Commun. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/CKMS.c170056
pISSN: 1225-1763 / eISSN: 2234-3024

# CERTAIN FRACTIONAL INTEGRALS AND IMAGE FORMULAS OF GENERALIZED k-BESSEL FUNCTION

Praveen Agarwal, Mehar Chand, Junesang Choi, and Gurmej Singh

Abstract. We aim to establish certain Saigo hypergeometric fractional integral formulas for a finite product of the generalized k-Bessel functions, which are also used to present image formulas of several integral transforms including beta transform, Laplace transform, and Whittaker transform. The results presented here are potentially useful, and, being very general, can yield a large number of special cases, only two of which are explicitly demonstrated.



## 1. Introduction and preliminaries

Díaz and Pariguan [4] introduced the k-Pochhammer symbol as follows:

$$(1.1)\quad (\gamma)_{n,\mathtt{k}} := \begin{cases} \dfrac{\Gamma_{\mathtt{k}}(\gamma + n\mathtt{k})}{\Gamma_{\mathtt{k}}(\gamma)} & (n \in \mathbb{N}; \mathtt{k} \in \mathbb{R}^+; \gamma \in \mathbb{C} \setminus \{0\}), \\ \gamma(\gamma+\mathtt{k})\cdots(\gamma+(n-1)\mathtt{k}) & (n \in \mathbb{N}; \gamma \in \mathbb{C}), \end{cases}$$

where $\Gamma_{\mathtt{k}}$ is the k-gamma function which has the following relation with the classical Euler's gamma function $\Gamma$ (see, e.g., [14, Section 1.1])

$$(1.2)\qquad \Gamma_{\mathtt{k}}(\gamma) = \mathtt{k}^{\frac{\gamma}{\mathtt{k}}-1}\Gamma\left(\frac{\gamma}{\mathtt{k}}\right).$$

Here and in the following, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{N}$, and $\mathbb{Z}_0^-$ be the sets of complex numbers, real numbers, positive real numbers, positive integers, and non-positive integers, respectively.

Obviously, the special case $\mathtt{k} = 1$ of (1.1) reduces to the familiar Pochhammer symbol (see, e.g., [14, pp. 2 and 5]):

$$(1.3)\quad (\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}). \end{cases}$$

Received February 19, 2017; Accepted March 28, 2017.

2010 *Mathematics Subject Classification.* Primary 26A33, 33C45; Secondary 33C60, 33C70.

*Key words and phrases.* k-Pochhammer symbol, k-gamma function, Fox-Wright function, generalized k-Bessel function, Laplace transform, beta transform, Whittaker transform, Saigo hypergeometric fractional integral operators.

Also, the following identity holds:

$$\Gamma_{\mathtt{s}}(\gamma) = \left(\frac{\mathtt{s}}{\mathtt{k}}\right)^{\frac{\gamma}{\mathtt{s}}-1} \Gamma_{\mathtt{k}}\left(\frac{\mathtt{k}\gamma}{\mathtt{s}}\right) \quad \left(\gamma \in \mathbb{C}; \mathtt{k}, s \in \mathbb{R}^+\right), \tag{1.4}$$

which, upon setting $\mathtt{k} = 1$ and replacing $\mathtt{s}$ by $\mathtt{k}$, yields (1.2). Further, the following relation holds:

$$(\gamma)_{nq,\frac{\mathtt{s}}{\mathtt{k}}} = \left(\frac{\mathtt{s}}{\mathtt{k}}\right)^{nq} \left(\frac{\mathtt{k}\gamma}{\mathtt{s}}\right)_{nq} \quad \left(n, q \in \mathbb{N}; \mathtt{s}, \mathtt{k} \in \mathbb{R}^+; \gamma \in \mathbb{C}\right), \tag{1.5}$$

which, upon setting $\mathtt{k} = 1$ and replacing $\mathtt{s}$ by $\mathtt{k}$, yields

$$(\gamma)_{nq,\mathtt{k}} = (\mathtt{k})^{nq} \left(\frac{\gamma}{\mathtt{k}}\right)_{nq} \quad \left(n, q \in \mathbb{N}; \mathtt{k} \in \mathbb{R}^+; \gamma \in \mathbb{C}\right). \tag{1.6}$$

For more details of $\mathtt{k}$-Pochhammer symbol with $\mathtt{k}$-special function and fractional Fourier transform, we refer the reader, for example, to [9, 10].

