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FUZZY REGRESSION MODEL WITH MONOTONIC RESPONSE FUNCTION

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Abstract. Fuzzy linear regression model has been widely studied with many successful applications but there have been only a few studies on the fuzzy regression model with monotonic response function as a generalization of the linear response function. In this paper, we propose the fuzzy regression model with the monotonic response function and the algorithm to construct the proposed model by using α-level set of fuzzy number and the resolution identity theorem. To estimate parameters of the proposed model, the least squares (LS) method and the least absolute deviation (LAD) method have been used in this paper. In addition, to evaluate the performance of the proposed model, two performance measures of goodness of fit are introduced. The numerical examples indicate that the fuzzy regression model with the monotonic response function is preferable to the fuzzy linear regression model when the fuzzy data represent the non-linear pattern.

1. Introduction

Regression analysis is a statistical technique for estimating relationship between explanatory variables and response variables by inducing a mathematical model. One of purposes of regression analysis is to predict the statistical relationship between input and output variables. However, in the real world, input-output variables have a vague relationship and can often be observed in imprecise and vague data. In the presence of vague data and structure, a new approach is needed for performing the data analysis and drawing statistical inference. For those reasons, the fuzzy set theory was introduced by Zadeh [22,23] and the fuzzy linear regression model based on fuzzy set theory was proposed by Tanaka [18, 19]. Also, the applications and the estimating method of the fuzzy linear regression model have been studied by many authors [4,5,9,13–16].

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However, a linear response function may not be enough to explain the causal relationship between the variables represented by the fuzzy number. In fuzzy regression model, the monotonic response function may be more effective than the linear response function when the dependent variable increases or decreases drastically with respect to the independent variable. Even if several authors have studied the fuzzy regression model with S-curve or non-linear as the response function, there have been only a few studies on fuzzy regression model with nonlinear response function until now [1, 2, 8, 15]. Aside from statistical methods, only numerical analysis or other approaches using fuzzy logic have been applied so far to analyze a nonlinear fuzzy regression model [24]. The reason is that it is difficult to represent the non-linear function using the mode and the spread with respect to fuzzy number. Since the monotonic response functions have served as a useful model for describing various physical and biological systems, it is necessary to study a fuzzy regression model with the monotonic response function to explain the causal relationship represented by fuzzy number occurring in many fields.

In this paper, we construct the fuzzy regression model with the monotonic response function by using definition of α-level set for monotonic function of fuzzy number and the resolution identity theorem. We propose a fuzzy regression model with the monotonic response function such as exponential function, power function, or reciprocal function. We also propose an algorithm to construct the proposed model using the α-level sets of fuzzy variables. We use the LS method and the LAD method to estimate the proposed model. The effectiveness of the proposed model is illustrated by numerical examples using the exponential response function.

The remainder of the paper is organized as follows. In Section 2, we review some preliminary concepts of fuzzy sets and briefly describe regression model and the fuzzy regression model. Construction of proposed model are introduced in Section 3. In Section 4, we illustrate the proposed model through the numerical examples and compare it with linear regression model. Section 5 is the conclusion.

2. Preliminaries

The fuzzy set theory provides a suitable framework for dealing with imprecise data. Following [6] and [20], we introduce some definitions regarding the fuzzy sets and the fuzzy numbers.

A fuzzy set A in $\mathbb{R}$ is a set of ordered pairs

$$A = \{(x, \mu_A(x)) : x \in \mathbb{R}\},$$

where $\mu_A : \mathbb{R} \to [0, 1]$ is a membership function of x in A.

