Commun. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/CKMS.c170080
pISSN: 1225-1763 / eISSN: 2234-3024

NON-ZERO CONSTANT CURVATURE FACTORABLE SURFACES IN PSEUDO-GALILEAN SPACE

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Abstract. Factorable surfaces, i.e. graphs associated with the product of two functions of one variable, constitute a wide class of surfaces in differential geometry. Such surfaces in the pseudo-Galilean space with zero Gaussian and mean curvature were obtained in [2]. In this study, we provide new results relating to the factorable surfaces with non-zero constant Gaussian and mean curvature.

1. Introduction

One of challenging problems in classical differential geometry has been obtaining surfaces with prescribed Gaussian (K) and mean curvature (H). Let $\mathbb{E}^3(x,y,z)$ be a Euclidean 3-space and $z=z(x,y)$ a real-valued smooth function of two independent variables. In particular, for the immersed graph of z into $\mathbb{E}^3$, such a problem is reduced to solve the *Monge-Ampère equation* given by ([25, 28])

$$\det\left(\frac{\partial z}{\partial u_i \partial u_j}\right)=K\left(1+|\nabla z|^2\right)^2,\ u_1=x,\ u_2=y$$

and the *equation of mean curvature type* in divergence form

$$\operatorname{div}\left(\frac{\nabla z}{\sqrt{1+|\nabla z|^2}}\right)=H,$$

where ∇ denotes the gradient of $\mathbb{E}^2$ ([17, 26, 27]). These equations are also related to the branches such as economics, meteorology, oceanography etc. [4, 5, 6, 7, 8].

Recall that the graph surfaces are also known as *Monge surfaces* (see [14], p. 398). In this study, we deal with a special Monge surface, namely *factorable surface* that is graph of the form $z(x,y)=f(x)g(y)$ for smooth functions f,g. Such surfaces in various ambient spaces with $K,H=const.$ have been classified

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Received March 3, 2017; Accepted June 29, 2017.
2010 *Mathematics Subject Classification.* 53A35, 53B25, 53B30, 53C42.
Key words and phrases. pseudo-Galilean space, factorable surface, Gaussian curvature, mean curvature.

in [3, 13, 15, 18, 19, 29, 32, 33]. Our purpose is to analyze the factorable surfaces in the pseudo-Galilean space $\mathbb{G}_3^1$ that is one of real Cayley-Klein spaces (for details, see [12, 16, 24, 30]). As distinct from the other ambient spaces, there exist two different kinds of factorable surfaces arising from the absolute figure of $\mathbb{G}_3^1$. Explicitly, a Monge surface in $\mathbb{G}_3^1$ is said to be *factorable* if it is given in one of the explicit forms

$$\Omega_1 : z(x,y) = f(x)\,g(y) \text{ and } \Omega_2 : x(y,z) = f(y)\,g(z).$$

We call Ω_1 and Ω_2 the *factorable surface of first* and *second kind*, respectively. Note that these surfaces have different geometric structures in $\mathbb{G}_3^1$ (such as metric, curvature etc.). Flat and minimal $(K, H = 0)$ factorable surfaces in $\mathbb{G}_3^1$ were presented in [2]. Still, obtaining such surfaces with $K, H = const. \neq 0$ is an open problem. The present paper is devoted to solve this problem.

2. Preliminaries

In this section, some basics of the pseudo-Galilean geometry shall be provided from [1, 9, 10, 11, 20, 21, 31]. In particular, the local theory of immersed surfaces into a pseudo-Galilean space was well-structured in [22].

Let $P_3(\mathbb{R})$ denote the real projective 3-space and $(u_0 : u_1 : u_2 : u_3)$ the homogeneous coordinates in $P_3(\mathbb{R})$. The *pseudo-Galilean 3-space* $\mathbb{G}_3^1$ is a metric space constructed within $P_3(\mathbb{R})$ having the absolute figure $\{\sigma, l, \epsilon\}$, where σ implies the *absolute plane* of $\mathbb{G}_3^1$, l *absolute line* in σ and ϵ is the *hyperbolic involution* of the points of l. These arguments are given by $\sigma : u_0 = 0$, $l : u_0 = u_1 = 0$ and