Recently, Romero et al. [11] (see also [2]) introduced the $\mathtt{k}$-Bessel function of the first kind

$$J^{\gamma,\lambda}_{\mathtt{k},\mu}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,\mathtt{k}}}{\Gamma_{\mathtt{k}}(\lambda n + \mu + 1)} \frac{(-1)^n}{(n!)^2} \left(\frac{z}{2}\right)^n \tag{1.7}$$

$$\left(\gamma, \lambda, \mu \in \mathbb{C} \text{ with } \min\{\Re(\lambda), \Re(\mu)\} > 0; \mathtt{k} \in \mathbb{R}^+\right).$$

A more generalized form of $\mathtt{k}$-Bessel function $\omega^{\gamma,\lambda}_{\mathtt{k},\nu,b,c}(z)$ is given as follows:

$$\omega^{\gamma,\lambda}_{\mathtt{k},\mu,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n c^n (\gamma)_{n,\mathtt{k}}}{\Gamma_{\mathtt{k}}(\lambda n + \mu + \frac{b+1}{2})} \frac{\left(\frac{z}{2}\right)^{\mu+2n}}{(n!)^2} \tag{1.8}$$

$$\left(b, c, \gamma, \lambda, \mu \in \mathbb{C} \text{ with } \min\{\Re(\lambda), \Re(\mu)\} > 0; \mathtt{k} \in \mathbb{R}^+\right).$$

In (1.7) and (1.8), $(\gamma)_{n,\mathtt{k}}$ and $\Gamma_{\mathtt{k}}$ are the $\mathtt{k}$-Pochhammer symbol and the $\mathtt{k}$-gamma function given in (1.1) and (1.2), respectively.

Consider the following product of (1.8)

$$\prod_{i=1}^{r} \omega^{\gamma_i,\lambda_i}_{\mathtt{k}_i,\mu_i,b_i,c_i}(z) = \prod_{i=1}^{r} \sum_{n=0}^{\infty} \frac{(-1)^n c_i^n (\gamma_i)_{n,\mathtt{k}_i}}{\Gamma_{\mathtt{k}_i}(\lambda_i n + \mu_i + \frac{b_i+1}{2})} \frac{\left(\frac{z}{2}\right)^{\mu_i+2n}}{(n!)^2} \tag{1.9}$$

$$\left(b_i, c_i, \gamma_i, \lambda_i, \mu_i \in \mathbb{C} \text{ with } \min\{\Re(\lambda_i), \Re(\mu_i)\} > 0; \mathtt{k}_i \in \mathbb{R}^+; i = 1, \ldots, r\right),$$

which, upon setting $b_i = c_i = 1$ $(i = 1, \ldots, r)$, yields

$$\begin{aligned} \prod_{i=1}^{r} \omega^{\gamma_i,\lambda_i}_{\mathtt{k}_i,\mu_i,1,1}(z) &= \prod_{i=1}^{r} \left(\frac{z}{2}\right)^{\mu_i} \sum_{n=0}^{\infty} \frac{(-1)^n (\gamma_i)_{n,\mathtt{k}_i}}{\Gamma_{\mathtt{k}_i}(\lambda_i n + \mu_i + 1)} \frac{\left(\frac{z^2}{4}\right)^n}{(n!)^2} \\ &= \prod_{i=1}^{r} \left(\frac{z}{2}\right)^{\mu_i} J^{\gamma_i,\lambda_i}_{\mathtt{k}_i,\mu_i}\left(\frac{z^2}{2}\right) \end{aligned} \tag{1.10}$$

$$\left(\gamma_i, \lambda_i, \mu_i \in \mathbb{C} \text{ with } \min\{\Re(\lambda_i), \Re(\mu_i)\} > 0; \mathtt{k}_i \in \mathbb{R}^+; i = 1, \ldots, r\right).$$

Setting $b_i = -1$ and $c_i = 1$ $(i = 1, \ldots, r)$, (1.9) reduces to product of the $\mathtt{k}$-Wright functions

$$\begin{aligned}\prod_{i=1}^{r} \omega^{\gamma_i,\lambda_i}_{\mathtt{k}_i,\mu_i,-1,1}(z) &= \prod_{i=1}^{r} \left(\frac{z}{2}\right)^{\mu_i} \sum_{n=0}^{\infty} \frac{(-1)^n (\gamma_i)_{n,\mathtt{k}_i}}{\Gamma_{\mathtt{k}_i}(\lambda_i n + \mu_i)} \frac{\left(\frac{z^2}{4}\right)^n}{(n!)^2} \\ &= \prod_{i=1}^{r} \left(\frac{z}{2}\right)^{\mu_i} W^{\gamma_i}_{\mathtt{k}_i,\lambda_i,\mu_i}\left(\frac{-z^2}{2}\right)\end{aligned} \tag{1.11}$$

$$\left(\gamma_i, \lambda_i, \mu_i \in \mathbb{C} \text{ with } \min\{\Re(\lambda_i), \Re(\mu_i)\} > 0; \mathtt{k}_i \in \mathbb{R}^+; i = 1, \ldots, r\right).$$