For any $\alpha \in [0, 1]$, the crisp set $A(\alpha) = \{x \in \mathbb{R} \,|\, \mu_A(x) \geq \alpha\}$ is called an α-level set of a fuzzy set A. The support of a fuzzy set A, $suppA$, is the crisp set of all $x \in \mathbb{R}$ such that $\mu_A(x) > 0$. A fuzzy number A is a fuzzy set A of the real line $\mathbb{R}$ such that (i) A is normal, i.e., $\mu_A(x_0) = 1$ for some $x_0 \in \mathbb{R}$ (ii)

$\mu_A(x)$ is convex. As a special case, a fuzzy number A is said to be an LR-fuzzy number if its membership function is defined by

$$\mu_A(x) = \begin{cases} L_A\left(\dfrac{a-x}{l_a}\right) & \text{for} \quad 0 \le a-x \le l_a, \\ R_A\left(\dfrac{x-a}{r_a}\right) & \text{for} \quad 0 \le x-a \le r_a, \\ 0 & \text{for otherwise,} \end{cases}$$

where a is the mode, and l_a and r_a are called the left and the right spreads, respectively. Symbolically, an LR-fuzzy number A is denoted by $A = (a, l_a, r_a)_{LR}$. L_A and R_A are functions verifying the properties of the class of fuzzy sets such that $L_A(0) = R_A(0) = 1$ and $L_A(x) = R_A(x) = 0$, $x \in \mathbb{R} \backslash [0,1)$. In particular, if $L_A(x) = R_A(x) = 1 - x$, then A is called a triangular fuzzy number and denoted by $A = (a, l_a, r_a)_T$. A fuzzy number A is said to be positive if $\mu_A(x) = 0$ for all $x \le 0$.

An α-level set of the fuzzy number is given by the closed interval

$$A(\alpha) = [l_{A(\alpha)}, r_{A(\alpha)}] = [\alpha - l_a L_A^{-1}(\alpha), \alpha + r_a R_A^{-1}(\alpha)],$$

where $l_{A(\alpha)}$ and $r_{A(\alpha)}$ are the left end point and right end point of $A(\alpha)$. The resolution identity theorem proposed by Zadeh [23] states that a fuzzy set can be represented either by its α-level sets or by its membership function. Let A be a fuzzy number with the membership function μ_A and α-level set $A(\alpha)$. Then we have

$$\mu_A(x) = \sup\{\alpha \cdot I_{A(\alpha)}(x) : \alpha \in [0,1]\},$$

where sup stands for supremum (that is, for the least upper bound) and an indicator function of the set $A(\alpha)$, denoted by $I_{A(\alpha)}(\cdot)$, is defined as follows:

$$I_{A(\alpha)}(x) = \begin{cases} 1 & x \in A(\alpha), \\ 0 & x \notin A(\alpha). \end{cases}$$

Let f be a continuous real valued function on $\mathbb{R}$, $D(f)$ be a domain of f, and $I = [a,b] (\subseteq D(f) \subseteq \mathbb{R})$ be a closed and bounded interval. The image of a subset $I \subseteq D(f)$ under f is a subset $f(I) \subseteq \mathbb{R}$ defined by $f(I) = \{y \,|\, y = f(x), x \in I\}$. Let A be a positive fuzzy number on $\mathbb{R}$ and f be a monotonic(strictly increasing or decreasing) function. Then an α-level set of $f(A)$ is defined by the following:

$$f(A)(\alpha) = \begin{cases} [f(l_{A(\alpha)}), f(r_{A(\alpha)})], & f : \nearrow, \\ [f(r_{A(\alpha)}), f(l_{A(\alpha)})], & f : \searrow, \end{cases}$$

where $\nearrow$ and $\searrow$ stand for an increasing function and a decreasing function, respectively. Using the above definition and simple algebra on interval, we can show that α-level set of fuzzy number $f(A)$ is equal to the value of function f on α-level set of fuzzy number A. The proof of this, which can be completed similarly to the method introduced by Ruoning [21], is omitted in this paper.