$$\epsilon : (u_0 : u_1 : u_2 : u_3) \longmapsto (u_0 : u_1 : u_3 : u_2).$$

The affine model of $\mathbb{G}_3^1$ can be introduced by changing homogenous coordinates with affine coordinates:

$$(u_0 : u_1 : u_2 : u_3) = (1 : x : y : z).$$

In terms of the affine coordinates, the *group of motions* is defined by

$$\left\{ \begin{array}{l} x' = a_1 + x, \\ y' = a_2 + a_3 x + (\cosh\theta)\,y + (\sinh\theta)\,z, \\ z' = a_4 + a_5 x + (\sinh\theta)\,y + (\cosh\theta)\,z, \end{array} \right. \tag{2.1}$$

where a_i, $i \in \{1, \ldots, 5\}$ and θ are some constants. The *pseudo-Galilean distance* is introduced with respect to the absolute figure, namely

$$d(x,y) = \left\{ \begin{array}{ll} |x_2 - x_1|, & \text{if } x_1 \neq x_2, \\ \sqrt{\left|(y_2 - y_1)^2 - (z_2 - z_1)^2\right|}, & \text{if } x_1 = x_2, \end{array} \right.$$

where $x = (x_1, y_1, z_1)$ and $y = (x_2, y_2, z_2)$. Note that this metric (also the absolute figure) is invariant under (2.1).

A plane is said to be *pseudo-Euclidean* if it satisfies the equation $x = const.$ Otherwise, it is called *isotropic plane.* A pseudo-Euclidean plane basically has

Minkowskian metric while an isotropic plane has Galilean metric, i.e., parabolic measures of distances and angles. Contrary to its denotation, the *isotropic vectors* are contained in the pseudo-Euclidean plane $x = 0$ and, up to the induced Minkowskian metric on this plane, such vectors are categorized by their causal characters, i.e., *spacelike*, *timelike* and *lightlike*. For further details of the Minkowskian geometry, see [23].

An immersed surface into $\mathbb{G}_3^1$ is given by the mapping

$$r : D \subseteq \mathbb{R}^2 \longrightarrow \mathbb{G}_3^1, \;\; (u_1, u_2) \longmapsto (x(u_1,u_2), y(u_1,u_2), z(u_1,u_2))$$

and such a surface is said to be *admissible* (i.e., without pseudo-Euclidean tangent plane) if $x_{,i} = \frac{\partial x}{\partial u_i} \neq 0$ for some $i = 1, 2$. The first fundamental form is given by

$$ds^2 = (\mathfrak{g}_1 du_1 + \mathfrak{g}_2 du_2)^2 + \omega \left(\mathfrak{h}_{11} du_1^2 + 2\mathfrak{h}_{12} du_1 du_2 + \mathfrak{h}_{22} du_2^2\right),$$

where $\mathfrak{g}_i = x_{,i}$, $\mathfrak{h}_{ij} = y_{,i} y_{,j} + z_{,i} z_{,j}$, $i, j = 1, 2$, and

$$\omega = \begin{cases} 0, & \text{if } du_1 : du_2 \text{ is non-isotropic direction,} \\ 1, & \text{if } du_1 : du_2 \text{ is isotropic direction.} \end{cases}$$

A side tangent vector field in the tangent plane of the surface r is of the form $x_{,1} r_{,2} - x_{,2} r_{,1}$. Its pseudo-Galilean norm corresponds to

$$W = \sqrt{\left|(x_{,1} y_{,2} - x_{,2} y_{,1})^2 - (x_{,1} z_{,2} - x_{,2} z_{,1})^2\right|}.$$

A surface with $W = 0$ is said to be *lightlike*. Throughout the study, all immersed admissible surfaces shall be assumed to be non-lightlike. Then the vector given by

$$S = \frac{x_{,1} r_{,2} - x_{,2} r_{,1}}{W} = \frac{1}{W}(0, x_{,1} y_{,2} - x_{,2} y_{,1}, x_{,1} z_{,2} - x_{,2} z_{,1}),$$

satisfies $S \cdot S = \varepsilon = \{-1, 1\}$, where " $\cdot$ " denotes the Minkowskian scalar product. Hence a surface is said to be *spacelike* (*timelike*) if $\varepsilon = 1$ ($\varepsilon = -1$). The normal vector field is defined as

$$N = \frac{1}{W}(0, x_{,1} z_{,2} - x_{,2} z_{,1}, x_{,1} y_{,2} - x_{,2} y_{,1})$$