The Fox-Wright function ${}_p\Psi_q$ is defined as follows (see, e.g., [15, p. 21]):

$${}_p\Psi_q \left[\begin{matrix}(\alpha_1, A_1), \ldots, (\alpha_p, A_p); \\ (\beta_1, B_1), \ldots, (\beta_q, B_q);\end{matrix} z\right] = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j + A_j k)}{\prod_{j=1}^{q} \Gamma(\beta_j + B_j k)} \frac{z^k}{k!}, \tag{1.12}$$

where the coefficients $A_1, \ldots, A_p \in \mathbb{R}^+$ and $B_1, \ldots, B_q \in \mathbb{R}^+$ satisfying

$$1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geqq 0. \tag{1.13}$$

A special case of (1.12) is

$${}_p\Psi_q \left[\begin{matrix}(\alpha_1, 1), \ldots, (\alpha_p, 1); \\ (\beta_1, 1), \ldots, (\beta_q, 1);\end{matrix} z\right] = \frac{\prod_{j=1}^{p} \Gamma(\alpha_j)}{\prod_{j=1}^{q} \Gamma(\beta_j)} {}_pF_q \left[\begin{matrix}\alpha_1, \ldots, \alpha_p; \\ \beta_1, \ldots, \beta_q;\end{matrix} z\right], \tag{1.14}$$

where ${}_pF_q$ is the generalized hypergeometric function (see, e.g., [14, Section 1.5]).

Very recently, Chand et al. [3] have used the following generalized $\mathtt{k}$-Mittag-Leffler function (see also [1])

$$E^{\gamma,q}_{\mathtt{k},\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,\mathtt{k}} z^n}{\Gamma_{\mathtt{k}}(n\alpha + \beta) n!} \tag{1.15}$$

$$\left(\mathtt{k}, q \in \mathbb{R}^+, \min\{\Re(\alpha), \Re(\beta), \Re(\gamma)\} > 0\right)$$

to present some Saigo hypergeometric fractional integral formulas for a finite product of the generalized $\mathtt{k}$-Mittag-Leffler functions (1.15) and gave image formulas of several integral transforms including Beta transform, Laplace transform, and Whittaker transform. In this sequel, we aim to establish certain Saigo hypergeometric fractional integral formulas for a finite product of the generalized $\mathtt{k}$-Bessel functions (1.8), which are also used to present image formulas of several integral transforms such as Beta transform, Laplace transform, and Whittaker transform.

Here we note the following points: The generalized $\mathtt{k}$-Mittag-Leffler function (1.15) cannot be reducible to give the generalized $\mathtt{k}$-Bessel function (1.8) and *vice versa*; At best, the generalized $\mathtt{k}$-Bessel function (1.8) is seen to reduce to

$$\omega^{\gamma,\lambda}_{\mathtt{k},0,-1,-4}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n,\mathtt{k}}}{\Gamma_{\mathtt{k}}(\lambda n + \mu)} \frac{z^{2n}}{(n!)^2}.$$

## 2. Fractional integration

Here, we establish certain fractional integral formulas for the generalized $\mathtt{k}$-Bessel function. For the basic concepts and detailed results about fractional calculus, among many other things, we refer the reader, for example, to Kiryakova [7], Miller and Ross [8], and Srivastava et al. [16].

