bar{g}(X,Z) - \bar{\eta}(X)\eta(Z). \tag{31}$$

Subtracting (31) from (30) yields

$$\eta(Z)g(\bar{\xi},X) - \eta(X)g(\bar{\xi},Z) = \bar{\eta}(X)\eta(Z) - \bar{\eta}(Z)\eta(X). \tag{32}$$

Putting $Z = \xi$ in (32) implies

$$g(\bar{\xi},X) - \eta(X)g(\bar{\xi},\xi) = \bar{\eta}(X) - \bar{\eta}(\xi)\eta(X). \tag{33}$$

Also $\bar{S}(X,Y) = S(X,Y)$ and hence $\bar{S}(\xi,\bar{\xi}) = S(\xi,\bar{\xi})$, which implies $\bar{\xi} = \eta(\bar{\xi})$. Using this relation in (33) we obtain

$$\bar{\eta}(X) = g(\bar{\xi},X). \tag{34}$$

By virtue of (34) we get from (31)

$$[g(X,Z) - \bar{g}(X,Z)]\eta(\bar{\xi}) = 0. \tag{35}$$

This implies

$$g(X,Z) = \bar{g}(X,Z)$$

for all smooth vector fields $X, Z$ provided $\eta(\bar{\xi}) \neq 0$.

Hence we can state the following:

**Theorem 5.1.** *In an almost Kenmotsu manifold with conformal Reeb foliation the transformation $\mu$ which leaves the curvature tensor invariant and $\eta(\bar{\xi}) \neq 0$ is an isometry.*

Let us now suppose that in an almost Kenmotsu manifold with Reeb foliation, the infinitesimal contact transformation leaves the Ricci tensor invariant. Then we have

$$(\mathcal{L}_V S)(X,Y) = 0, \tag{36}$$

where $\mathcal{L}$ is the lie derivative along $V$.

Substituting $Y = \xi$ in (36) we get

$$(\mathcal{L}_V S)(X,\xi) = 0. \tag{37}$$

Now

$$(\mathcal{L}_V S)(X,\xi) = \mathcal{L}_V(S(X,\xi)) - S(\mathcal{L}_V X,\xi) - S(X,\mathcal{L}_V \xi). \tag{38}$$

Using (17) in (38) yields

$$2n(\mathcal{L}_V \eta)(X) + S(X,\mathcal{L}_V \xi) = 0. \tag{39}$$

Putting $(\mathcal{L}_V \eta)X = \sigma\eta(X)$ in (39), we obtain

$$S(X,\mathcal{L}_V \xi) = -2n\sigma\eta(X). \tag{40}$$

Replacing $X$ by $\xi$ in (40) and then using (17) implies

$$S(\xi, \mathcal{L}_V \xi) = -2n\sigma. \tag{41}$$

Using (17) in (41) we have

$$\eta(\mathcal{L}_V \xi) = \sigma. \tag{42}$$

Again

$$(\mathcal{L}_V \eta)X = \sigma \eta(X). \tag{43}$$

Putting $X = \xi$ in the above equation we get

$$(\mathcal{L}_V \eta)\xi = \sigma, \tag{44}$$

which implies

$$\mathcal{L}_V(\eta(\xi)) - \eta(\mathcal{L}_V \xi) = \sigma. \tag{45}$$

Using (42) and (45),

$$\sigma = 0.$$

Thus we can state the following:

**Theorem 5.2.** *In an almost Kenmotsu manifold with conformal Reeb foliation the infinitesimal almost contact transformation which leaves the Ricci tensor invariant is an infinitesimal strict contact transformation.*

## 6. Yamabe soliton on an almost Kenmotsu manifolds with conformal Reeb foliation

In this section we characterize Yamabe soliton on an almost Kenmotsu manifolds with conformal Reeb foliation.

Suppose the potential vector field $V$ is pointwise collinear with $\xi$, that is, $V = b\xi$. Then

$$\begin{aligned} \nabla_X V &= \nabla_X b\xi \\ &= (Xb)\xi + b\nabla_X \xi \\ &= (Xb)\xi + b(-\phi^2 X). \end{aligned} \tag{46}$$

Now,

$$(\mathcal{L}_V g)(X, Y) = g(\nabla_X V, Y) + g(\nabla_Y V, X). \tag{47}$$

Using (8) and (46) in (47) we obtain

$$(\mathcal{L}_V g)(X, Y) = (Xb)\eta(Y) + (Yb)\eta(X) + 2b[g(X, Y) - \eta(X)\eta(Y)]. \tag{48}$$

Using (5) in (48) it follows that

$$(Xb)\eta(Y) + (Yb)\eta(X) + 2b[g(X, Y) - \eta(X)\eta(Y)] = (\lambda - r)g(X, Y). \tag{49}$$

Contracting $X$ and $Y$ in (49) yields

$$(\xi b) = \frac{1}{2}(\lambda - r)(2n + 1) - 2nb. \tag{50}$$

Substituting $Y = \xi$ in (49) implies

$$(Xb) = [(\lambda - r)(\frac{1}{2} - \lambda) + 2nb]\eta(X). \tag{51}$$

Replacing $X$ by $\xi$ in (51) we get

$$(\xi b) = [(\lambda - r)(\frac{1}{2} - \lambda) + 2nb]. \tag{52}$$

Therefore, from (50) and (52) we have

$$[(\lambda - r)(\frac{1}{2} - \lambda) + 2nb] = \frac{1}{2}(\lambda - r)(2n + 1) - 2nb. \tag{53}$$

From (53) we obtain

$$\lambda = (r + 2b). \tag{54}$$

Using (54) in (51) yields

$$(Xb) = b\eta(X), \tag{55}$$

that is,

$$g(Db, X) = bg(X, \xi). \tag{56}$$

Removing $X$ in the above equation we have

$$Db = b\xi,$$

and hence

$$Db = V.$$

Thus we can state the following:

**Theorem 6.1.** *If an almost Kenmotsu manifold $M^{2n+1}(\phi, \xi, \eta, g)$ with conformal Reeb foliation admits a Yamabe soliton then the Potential vector field is the gradient of the scalar $b$.*

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## References

[1] C. L. Bejan and M. Crasmareanu, *Second order parallel tensors and Ricci solitons in 3-dimensional normal paracontact geometry*, Ann. Glob. Anal. Geom. **4 6** (2014), no. 2, 117–127.