such that $N \cdot N = -\varepsilon$. The second fundamental form is $II = \sum_{i,j=1}^{2} L_{ij} du_i du_j$, where if $\mathfrak{g}_1 \neq 0$

$$L_{ij} = \frac{\varepsilon}{\mathfrak{g}_1}(\mathfrak{g}_1 (0, y_{,ij}, z_{,ij}) - \mathfrak{g}_{i,j} (0, y_{,1}, z_{,1})) \cdot N,$$

otherwise

$$L_{ij} = \frac{\varepsilon}{\mathfrak{g}_2}(\mathfrak{g}_2 (0, y_{,ij}, z_{,ij}) - \mathfrak{g}_{i,j} (0, y_{,2}, z_{,2})) \cdot N$$

for $y_{,ij} = \frac{\partial^2 y}{\partial u_i \partial u_j}$, $1 \leq i, j \leq 2$. Consequently, the *Gaussian* and *mean curvature* are defined as

$$K = -\varepsilon \frac{L_{11} L_{22} - L_{12}^2}{W^2} \text{ and } H = -\varepsilon \frac{\mathfrak{g}_2^2 L_{11} - 2\mathfrak{g}_1 \mathfrak{g}_2 L_{12} + \mathfrak{g}_1^2 L_{22}}{2W^2}.$$

A surface is said to have *constant Gaussian* (resp. *mean*) *curvature* if K (resp. H) is a constant function identically. In particular, it is said to be *flat* (resp. *minimal*) if the constant function vanishes.

3. Factorable surfaces of first kind

Let us consider the factorable surface of first kind in $\mathbb{G}_3^1$ given in explicit form $\Omega_1 : z(x,y) = f(x)\, g(y)$. Our purpose is to describe such a surface with $K = const. \neq 0$ and $H = const. \neq 0$. For this, firstly we can give the following result:

Theorem 3.1. *Let a factorable surface of first kind in $\mathbb{G}_3^1$ have non-zero constant Gaussian curvature K_0. Then we have*:

$$z(x,y) = \tanh\left(\pm\sqrt{|K_0|}x + \lambda_1\right)(y + \lambda_2), \ \lambda_1, \lambda_2 \in \mathbb{R}.$$

Proof. Assume that Ω_1 has non-zero constant Gaussian curvature K_0. Hence, we get a relation as follows:

$$K_0 = \frac{fgf''g'' - (f'g')^2}{\left[1 - (fg')^2\right]^2}, \tag{3.1}$$

where $f' = \frac{df}{dx}$, $g' = \frac{dg}{dy}$, etc. K_0 vanishes identically when f or g is a constant function. Then f and g must be non-constant functions. We distinguish two cases for the equation (3.1):

Case a. $f' = f_0, f_0 \in \mathbb{R} - \{0\}$. Thereby (3.1) turns into the following polynomial equation on (g'):

$$K_0 + \left(f_0^2 - 2K_0 f^2\right)(g')^2 + K_0 f^4 (g')^4 = 0,$$

which yields a contradiction.

Case b. $f'' \neq 0$. We have again two cases:

Case b.1. $g' = g_0$, $g_0 \in \mathbb{R} - \{0\}$. Then (3.1) leads to

$$\pm\sqrt{|K_0|} = \frac{g_0 f'}{1 - (g_0 f)^2}. \tag{3.2}$$

After solving (3.2), we obtain

$$f(x) = \frac{1}{g_0}\tanh\left(\pm\sqrt{|K_0|}x + \lambda_1\right), \ \lambda_1 \in \mathbb{R}.$$

Case b.2. $g'' \neq 0$. Then (3.1) can be arranged as follows:

$$\frac{K_0\left[1 - (fg')^2\right]^2}{ff''(g')^2} = \frac{gg''}{(g')^2} - \frac{(f')^2}{ff''}. \tag{3.3}$$

The partial derivative of (3.2) with respect to x and y leads to a polynomial equation on (g'):

$$-\left(\frac{1}{ff''}\right)' + \left(\frac{f^3}{f''}\right)' (g')^4 = 0. \tag{3.4}$$

Since all coefficients must vanish in (3.4), the contradiction $f' = 0$ is obtained. Therefore the proof is completed. □

Theorem 3.2. *Let a factorable surface of first kind in $\mathbb{G}_3^1$ have non-zero constant mean curvature H_0. Then the following occurs:*

$$z(x,y) = f_0 g(y) = \frac{1}{2H_0}\sqrt{(2H_0 y + \lambda_1)^2 \pm 1} + \lambda_2,$$

where "$\pm$" happens plus (resp. minus) when the surface is spacelike (resp. timelike). Further, f_0 is non-zero constant and λ_1, λ_2 some constants.