For our purpose, we recall the following pair of Saigo hypergeometric fractional integral operators. For $x \in \mathbb{R}^+$, $\sigma, \vartheta, \lambda \in \mathbb{C}$ with $\Re(\lambda) > 0$, we have

$$\begin{aligned} &\left(I^{\lambda,\sigma,\vartheta}_{0,x} f(t)\right)(x) \\ &= \frac{x^{-\lambda-\sigma}}{\Gamma(\lambda)} \int_0^x (x-t)^{\lambda-1} {}_2F_1\left(\lambda+\sigma, -\vartheta; \lambda; 1-\frac{t}{x}\right) f(t)dt \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} &\left(J^{\lambda,\sigma,\vartheta}_{x,\infty} f(t)\right)(x) \\ &= \frac{1}{\Gamma(\lambda)} \int_x^{\infty} (t-x)^{\lambda-1} t^{-\lambda-\sigma} {}_2F_1\left(\lambda+\sigma, -\vartheta; \lambda; 1-\frac{x}{t}\right) f(t)dt, \end{aligned} \tag{2.2}$$

where ${}_2F_1(\cdot)$ is the Gauss hypergeometric function (see, e.g., [14, Section 1.5]).

The operators $I^{\lambda,\sigma,\vartheta}_{0,x}(\cdot)$ and $J^{\lambda,\sigma,\vartheta}_{x,\infty}(\cdot)$ contain the familiar Riemann-Liouville fractional integral operators $R^{\lambda}_{0,x}(\cdot)$ and $W^{\lambda}_{x,\infty}(\cdot)$, as their respective special cases, by means of the following relationships:

$$\left(R^{\lambda}_{0,x} f(t)\right)(x) = \left(I^{\lambda,-\lambda,\vartheta}_{0,x} f(t)\right)(x) = \frac{1}{\Gamma(\lambda)} \int_0^x (x-t)^{\lambda-1} f(t)dt \tag{2.3}$$

and

$$\left(W^{\lambda}_{x,\infty} f(t)\right)(x) = \left(J^{\lambda,-\lambda,\vartheta}_{x,\infty} f(t)\right)(x) = \frac{1}{\Gamma(\lambda)} \int_x^{\infty} (t-x)^{\lambda-1} f(t)dt. \tag{2.4}$$

It is also noted that the operators (2.1) and (2.2) unify the Erdêlyi-Kober fractional integral operators as follows:

$$\left(E^{\lambda,\vartheta}_{0,x} f(t)\right)(x) = \left(I^{\lambda,0,\vartheta}_{0,x} f(t)\right)(x) = \frac{x^{-\lambda-\vartheta}}{\Gamma(\lambda)} \int_0^x (x-t)^{\lambda-1} t^{\eta} f(t)dt \tag{2.5}$$

and

$$\left(K^{\lambda,\vartheta}_{x,\infty} f(t)\right)(x) = \left(J^{\lambda,0,\vartheta}_{x,\infty} f(t)\right)(x) = \frac{x^{\vartheta}}{\Gamma(\lambda)} \int_x^{\infty} (t-x)^{\lambda-1} t^{-\lambda-\vartheta} f(t)dt. \tag{2.6}$$

We need to recall the following formulas in Lemmas 1 and 2 (see [6]).

**Lemma 1.** *Let $\lambda, \rho, \sigma, \vartheta \in \mathbb{C}$ such that $\Re(\lambda) > 0$ and $\Re(\rho) > \max[0, \Re(\sigma - \vartheta)]$. Then*

$$\left(I_{0,x}^{\lambda,\sigma,\vartheta} t^{\rho-1}\right)(x) = \frac{\Gamma(\rho)\Gamma(\rho+\vartheta-\sigma)}{\Gamma(\rho-\sigma)\Gamma(\rho+\lambda+\vartheta)} x^{\rho-\sigma-1}. \tag{2.7}$$

**Lemma 2.** *Let $\lambda, \rho, \sigma, \vartheta \in \mathbb{C}$ such that $\Re(\lambda) > 0$ and $\Re(\rho) < 1 + \min[\Re(\sigma), \Re(\vartheta)]$. Then*

$$\left(J_{x,\infty}^{\lambda,\sigma,\vartheta} t^{\rho-1}\right)(x) = \frac{\Gamma(\sigma-\rho+1)\Gamma(\vartheta-\rho+1)}{\Gamma(1-\rho)\Gamma(\lambda+\sigma+\vartheta-\rho+1)} x^{\rho-\sigma-1}. \tag{2.8}$$

Now we are ready to present our main results asserted by Theorems 3 and 4.