We can define the α-level set of the monotonic function with respect to fuzzy number as follows:

$$f(A)(\alpha) = f(A(\alpha)). \tag{1}$$

From (1), we can define the α-level sets of the monotonic functions with respect to fuzzy number such as exponential, power, and reciprocal functions as follows:

$$\begin{aligned} &\text{(i) } \exp(A)(\alpha) = [\exp(l_A(\alpha)), \exp(r_A(\alpha))], \\ &\text{(ii) } X^A(\alpha) = [\exp(l_A(\alpha)\ln l_X(\alpha)), \exp(r_A(\alpha)\ln r_X(\alpha))], \\ &\text{(iii) } A^{-1}(\alpha) = \left[\frac{1}{r_A(\alpha)}, \frac{1}{l_A(\alpha)}\right]. \end{aligned} \tag{2}$$

Also, the membership function $\mu_{f(A)}$ can be obtained by the set $f(A)(\alpha)$ in (1) and the resolution identity theorem.

We often encounter the vague relationship among variables and/or imprecisely observed data in the regression analysis. In addition, we often see that the increasing or decreasing pattern between dependent variable and independent variable is represented by monotonic function rather than by linear function. Using (1) and (2), we deal with the fuzzy regression model with monotonic response function to reflect these cases in the next section.

3. Fuzzy regression with monotonic response function

In this section, we propose a method for estimating a fuzzy regression model with monotonic response function. In addition, we estimate the fuzzy regression coefficients for proposed method by using the LS method and the LAD method. Let F be a known strictly monotonic function on $\mathbb{R}$. Then the fuzzy regression model with monotonic response function is given by

$$Y(X_i) = F(A, X_i) \oplus E_i, \tag{3}$$

where $X_i = (X_{ij})_{(1\times(p+1))}$ is the fuzzy input, $A = (A_j)_{(1\times(p+1))}$ is the fuzzy coefficients, $F(A, X_i)$ is the response function, $Y(X_i)$ is the fuzzy output, $\oplus$ denotes the addition of fuzzy numbers, and E_i is the fuzzy error associated with the fuzzy regression model. In addition, the left and the right end points of α-level set for the proposed fuzzy regression model are represented by

$$l_{Y_i(\alpha)} = f_l(l_{A(\alpha)}, l_{X_i(\alpha)}) + l_{E_i(\alpha)}$$

and

$$r_{Y_i(\alpha)} = f_r(r_{A(\alpha)}, r_{X_i(\alpha)}) + r_{E_i(\alpha)},$$

respectively, where f_l and f_r are the real valued functions and l_{E_i} and r_{E_i} are the left and the right end points of α-level set of the error term E_i, respectively.

Now, we propose an algorithm to construct the proposed fuzzy regression model with monotonic response function represented by α-level set. The most common method for estimating the parameters in the fuzzy regression model (3) is the minimization of the difference between the observed values and the predicted value. That is to estimate the parameters by using the fitting method that the predicted values best represent the pattern of the observed values. For this, we have to define the distance $d(Y_i, \hat{Y}_i)$ between the observed value Y_i and the predicted value $\hat{Y}_i$. In this paper, we estimate the model (3) by using the LS method to minimize the square of the difference between a value and the estimate and the LAD methods to minimize the absolute difference between a value and the estimate. In addition, we apply the resolution identity theorem to α-level set for the estimated fuzzy regression model to obtain the predicted fuzzy number. First, we consider the set

$$C = \{\alpha_j : j = 1, \ldots, s, \;\; 0 \le \alpha_j \le 1\}.$$

Then the following five steps are used to estimate the monotonic response function based on the set

$$\{(l_{Y_i(\alpha)}, l_{X_i(\alpha)}) : i = 1, \ldots, n\} \text{ or } \{(r_{Y_i(\alpha)}, r_{X_i(\alpha)}) : i = 1, \ldots, n\} \text{ for } \alpha \in C.$$

Algorithm to construct the fuzzy regression model by using α-level set

Step1.

The set C, ranked in ascending order, is represented by $C = \{\alpha_{(1)}, \ldots, \alpha_{(s)}\}$. Choose the minimum value $\alpha_{(1)}$ and estimate $\hat{l}_{Y_i(\alpha_{(1)})}$ by minimizing the objective function

$$\sum_{i=1}^{n} d(l_{Y_i(\alpha_{(1)})}, f_l(l_{A(\alpha_{(1)})}, l_{X_i(\alpha_{(1)})})).$$

Step2.