Proof. Relating to the mean curvature, we get

$$H_0 = \frac{fg''}{2\left|1-(fg')^2\right|^{\frac{3}{2}}}. \tag{3.5}$$

It is clear from (3.5) that g is a non-linear function. By taking partial derivative of (3.5) with respect to x, we deduce

$$f'\left|1-(fg')^2\right| - 3f\left|ff'(g')^2\right| = 0, \tag{3.6}$$

which yields two cases:

Case a. $f = f_0 \neq 0,\ f_0 \in \mathbb{R}$, is a solution for (3.6). If the surface is spacelike, then (3.5) turns to

$$-2H_0 = \frac{f_0 g''}{\left[1-(f_0 g')^2\right]^{\frac{3}{2}}}. \tag{3.7}$$

By solving (3.7), we find

$$g(y) = \frac{1}{2f_0 H_0}\sqrt{(2H_0 y + \lambda_1)^2 + 1} + \lambda_2,$$

where λ_1 and λ_2 are some constants. Otherwise, i.e., timelike situation yields

$$2H_0 = \frac{f_0 g''}{\left[(f_0 g')^2 - 1\right]^{\frac{3}{2}}}. \tag{3.8}$$

After solving (3.8), we obtain

$$g(y) = \frac{1}{2f_0 H_0}\sqrt{(2H_0 y + \lambda_3)^2 - 1} + \lambda_4$$

for some constants λ_3, λ_4.

Case b. $f' \neq 0$. If the surface is spacelike or timelike, then (3.6) implies

$$1 + 2\left(fg'\right)^2 = 0,$$

which is not possible. □

4. Factorable surfaces of second kind

As in previous section we try to describe the factorable graph surfaces of second kind in $\mathbb{G}_3^1$ given in explicit form $\Omega_2 : x\left(y,z\right) = f\left(y\right) g\left(z\right)$, assuming $K = const. \neq 0$ and $H = const. \neq 0$. Therefore the following non-existence result can be stated:

Theorem 4.1. *There does not exist a factorable surface of second kind in* $\mathbb{G}_3^1$ *having non-zero constant Gaussian curvature.*

Proof. It is proved by contradiction. Then we suppose that Ω_2 has the Gaussian curvature $K_0 \neq 0$ in $\mathbb{G}_3^1$. By a calculation, relating to the Gaussian curvature, we get

$$K_0 = \frac{fgf''g'' - \left(f'g'\right)^2}{\left[\left(fg'\right)^2 - \left(f'g\right)^2\right]^2}, \tag{4.1}$$

where $f' = \frac{df}{dy}$, $g' = \frac{dg}{dz}$ and so on. Hereinafter f and g must be non-constant functions so that K_0 does not vanish. Point that the roles of f and g are symmetric and it is sufficient to discuss the cases depending on f. Thus, if $f'' = 0$, i.e., $f' = f_0 \neq 0$, then (4.1) turns to a polynomial equation on (f):

$$\left[K_0\left(g'\right)^4\right] f^4 - \left[2K_0\left(f_0gg'\right)^2\right] f^2 + K_0\left(f_0g\right)^4 + \left(f_0g'\right)^2 = 0. \tag{4.2}$$

The fact that the coefficients must be zero yields the contradiction $g' = 0$. Hence f is a non-linear function and by symmetry, so is g. By dividing (4.1) with $ff''\left(g'\right)^2$, we can write

$$K_0\left[\frac{f^3}{f''}\left(g'\right)^2 - 2\frac{f\left(f'\right)^2}{f''}g^2 + \frac{\left(f'\right)^4}{ff''}\left(\frac{g^2}{g'}\right)^2\right] = \frac{gg''}{\left(g'\right)^2} - \frac{\left(f'\right)^2}{ff''}. \tag{4.3}$$