**Theorem 3.** *Let $x, \mathtt{k}_i, q_i \in \mathbb{R}^+$ $(i = 1, \ldots, r)$. Also let $\lambda, \sigma, \vartheta, \rho, \gamma_i, \mu_i, b_i, c_i \in \mathbb{C}$ such that $\Re(\lambda) > 0$, $\Re(\rho) > \max\{0, \Re(\sigma - \vartheta)\}$, and $\Re(\mu_i) > 0$ $(i = 1, \ldots, r)$. Then*

$$\begin{aligned} &\left(I_{0,x}^{\lambda,\sigma,\vartheta} t^{\rho-1} \prod_{i=1}^{r} \omega_{\mathtt{k}_i,\mu_i,b_i,c_i}^{\gamma_i,q_i}(t)\right)(x) \\ &= x^{\rho-\sigma-1} \prod_{i=1}^{r} \frac{\mathtt{k}_i^{1-\frac{\mu_i}{\mathtt{k}_i}-\frac{b_i+1}{2\mathtt{k}_i}}}{\Gamma\left(\frac{\gamma_i}{\mathtt{k}_i}\right)} \left(\frac{x}{2}\right)^{\mu_i} \\ &\times {}_{r+2}\Psi_{r+3}\left[\begin{matrix} (\gamma_1/\mathtt{k}_1, 1), \ldots, (\gamma_r/\mathtt{k}_r, 1), \\ (\mu_1/\mathtt{k}_1 + (b_1+1)/\mathtt{k}_1, q_1/\mathtt{k}_1), \ldots, (\mu_r/\mathtt{k}_r + (b_r+1)/\mathtt{k}_r, q_r/\mathtt{k}_r), \end{matrix}\right. \\ &\qquad \begin{matrix} (\mathbf{u}+\rho, 2r), (\mathbf{u}+\rho+\vartheta-\sigma, 2r); \\ (\mathbf{u}+\rho-\sigma, 2r), (\mathbf{u}+\rho+\lambda+\vartheta, 2r), (1,1); \end{matrix} \\ &\qquad \left. \tfrac{1}{4}(-c_1)\mathtt{k}_1^{(1-q_1/\mathtt{k}_1)} \cdots (-c_r)\mathtt{k}_r^{(1-q_r/\mathtt{k}_r)} x^{2r} \right], \end{aligned} \tag{2.9}$$

*where*

$$\mathbf{u} := \sum_{i=1}^{r} \mu_i \quad (r \in \mathbb{N}). \tag{2.10}$$

*Proof.* Let $\mathscr{I}$ be the left-hand side of (2.9). Using (1.9) and changing the order of integration and summation, which is verified under the conditions given in the theorem, we have

$$\begin{aligned} \mathscr{I} = \prod_{i=1}^{r} &\left\{ \sum_{n=0}^{\infty} \frac{(-1)^n c_i^n (\gamma_i)_{n,\mathtt{k}_i}}{\Gamma_{\mathtt{k}_i}(q_i n + \mu_i + \frac{b_i+1}{2})} \frac{\left(\frac{1}{2}\right)^{\mu_i+2n}}{(n!)^2} \right\} \\ &\times \left(I_{0,x}^{\lambda,\sigma,\vartheta} t^{\mu_1+\cdots+\mu_r+2nr+\rho-1}\right)(x), \end{aligned} \tag{2.11}$$

which, upon applying (2.7), yields

$$\mathscr{I} = \prod_{i=1}^{r} \left\{ \sum_{n=0}^{\infty} \frac{(-1)^n c_i^n (\gamma_i)_{n,\mathtt{k}_i}}{\Gamma_{\mathtt{k}_i}(q_i n + \mu_i + \frac{b_i+1}{2})} \frac{\left(\frac{1}{2}\right)^{\mu_i+2n}}{(n!)^2} \right\}$$

$$\times \frac{\Gamma(\mu_1+\cdots+\mu_r+2nr+\rho)\Gamma(\mu_1+\cdots+\mu_r+2nr+\rho+\vartheta-\sigma)}{\Gamma(\mu_1+\cdots+\mu_r+2nr+\rho-\sigma)\Gamma(\mu_1+\cdots+\mu_r+2nr+\rho+\lambda+\vartheta)} \times x^{\mu_1+\cdots+\mu_r+2nr+\rho-\sigma-1}. \tag{2.12}$$