For $j = 2, 3, \ldots, s$, estimate the intermediate estimator $\bar{l}_{Y_i(\alpha_{(j)})}$ based on $\{(l_{Y_i(\alpha_{(j)})}, l_{X_i(\alpha_{(j)})}) : i = 1, \ldots, n\}$ by minimizing the objective function

$$\sum_{i=1}^{n} d(l_{Y_i(\alpha_{(j)})}, f_l(l_{A(\alpha_{(j)})}, l_{X_i(\alpha_{(j)})})).$$

Step3.

For $j = 2, 3, \ldots, s$, estimate $\hat{l}_{Y_i(\alpha_{(j)})}$ based on

$$\hat{l}_{Y_i(\alpha_{(j)})} = \max\{\bar{l}_{Y_i(\alpha_{(j)})}, \hat{l}_{Y_i(\alpha_{(j-1)})}\}.$$

Step4.

Estimate the membership function $L_{\hat{Y}_i}(\cdot)$ based on $\{(\hat{l}_{Y_i(\alpha_j)}, \alpha_j) : j=1, \ldots, s\}$ by minimizing the objective function

$$\sum_{j=1}^{s} d(\alpha_j, L_{Y_i}(\hat{l}_{Y_i(\alpha_j)})) \quad \text{subject to} \quad L_{Y_i}(\hat{l}_{Y_i(1)}) = 1.$$

Step5.

The left reference function $L_{\hat{Y}_i}(\cdot)$ for the predicted fuzzy number $\hat{Y}_i$ is defined from Step 1 and Step 4.

In the above steps, the symbol $d(a,b)$ stands for the distance between a and b. In this paper, we use two distances to estimate the parameter of fuzzy regression model as follows:

$$d(a,b) = (a-b)^2 \quad \text{for LS method and } d(a,b) = |a-b| \quad \text{for LAD method.}$$

Using the above five steps and the following $\hat{r}_{Y_i(\alpha_{(j)})}$, we can also obtain the right reference function $R_{\hat{Y}_i}(\cdot)$ of the predicted fuzzy number $\hat{Y}_i$

$$\hat{r}_{Y_i(\alpha_{(j)})} = \min\{\bar{r}_{Y_i(\alpha_{(j)})}, \hat{r}_{Y_i(\alpha_{(j-1)})}\}, \quad j = 2, 3, \ldots, s.$$

The method used above can be extended to the robust estimation method, which is commonly used in regression analysis. In addition, we estimate the membership function $\mu_{A_k}(\cdot)$ of the fuzzy coefficient A_k for the fuzzy regression model (3) by using the above five steps. Estimated regression models can be different from each other based on the estimation methods and the types of response functions. Therefore, it is important to compare the efficiencies between the estimation results from various kinds of estimation methods and various types of response functions. The efficiencies can be obtained from the difference between the observed values and the estimated values. To evaluate the performance of the fuzzy regression model with monotonic response function, we use the two performance measures. One is M_d based on the difference between the predicted value and the observed value and the other is M_s based on the similarity between the predicted value and the observed value [3,11,12,17]. The measure M_d based on the difference of two fuzzy numbers is defined as follows:

$$M_d(Y, \hat{Y}) = \sum_{i=1}^{n} m_d(Y, \hat{Y}_i),$$

where

$$m_d(Y_i, \hat{Y}_i) = \frac{\int_{-\infty}^{\infty} \left|\mu_{Y_i}(x) - \mu_{\hat{Y}_i}(x)\right| dx}{\int_{-\infty}^{\infty} \left|\mu_{Y_i}(x)\right| dx} + h_d(Y_i(0), \hat{Y}_i(0)).$$

Here, $h_d(Y_i(0), \hat{Y}_i(0)) = \inf\{\inf\{|a-b| : a \in Y_i(0)\} : b \in \hat{Y}_i(0)\}$, where inf stands for infimum (that is, for the greatest lower bound). The more efficient model has the smaller value of M_d.