Put $f' = p$, $\dot{p} = \frac{dp}{df} = \frac{f''}{f'}$ and $g' = r$, $\dot{r} = \frac{dr}{dg} = \frac{g''}{g'}$ in (4.3). Then the partial derivative of (4.3) with respect to g gives

$$K_0\left[2\frac{f^3}{p\dot{p}}r\dot{r} - 4\frac{fp}{\dot{p}}g + 2\frac{p^3}{f\dot{p}}\left(\frac{g^2}{r}\right)\left\{\frac{d}{dg}\left(\frac{g^2}{r}\right)\right\}\right] = \frac{d}{dg}\left(\frac{g\dot{r}}{r}\right). \tag{4.4}$$

The partial derivative of (4.4) with respect to f yields

$$r\dot{r}\left[\frac{d}{df}\left(\frac{f^3}{p\dot{p}}\right)\right] - 2g\left[\frac{d}{df}\left(\frac{fp}{\dot{p}}\right)\right] + \left[\frac{d}{df}\left(\frac{p^3}{f\dot{p}}\right)\right]\left[\left(\frac{g^2}{r}\right)\frac{d}{dg}\left(\frac{g^2}{r}\right)\right] = 0. \tag{4.5}$$

By dividing (4.5) with g and taking partial derivative with respect to g, we derive

$$\underbrace{\left[\frac{d}{df}\left(\frac{f^3}{p\dot{p}}\right)\right]}_{F_1(f)}\overbrace{\left[\frac{d}{dg}\left(\frac{r\dot{r}}{g}\right)\right]}^{G_1(g)}+\underbrace{\left[\frac{d}{df}\left(\frac{p^3}{f\dot{p}}\right)\right]}_{F_2(f)}\overbrace{\left[\frac{d}{dg}\left\{\left(\frac{g}{r}\right)\frac{d}{dg}\left(\frac{g^2}{r}\right)\right\}\right]}^{G_2(g)}=0. \tag{4.6}$$

We have to distinguish several cases:

Case a. $F_1=0$. Then $f^3=\lambda_1 p\dot{p}$, $\lambda_1\in\mathbb{R}$, $\lambda_1\neq 0$. We have again two cases:

Case a.1. $F_2=0$, namely $p^3=\lambda_2 f\dot{p}$, $\lambda_2\in\mathbb{R}$, $\lambda_2\neq 0$. Considering these in (4.5) implies $fp=\lambda_3\dot{p}$, $\lambda_3\in\mathbb{R}$, $\lambda_3\neq 0$. Substituting these into (4.3) yields

$$K_0\left[\lambda_1 r^2-2\lambda_3 g^2+\lambda_2\left(\frac{g^2}{r}\right)^2\right]-\frac{g\dot{r}}{r}=\frac{-\lambda_2}{p^2}. \tag{4.7}$$

The left side of (4.7) is either a function of g or a constant, however other side is a non-constant function of f. This is not possible.

Case a.2. $G_2=0$. It implies $\frac{d}{dg}\left(\frac{g^2}{r}\right)=\frac{\lambda_4 r}{g}$, $\lambda_4\in\mathbb{R}$. By considering this one into (4.5) together with the assumption of Case a, we conclude

$$\left[\frac{d}{df}\left(\frac{p}{f}\right)\right]\left[1-\lambda_4\left(\frac{p}{f}\right)^2\right]=0. \tag{4.8}$$

If $p=\lambda_5 f$, $\lambda_5\in\mathbb{R}$, $\lambda_5\neq 0$, in (4.8) then we have $\dot{p}=\lambda_5$. Combining it with the assumption of Case a gives $f^2=\lambda_1\lambda_5^2$ that contradicts with $K_0\neq 0$.

Case b. $G_1=0$. Hence $r\dot{r}=\lambda_1 g$, $\lambda_1\in\mathbb{R}$, $\lambda_1\neq 0$. We have two cases:

Case b.1. $F_2=0$, i.e., $p^3=\lambda_2 f\dot{p}$, $\lambda_2\in\mathbb{R}$, $\lambda_2\neq 0$. Then (4.5) follows

$$\left[\frac{d}{df}\left(\frac{f}{p}\right)\right]\left[\lambda_1\left(\frac{f}{p}\right)^2-1\right]=0. \tag{4.9}$$

If $p=\lambda_3 f$, $\lambda_3\in\mathbb{R}$, $\lambda_3\neq 0$, in (4.8) then we get $\dot{p}=\lambda_3$. Comparing this one with the assumption of Case b.1 gives $f^2=\frac{\lambda_2}{\lambda_3^2}$, which is no possible since $K_0\neq 0$.