We simplify (2.12) to give

$$\begin{aligned}\mathscr{I} = x^{\rho-\sigma-1}\prod_{i=1}^{r}\frac{\mathtt{k}_i^{1-\frac{\mu_i}{\mathtt{k}_i}-\frac{b_i+1}{2\mathtt{k}_i}}}{\Gamma\left(\frac{\gamma_i}{\mathtt{k}_i}\right)}\left(\frac{x}{2}\right)^{\mu_i}&\left\{\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{\gamma_i}{\mathtt{k}_i}+n\right)}{\Gamma\left(\frac{\mu_i}{\mathtt{k}_i}+\frac{b_i+1}{\mathtt{k}_i}+\frac{q_i n}{\mathtt{k}_i}\right)}\right.\\ &\times\frac{\Gamma(\mu_1+\cdots+\mu_r+\rho+2nr)\Gamma(\mu_1+\cdots+\mu_r+\rho+\vartheta-\sigma+2nr)}{\Gamma(\mu_1+\cdots+\mu_r+\rho-\sigma+2nr)\Gamma(\mu_1+\cdots+\mu_r+\rho+\lambda+\vartheta+2nr)\Gamma(n+1)}\\ &\left.\times\frac{1}{n!}\left(\frac{x^{2r}(-c_i)\mathtt{k}_i^{(1-q_i/\mathtt{k}_i)}}{4}\right)^{n}\right\},\end{aligned} \tag{2.13}$$

which, with the help of (1.12), is expressed in terms of ${}_{r+2}\Psi_{r+3}$ function to yield the right-hand side of (2.9). □

**Theorem 4.** *Let $x, \mathtt{k}_i, q_i \in \mathbb{R}^+$ $(i=1,\ldots,r)$. Also let $\lambda,\sigma,\vartheta,\rho,\gamma_i,\mu_i,b_i,c_i \in \mathbb{C}$ such that $\Re(\lambda)>0$, $\Re(\rho)<1+\min\{\Re(\sigma),\Re(\vartheta)\}$, and $\Re(\mu_i)>0$ $(i=1,\ldots,r)$. Then*

$$\begin{aligned}&\left(J_{x,\infty}^{\lambda,\sigma,\vartheta}t^{\rho-1}\prod_{i=1}^{r}\omega_{\mathtt{k}_i,\mu_i,b_i,c_i}^{\gamma_i,q_i}(1/t)\right)(x)\\ &= x^{\rho-\sigma-1}\prod_{i=1}^{r}\frac{\mathtt{k}_i^{1-\frac{\mu_i}{\mathtt{k}_i}-\frac{b_i+1}{2\mathtt{k}_i}}}{\Gamma\left(\frac{\gamma_i}{\mathtt{k}_i}\right)}\left(\frac{1}{2x}\right)^{\mu_i}\\ &\quad\times {}_{r+2}\Psi_{r+3}\left[\begin{matrix}(\gamma_1/\mathtt{k}_1,1),\ldots,(\gamma_r/\mathtt{k}_r,1),\\ (\mu_1/\mathtt{k}_1+(b_1+1)/\mathtt{k}_1,q_1/\mathtt{k}_1),\ldots,(\mu_r/\mathtt{k}_r+(b_r+1)/\mathtt{k}_r,q_r/\mathtt{k}_r),\end{matrix}\right.\\ &\qquad\begin{matrix}(1+\sigma+\mathbf{u}-\rho,2r),(1+\vartheta+\mathbf{u}-\rho,2r);\\ (1+\mathbf{u}-\rho,2r),(1+\lambda+\sigma+\vartheta+\mathbf{u}-\rho,2r),(1,1);\end{matrix}\\ &\qquad\left.\frac{(-c_1)\mathtt{k}_1^{(1-q_1/\mathtt{k}_1)}\cdots(-c_r)\mathtt{k}_r^{(1-q_r/\mathtt{k}_r)}}{4x^{2r}}\right],\end{aligned} \tag{2.14}$$

*where $\mathbf{u}$ is given as in (2.10).*

*Proof.* The proof is parallel to that of Theorem 3. We omit the details. □

The results given in (2.9) and (2.14), being very general, can yield a large number of special cases by assigning some suitable values to the involved parameters. We demonstrate only two formulas as in Corollaries 1 and 2.

Setting $r=1$ and $\mathtt{k}_i=\mathtt{k}$, $\mu_i=\mu$, $\gamma_i=\gamma$, $q_i=q$, $b_i=b$, and $c_i=c$ in (2.9) and (2.14), we obtain the following two formulas in Corollaries 1 and 2.