[2] D. E. Blair, *Contact manifold in Reimannian Geometry*, Lecture Notes on Mathematics, Springer, Berlin, **509**, 1976.

[3] ______, *Riemannian Geomatry on contact and sympletic manifolds*, Progr. Math. Birkhäuser, 2010.

[4] D. E. Blair, T. Koufogiorgos, and B. J. Papantoniou, *Contact metric manifolds satisfying a nullitty condition*, Israel J. Math. **91** (1995), no. 1-3, 189–214.

[5] B. Chow, P. Lu, and L. Ni, *Hamilton's Ricci flow*, Graduate Studies in Mathematics, Volume 77, American Mathematical Society, Science Press, 2006.

[6] U. C. De and K. Mandal, *On a type of almost Kenmotsu manifolds with nullity distribution*, Arab J. Math. Sci.; doi. org/ 10.2016/j.ajmsc.2016.04.001.

[7] ______, *On $\phi$-Ricci recurrent almost Kenmotsu manifolds with nullity distribution*, Int. Electron. J. Geom. **9** (2016), no. 2, 70–79.
[8] ______, *On locally $\phi$-conformally symmetric almost Kenmotsu manifolds with nullity distributions*, Commun. Korean Math Soc. **32** (2017), no. 2, 401–416.
[9] U. C. De and G. Pathak, *On a type of Kenmotsu manifolds*, J. Pure Math. **18** (2001), 79–83.
[10] G. Dileo and A. M. Pastore, *Almost Kenmotsu manifolds and local symmetry*, Bull. Belg. Math. Soc. Simon Stevin **14** (2007), no. 2, 343–354.
[11] R. S. Hamilton, *The Ricci flow on surfaces*, Mathematics and General Relativity (Santa Cruz, CA, 1986), 237–262, Contemp. Math., 71, Amer. Math. Soc., Providence, RI, 1988.
[12] D. Janssens and L. Vanhecke, *Almost contact structures and curvature tensors*, Kodai Math J. Geom. **4** (1981), no. 1, 1–27.
[13] K. Kenmotsu, *A class of almost contact Riemannian manifolds*, Tohoku Math. J. **24** (1972), 93–103.
[14] K. Mandal and U. C. De, *Ricci solitons on almost Kenmotsu manifolds*, An. Univ. Oradea Fasc. Mat. **2** (2016), no. 2, 109–116.
[15] A. M. Pastore and V. Saltarelli, *Almost Kenmotsu manifolds with conformal Reeb foliation*, Bull. Belg. Math. Soc. Simon Stevin **18** (2011), no. 4, 655–666.
[16] G. Pitis, *Geometry of Kenmotsu manifolds*, Brasov, 2007.
[17] S. Sasaki, *Almost contact manifolds Lecture Notes*, Tohoku Univ. **1** (1965), **2** (1967); **3** (1968).
[18] P. A. Shirokov, *Constant vector fields and tensor fields of second order in Riemannian spaces*, Izv. Kazan Fiz. Mat. Obshchestva Ser. **25** (1925), 86–114.
[19] Z. I. Szabo, *Structures theorems on Riemannian spaces satisfying $R(X,Y)\cdot R = 0$ I the local version*, J. Differential Geom. **17** (1982), no. 4, 531–582.
[20] L. Verstraelen, *Comments on pseudosymmetry in the sense of Ryszard Deszcz*, In: Geometry and Topology of submanifolds, VI. River Edge, NJ: World Sci. Publishing, 1994, 199–209.
[21] Y. Wang, *Yamabe solitons in three dimensional kenmotsu manifolds*, Bull. Belg. Math. Soc. Stenvin **23** (2016), 345–355.
[22] ______, *Gradient Ricci almost solitons on two classes of almost Kenmotsu manifolds*, J. Korean Math. Soc. **53** (2016), no. 5, 1101–1114.
[23] Y. Wang, U. C. De, and X. Liu, *Gradient Ricci solitons on almost Kenmotsu manifolds*, Publ. Inst. Math. (Beograd) (N.S) **98(112)** (2015), 227–235.
[24] Y. Wang and X. Liu, *Riemannian semisymmetric almost Kenmotsu manifolds and nullity distributions*, Ann. Polon. Math. **112** (2014), no. 1, 37–46.
[25] ______, *Locally symmetric CR-integrable almost Kenmotsu manifolds*, Mediterr. J. Math. **12** (2015), no. 1, 159–171.
[26] ______, *On $\phi$-recurrent almost Kenmotsu manifolds*, Kuwait J. Sci. **42** (2015), no. 1, 65–77.
[27] ______, *Ricci solitons on three dimensional $\eta$-Einstein almost Kenmotsu manifolds*, Taiwanese J. Math. **19** (2015), no. 1, 91–100.
[28] ______, *On almost Kenmotsu manifolds satisfying some nullity distributions*, Proc. Nat. Acad. Sci. India Sect. A **86** (2016), no. 3, 347–353.
[29] Y. Wang and W. Wang, *Curvature properties of almost Kenmotsu manifolds with generalized nullity conditions*, Filomat **30** (2016), no. 14, 3807–3816.

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