Case b.2. $G_2=0$. It follows

$$\left(\frac{g}{r}\right)\frac{d}{dg}\left(\frac{g^2}{r}\right)=\lambda_4,\ \lambda_4\in\mathbb{R}. \tag{4.10}$$

An integration of (4.10) with respect to g gives

$$r=\pm\frac{g^2}{\sqrt{\lambda_4 g^2+\lambda_5}},\ \lambda_5\in\mathbb{R}, \tag{4.11}$$

where λ_4 and λ_5 are not equal to zero together. After taking derivative of (4.11) with respect to g and producting with r, we conclude

$$r\dot{r}=\frac{\lambda_4 g^5+2\lambda_5 g^3}{\left(\lambda_4 g^2+\lambda_5\right)^2}. \tag{4.12}$$

Due to the assumption of the Case b, (4.12) turns to the following polynomial equation on g :

$$\left(\lambda_4-\lambda_1\lambda_4^2\right)g^5+2\left(\lambda_5-\lambda_1\lambda_4\lambda_5\right)g^3-\left(\lambda_1\lambda_5^2\right)g=0. \tag{4.13}$$

Since $\lambda_1\neq 0$, we get $1=\lambda_1\lambda_4$ and $\lambda_5=0$. It follows from (4.11) that $r=\left(\lambda_4\right)^{\frac{-1}{2}}g$. Then by substituting it into (4.3), we obtain

$$K_0\left[\frac{f^3}{\lambda_4 p\dot{p}}-2\frac{fp}{\dot{p}}+\lambda_4\frac{p^3}{f\dot{p}}\right]g^2+\frac{p}{f\dot{p}}-1=0. \tag{4.14}$$

This polynomial equation leads to

$$f\dot{p}=p \tag{4.15}$$

and

$$\frac{f^3}{\lambda_4 p\dot{p}}-2\frac{fp}{\dot{p}}+\lambda_4\frac{p^3}{f\dot{p}}=0. \tag{4.16}$$

Substituting (4.15) into (4.16) gives $p=\pm\left(\lambda_4\right)^{\frac{-1}{2}}f$ or $f\left(y\right)=\lambda_6\exp\left(\pm\left(\lambda_4\right)^{\frac{-1}{2}}y\right)$, $\lambda_6\in\mathbb{R}$, $\lambda_6\neq 0$. Further, since $r=\left(\lambda_4\right)^{\frac{-1}{2}}g$, we have $g\left(z\right)=\lambda_7\exp\left(\left(\lambda_4\right)^{\frac{-1}{2}}z\right)$, $\lambda_7\in\mathbb{R}$, $\lambda_7\neq 0$. However these lead the surface to be flat, i.e., $K_0=0$, which is not our case.

Case c. $F_1G_1\neq 0$. Then (4.6) can be rewritten as

$$\frac{F_1\left(f\right)}{F_2\left(f\right)}+\frac{G_2\left(g\right)}{G_1\left(g\right)}=0, \tag{4.17}$$

which implies

$$\frac{f^3}{p\dot{p}}=\lambda_1\frac{p^3}{f\dot{p}}+\lambda_2,\ \text{and}\ \left(\frac{g}{r}\right)\frac{d}{dg}\left(\frac{g^2}{r}\right)=-\lambda_1\frac{r\dot{r}}{g}+\lambda_3, \tag{4.18}$$

where $\lambda_1,\lambda_2,\lambda_3\in\mathbb{R}$, $\lambda_1\neq 0$. Substituting (4.18) into (4.5) gives

$$2\frac{fp}{\dot{p}}=\lambda_3\frac{p^3}{f\dot{p}}+\lambda_4,\ \lambda_4\in\mathbb{R}. \tag{4.19}$$

Comparing (4.19) with the first equality in (4.18) leads to

$$\left\{\begin{array}{l} f^4-\lambda_1 p^4=\lambda_2 fp\dot{p} \\ 2f^2p^2-\lambda_3 p^4=\lambda_4 fp\dot{p}. \end{array}\right. \tag{4.20}$$