One more measure is M_s, which is the measure of how much overlapped is in the membership. It is defined as follows:

$$M_s(Y, \hat{Y}) = \sum_{i=1}^{n} m_s(Y, \hat{Y}_i),$$

where

$$m_s(Y_i, \hat{Y}_i) = \frac{\int_{-\infty}^{\infty} \text{Min}(\hat{Y}_i, Y_i(x)) dx}{\int_{-\infty}^{\infty} \text{Max}(\hat{Y}_i, Y_i(x)) dx}.$$

4. Numerical examples

In this section, we illustrate the numerical examples to show that the fuzzy regression model with a monotonic response function is better than linear regression model in the non-linear phenomena. In this paper, two examples described in other literatures having the monotonic increasing and decreasing response functions are used. In this paper, we only deal with two types of the monotonic response functions but the fuzzy regression model using the other types of monotonic functions can also be estimated similarly to our proposed method.

The following example represents the efficiency of the fuzzy regression model with response function that increases monotonically.

Example 1. The fuzzified data of the temperature and pressure in a saturated steam quoted in Draper and Smith [7] are listed in Table 1. To estimate the fuzzy regression model for temperature, the linear (Y_L) and exponential (Y_E) response functions are applied. To estimate the parameters for the linear and exponential response functions the LS method ($\hat{Y}$) and the LAD method($\tilde{Y}$) are used.

Table 1. Dataset for Example 1

(x=temperature, Y=pressure)

x	Y
0	$(4.14, 3.2, 2.1)_T$
10	$(8.52, 2.2, 3.8)_T$
20	$(16.31, 3.4, 4.1)_T$
30	$(32.18, 3.4, 3.4)_T$
40	$(64.62, 0.7, 1.9)_T$
50	$(98.76, 4.1, 1.7)_T$
60	$(151.13, 4.5, 3)_T$
70	$(224.74, 4.6, 3.3)_T$
80	$(341.35, 4.7, 3)_T$
85	$(423.36, 1.7, 3.1)_T$
90	$(522.78, 0.2, 1.5)_T$
95	$(674.32, 2, 2.6)_T$
100	$(782.04, 1.4, 0.4)_T$
105	$(920.01, 3.1, 2.7)_T$

We used the set of finite α-level $C = \{0, 0.25, 0.5, 0.75, 1\}$ to estimate the fuzzy regression model for the pressure by using the algorithm introduced in Section 3. The estimation results by using the LS method based on linear and exponential response functions are

$$\hat{Y}_L = (-175.503, 3.274, 3.286)_T \oplus 8.049x$$

and

$$\hat{Y}_E = (-19.939, 3.236, 3.052)_T \oplus 17.78 \exp(0.038x),$$

respectively.

By the same procedure, the estimation results by using the LAD method are

$$\tilde{Y}_L = (-213, 0.9, 4.2)_T \oplus 8.049x$$

and

$$\tilde{Y}_E = (-17.97, 4.598, 0.908)_T \oplus (17.496, 0.038, 0.037)_T \otimes \exp(0.038x),$$

where $\otimes$ denotes the multiplication of fuzzy numbers.

Table 2 shows that the exponential response function is more efficient than the linear response function in the fuzzy regression model for the pressure.

Table 2. Results of performance

Measure	$\hat{Y}_L$	$\hat{Y}_E$	$\tilde{Y}_L$	$\tilde{Y}_E$
M_d	55.26	5.88	58.54	10.03
M_s	0	0.12	0.04	0.23

In the following example, we estimate the fuzzy regression model with response function that decreases. Also, we verify that the fuzzy regression model having the sum of the exponential function as the response function is more efficient than fuzzy linear regression model.

Example 2. The following data presented in Jennrich [10], obtained by measuring a radioactive trace, record the retention of a drug aurothiomalate used in the treatment of arthritic.