**Corollary 1.** *Let $x,\mathtt{k},q\in\mathbb{R}^+$. Also let $\lambda,\sigma,\vartheta,\rho,\gamma,\mu,b,c\in\mathbb{C}$ such that $\min\{\Re(\lambda),\Re(\mu)\}>0$ and $\Re(\rho)>\max\{0,\Re(\sigma-\vartheta)\}$. Then*

$$\left(I_{0,x}^{\lambda,\sigma,\vartheta}t^{\rho-1}\omega_{\mathtt{k},\mu,b,c}^{\gamma,q}(t)\right)(x) \tag{2.15}$$

$$= x^{\rho-\sigma-1}\frac{\mathtt{k}^{1-\frac{\mu}{\mathtt{k}}-\frac{b+1}{2\mathtt{k}}}}{\Gamma\left(\frac{\gamma}{\mathtt{k}}\right)}\left(\frac{x}{2}\right)^{\mu}$$

$$\times {}_3\Psi_4\left[\begin{matrix}(\gamma/\mathtt{k},1),(\mu+\rho,2),(\mu+\rho+\vartheta-\sigma,2);\\ (\mu/\mathtt{k}+(b+1)/\mathtt{k},q/\mathtt{k}),(\mu+\rho-\sigma,2),(\mu+\rho+\lambda+\vartheta,2),(1,1);\end{matrix}\ \frac{(-c)\mathtt{k}^{(1-q/\mathtt{k})}x^2}{4}\right].$$

**Corollary 2.** *Let $x, q, \mathtt{k} \in \mathbb{R}^+$. Also let $\lambda, \sigma, \vartheta, \rho, \gamma, \mu, b, c \in \mathbb{C}$ such that $\min\{\Re(\lambda), \Re(\mu)\} > 0$ and $\Re(\rho) < 1 + \min\{\Re(\sigma), \Re(\vartheta)\}$. Then*

$$\left(J^{\lambda,\sigma,\vartheta}_{x,\infty}t^{\rho-1}\omega^{\gamma,q}_{\mathtt{k},\mu,b,c}(1/t)\right)(x) \tag{2.16}$$

$$= x^{\rho-\sigma-1}\frac{\mathtt{k}^{1-\frac{\mu}{\mathtt{k}}-\frac{b+1}{2\mathtt{k}}}}{\Gamma\left(\frac{\gamma}{\mathtt{k}}\right)}\left(\frac{1}{2x}\right)^{\mu}$$

$$\times {}_3\Psi_4\left[\begin{matrix}(\gamma/\mathtt{k},1),(1+\sigma+\mu-\rho,2),(1+\vartheta+\mu-\rho,2);\\ (\mu/\mathtt{k}+(b+1)/\mathtt{k},q/\mathtt{k}),(1+\mu-\rho,2),(1+\lambda+\sigma+\vartheta+\mu-\rho,2),(1,1);\end{matrix}\ \frac{(-c)\mathtt{k}^{(1-q/\mathtt{k})}}{4x^2}\right].$$

## 3. Image formulas associated with integral transforms

Here, by using the results in previous section, we establish certain interesting image formulas associated with such integral transforms as Beta transform, Laplace transform, and Whittaker transform.

We recall the Beta transform of $f(z)$ defined by (see [12])

$$B\{f(z):a,b\} = \int_0^1 z^{a-1}(1-z)^{b-1}f(z)dz. \tag{3.1}$$

**Theorem 5.** *Let $x, \mathtt{k}_i, q_i \in \mathbb{R}^+$ $(i = 1, \ldots, r)$. Also let $\lambda, \sigma, \vartheta, \rho, \mu_i, \gamma_i, b_i, c_i \in \mathbb{C}$ such that $\Re(\lambda) > 0$, $\Re(\rho) > \max\{0, \Re(\sigma - \vartheta)\}$ and $\Re(\mu_i) > 0$ $(i = 1, \ldots, r)$. Further let $l, m \in \mathbb{C}$ with $\Re(m) > 0$ and $\Re\Big(l + \sum_{i=1}^{r}\mu_i\Big) > 0$. Then*

$$B\left\{\left(I^{\lambda,\sigma,\vartheta}_{0,x}t^{\rho-1}\prod_{i=1}^{r}\omega^{\gamma_i,q_i}_{\mathtt{k}_i,\mu_i,b_i,c_i}(t)\right)(x):l,m\right\} \tag{3.2}$$

$$= \Gamma(m)x^{\rho-\sigma-1}\prod_{i=1}^{r}\frac{\mathtt{k}_i^{1-\frac{\mu_i}{\mathtt{k}_i}-\frac{b_i+1}{2\mathtt{k}_i}}}{\Gamma\left(\frac{\gamma_i}{\mathtt{k}_i}\right)}\left(\frac{x}{2}\right)^{\mu_i}$$