By (4.20), we derive an equation as follows:

$$\lambda_4 f^4-2\lambda_2 f^2p^2+\left(\lambda_2\lambda_3-\lambda_1\lambda_4\right)p^4=0 \tag{4.21}$$

or

$$\lambda_4\left(\frac{f}{p}\right)^2+\left(\lambda_2\lambda_3-\lambda_1\lambda_4\right)\left(\frac{f}{p}\right)^{-2}=2\lambda_2. \tag{4.22}$$

Taking derivative of (4.22) with respect to f leads to

$$\frac{d}{df}\left(\frac{f}{p}\right)\left[1-\left(\lambda_2\lambda_3-\lambda_1\right)\left(\frac{f}{p}\right)^{-4}\right]=0, \tag{4.23}$$

which yields that the ratio f/p is constant, i.e., $p=\lambda_5 f$, $\lambda_5\in\mathbb{R}$, $\lambda_5\neq 0$. Substituting this one into (4.3) gives the following polynomial equation on (f) :

$$K_0\left[\frac{r^2}{\lambda_5^2}-2g^2+\lambda_5^2\left(\frac{g^2}{r}\right)^2\right]f^2-\frac{g\dot{r}}{r}+1=0,$$

which implies

$$\frac{r^2}{\lambda_5^2}-2g^2+\lambda_5^2\left(\frac{g^2}{r}\right)^2=0$$

and $r=g\dot{r}$. Solving this one leads to $r=\lambda_6 g$, $\lambda_6\in\mathbb{R}$, $\lambda_6\neq 0$. However, this is not possible since $K_0\neq 0$. This completes the proof. □

Theorem 4.2. *Let a factorable surface of second kind in* $\mathbb{G}_3^1$ *have non-zero constant mean curvature* H_0. *Then we have*:

$$x\left(y,z\right)=\lambda_1\exp\left(\lambda_2 y+\frac{\lambda_2}{2H_0}\sqrt{\left(2H_0 z+\lambda_3\right)^2\pm 1}\right),$$

where " ± " *happens plus* (*resp. minus*) *when the surface is timelike* (*resp. spacelike*). *Further,* λ_1,λ_2 *are non-zero constants and* λ_3 *some constant.*

Proof. It is only proved for spacelike situation since the calculations are almost same for other situation. Then we have

$$\left(fg'\right)^2-\left(f'g\right)^2>0$$

for all pairs (y,z). Since the mean curvature is constant $H_0\neq 0$, by a calculation, we deduce

$$2H_0\left[\left(fg'\right)^2-\left(f'g\right)^2\right]^{\frac{3}{2}}=\left(fg'\right)^2f''g-2fg\left(f'g'\right)^2+\left(f'g\right)^2fg''. \tag{4.24}$$

Note that f is not a constant function since $H_0\neq 0$ and, by symmetry, neither is g. Then dividing (4.24) with $fg\left(f'g'\right)^2$ yields

$$2H_0\left[\left(\frac{f}{f'}\right)^{\frac{4}{3}}\left(\frac{g'}{g}\right)^{\frac{2}{3}}-\left(\frac{f'}{f}\right)^{\frac{2}{3}}\left(\frac{g}{g'}\right)^{\frac{4}{3}}\right]^{\frac{3}{2}}=\frac{ff''}{\left(f'\right)^2}+\frac{gg''}{\left(g'\right)^2}-2. \tag{4.25}$$

Let us put $f' = p$, $\dot{p} = \frac{dp}{df} = \frac{f''}{f'}$ and $g' = r$, $\dot{r} = \frac{dr}{dg} = \frac{g''}{g'}$ in (4.25). Thus (4.25) can be rewritten as

$$2H_0\left[\left(\frac{f}{p}\right)^{\frac{4}{3}}\left(\frac{r}{g}\right)^{\frac{2}{3}} - \left(\frac{p}{f}\right)^{\frac{2}{3}}\left(\frac{g}{r}\right)^{\frac{4}{3}}\right]^{\frac{3}{2}} = \frac{f\dot{p}}{p} + \frac{g\dot{r}}{r} - 2. \tag{4.26}$$