Table 3. Dataset for Example 2

(x=days after treatment, Y=tracer retained)

x	Y
0	$(100, 1.07, 1.07)_T$
2	$(86, 0.99, 0.99)_T$
4	$(78, 1.1, 1.1)_T$
14	$(60, 1.35, 1.35)_T$
21	$(53, 1.74, 1.74)_T$
42	$(47, 2.26, 2.26)_T$
63	$(42, 3.1, 3.1)_T$
97	$(40, 3.87, 3.87)_T$
155	$(35, 5.51, 5.51)_T$
217	$(33, 7.36, 7.36)_T$

Using the LS method and the proposed algorithm, the estimated fuzzy regression model of three response functions, linear (Y_L), exponential (Y_E) and additive exponential function (Y_{AE}), are given as follows:

$$\begin{aligned}\hat{Y}_L =&(71.1182, 1.0371, 1.0372)_T \oplus (-0.2399, 0.0293, 0.0292)_T \otimes x,\\ \hat{Y}_E =&(70.2514, 0.5542, 0.7665)_T \otimes \exp(-0.0044, 0.0011, 0.0009)_T \otimes x,\\ \hat{Y}_{AE} =&(53, 0.0749, 0.0979)_T \otimes \exp(-0.088, 0.0749, 0)_T \otimes x\\ &\oplus (44, 3.3333, 1.0901)_T \otimes \exp(-0.001, 0.0749, 0)_T \otimes x.\end{aligned}$$

Using the LAD method and the proposed algorithm, the estimated fuzzy regression model of three response functions, linear (Y_L), exponential (Y_E) and additive exponential function (Y_{AE}), are given as follows:

$$\begin{aligned}\hat{Y}_L =&(62.48, 0.5618, 0.5682)_T \oplus (-0.1773, 0.0177, 0.0177)_T \otimes x,\\ \hat{Y}_E =&(79.995, 0.0352, 0.5319)_T \otimes \exp(-0.0044, 0.0011, 0.0009)_T \otimes x,\\ \tilde{Y}_{AE} =&(44.2, 0.5556, 0.8)_T \otimes \exp(-0.0015, 0.01, 0.01)_T \otimes x\\ &\oplus (53.2, 0.0749, 0.0979)_T \otimes \exp(-0.075, 0, 0)_T \otimes x.\end{aligned}$$

The results of two performance measures M_d and M_s are presented in Table 4. The smaller value of M_d and the larger value of M_s imply that a model is more likely to be appropriate for the given sample data. Table 4 indicates that the best-fitting model is an additive exponential model.

Table 4. Results of performance measures

Model	$\hat{Y}_L$	$\hat{Y}_E$	$\hat{Y}_{AE}$	$\tilde{Y}_L$	$\tilde{Y}_E$	$\tilde{Y}_{AE}$
M_d	10.93	10.30	3.50	10.64	8.86	2.10
M_s	0.1	0.11	0.25	0.12	0.12	0.37

Examples 1 and 2 show that the fuzzy regression model using the nonlinear monotonic function may be more effective than the fuzzy linear regression model when the response function increases or decreases drastically.

5. Conclusion

Fuzzy linear regression model has been studied by many authors and it is one of the popular fuzzy statistical analysis. In the classical regression model, the monotonic response functions as well as the linear response function have served as useful models describing various physical and biological systems but they have rarely been used in fuzzy regression.

In this paper, we propose the fuzzy regression model with the monotonic response function. Using α-level set, we represent the fuzzy regression model with the monotonic response function and propose an algorithm to construct the fuzzy regression model. To illustrate the proposed algorithm, we use two monotonic response functions, that is, exponential and additive exponential response functions. The two measures of performance are introduced to evaluate the proposed fuzzy regression model and to compare it with the fuzzy linear regression model. Through the numerical examples, we demonstrate that the proposed model is superior to the fuzzy linear regression model when data have the non-linear data structure.

In our future studies, we will propose the fuzzy regression models with the various monotonic response functions such as logarithmic and arc trigonometric and study the empirical analyses to illustrate such models. We will also use a robust estimation method such as an M-estimator or a rank transformation method to estimate a non-linear fuzzy regression model as an extension of the method presented in this study. In addition, an asymptotic properties of fuzzy regression model with monotonic response function will be investigated.

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