The partial derivative of (4.26) with respect to f gives

$$\begin{aligned} &2H_0\left[\left(\frac{f}{p}\right)^{\frac{4}{3}}\left(\frac{r}{g}\right)^{\frac{2}{3}} - \left(\frac{p}{f}\right)^{\frac{2}{3}}\left(\frac{g}{r}\right)^{\frac{4}{3}}\right]^{\frac{1}{2}}\left[2\left(\frac{r}{g}\right)^{\frac{2}{3}} + \left(\frac{p}{f}\right)^{2}\left(\frac{g}{r}\right)^{\frac{4}{3}}\right]\frac{d}{df}\left(\frac{f}{p}\right) \\ &= \frac{d}{df}\left(\frac{f\dot{p}}{p}\right)\left(\frac{p}{f}\right)^{\frac{1}{3}}. \end{aligned} \tag{4.27}$$

If $\dot{p} = 0$, then (4.27) reduces to

$$2 + \left(\frac{p}{f}\right)^{2}\left(\frac{g}{r}\right)^{2} = 0,$$

which is not possible. Thus p is not constant function and, by symmetry, so is r. In addition, we have to consider two cases in order to solve (4.27):

Case a. $p = \lambda_1 f$, $\lambda_1 \neq 0$, is a solution for (4.27). Substituting this one into (4.26) gives

$$2H_0\left[\lambda_1^{-\frac{4}{3}}\left(\frac{r}{g}\right)^{\frac{2}{3}} - \lambda_1^{\frac{2}{3}}\left(\frac{g}{r}\right)^{\frac{4}{3}}\right]^{\frac{3}{2}} = \frac{g\dot{r}}{r} - 1$$

or

$$2H_0\left[\left(\frac{r}{g}\right)^{2} - \lambda_1^2\right]^{\frac{3}{2}} = \lambda_1^2\left[\frac{r\dot{r}}{g} - \left(\frac{r}{g}\right)^{2}\right].$$

The last equality can be rearranged as

$$2H_0 = \frac{\lambda_1^2\left(\frac{g'}{g}\right)'}{\left[\left(\frac{g'}{g}\right)^{2} - \lambda_1^2\right]^{\frac{3}{2}}}. \tag{4.28}$$

An integration of (4.28) with respect to z yields

$$2H_0 z + \lambda_2 = \frac{-\frac{g'}{g}}{\sqrt{\left(\frac{g'}{g}\right)^{2} - \lambda_1^2}}$$

or

$$\frac{g'}{g} = \frac{\lambda_1 \left(2H_0 z + \lambda_2\right)}{\sqrt{\left(2H_0 z + \lambda_2\right)^2 - 1}}. \tag{4.29}$$

An again integration of (4.29) with respect to z leads to

$$g\left(z\right) = \lambda_3 \exp\left(\frac{\lambda_1}{2H_0}\sqrt{\left(2H_0 z + \lambda_2\right)^2 - 1}\right),\ \lambda_3 \neq 0.$$

Due to the assumption of Case a, we conclude $f\left(y\right) = \lambda_4 \exp\left(\lambda_1 y\right)$, $\lambda_4 \neq 0$, which gives the assertion of the theorem.

Case b. $\frac{d}{df}\left(\frac{f}{p}\right) \neq 0$. By symmetry, we deduce $\frac{d}{dg}\left(\frac{g}{r}\right) \neq 0$. Then (4.27) can be rewritten as

$$\left[\left(\frac{r}{g}\right)^2 - \left(\frac{p}{f}\right)^2\right]^{\frac{1}{2}} \left[2 + \left(\frac{p}{f}\right)^2 \left(\frac{r}{g}\right)^{-2}\right] = \frac{\frac{d}{df}\left(\frac{f\dot{p}}{p}\right)\left(\frac{p}{f}\right)}{2H_0 \frac{d}{df}\left(\frac{f}{p}\right)}. \tag{4.30}$$

The partial derivative of (4.30) with respect to g leads to

$$2\left(\frac{r}{g}\right)^4 - \left(\frac{r}{g}\right)^2 \left(\frac{p}{f}\right)^2 + 2\left(\frac{p}{f}\right)^4 = 0,$$

or

$$\left[\left(\frac{r}{g}\right)^2 - \left(\frac{p}{f}\right)^2\right]^2 + \frac{3}{2}\left(\frac{r}{g}\right)^2 \left(\frac{p}{f}\right)^2 = 0,$$

which yields a contradiction. Therefore the proof is completed. ☐

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