$$\times {}_{r+3}\Psi_{r+4}\left[\begin{matrix}(\gamma_1/\mathtt{k}_1,1),\ldots,(\gamma_r/\mathtt{k}_r,1),\\ (\mu_1/\mathtt{k}_1+(b_1+1)/\mathtt{k}_1,q_1/\mathtt{k}_1),\ldots,(\mu_r/\mathtt{k}_r+(b_r+1)/\mathtt{k}_r,q_r/\mathtt{k}_r),\end{matrix}\right.$$

$$\begin{matrix}(\mathbf{u}+\rho,2r),(\mathbf{u}+\rho+\vartheta-\sigma,2r),(l+\mathbf{u},2r);\\ (\mathbf{u}+\rho-\sigma,2r),(\mathbf{u}+\rho+\lambda+\vartheta,2r),(l+m+\mathbf{u},2r),(1,1);\end{matrix}$$

$$\left.\frac{(-c_1)\mathtt{k}_1^{(1-q_1/\mathtt{k}_1)}\cdots(-c_r)\mathtt{k}_r^{(1-q_r/\mathtt{k}_r)}x^{2r}}{4}\right],$$

*where* $\mathbf{u}$ *is given as in* (2.10).

*Proof.* Let $\mathscr{B}$ be the left-hand side of (3.2). Using the definition of Beta transform, we have

$$\mathscr{B}=\int_0^1 z^{l-1}(1-z)^{m-1}\left(I_{0,x}^{\lambda,\sigma,\vartheta}t^{\rho-1}\prod_{i=1}^{r}\omega_{\mathtt{k}_i,\mu_i,b_i,c_i}^{\gamma_i,q_i}(tz)\right)(x)dz, \tag{3.3}$$

which, using (1.9) and changing the order of integration and summation, which is valid under the conditions of Theorem 3, yields

$$\begin{aligned}\mathscr{B}=&\prod_{i=1}^{r}\left\{\sum_{n=0}^{\infty}\frac{(-1)^n c_i^n(\gamma_i)_{n,\mathtt{k}_i}}{\Gamma_{\mathtt{k}_i}(q_i n+\mu_i+\frac{b_i+1}{2})}\frac{\left(\frac{1}{2}\right)^{\mu_i+2n}}{(n!)^2}\right\}\\ &\times\left(I_{0,x}^{\lambda,\sigma,\vartheta}t^{\mu_1+\cdots+\mu_r+2nr+\rho-1}\right)(x)\int_0^1 z^{l+\mu_1+\cdots+\mu_r+2nr-1}(1-z)^{m-1}dz.\end{aligned} \tag{3.4}$$

Applying (2.7) to (3.4), after a little simplification, we obtain

$$\begin{aligned}\mathscr{B}=&\,x^{\rho-\sigma-1}\prod_{i=1}^{r}\frac{\mathtt{k}_i^{1-\frac{\mu_i}{\mathtt{k}_i}-\frac{b_i+1}{2\mathtt{k}_i}}}{\Gamma\left(\frac{\gamma_i}{\mathtt{k}_i}\right)}\left(\frac{x}{2}\right)^{\mu_i}\left\{\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{\gamma_i}{\mathtt{k}_i}+n\right)}{\Gamma\left(\frac{\mu_i}{\mathtt{k}_i}+\frac{b_i+1}{\mathtt{k}_i}+\frac{q_i n}{\mathtt{k}_i}\right)}\right.\\ &\times\frac{\Gamma(\mu_1+\cdots+\mu_r+\rho+2nr)\Gamma(\mu_1+\cdots+\mu_r+\rho+\vartheta-\sigma+2nr)}{\Gamma(\mu_1+\cdots+\mu_r+\rho-\sigma+2nr)\Gamma(\mu_1+\cdots+\mu_r+\rho+\lambda+\vartheta+2nr)\Gamma(n+1)}\\ &\left.\times\frac{1}{n!}\left(\frac{x^{2r}(-c_i)\mathtt{k}_i^{(1-q_i/\mathtt{k}_i)}}{4}\right)^n\right\}\int_0^1 z^{l+\mu_1+\cdots+\mu_r+2nr-1}(1-z)^{m-1}dz.\end{aligned} \tag{3.5}$$

Applying the beta function $B(\alpha,\beta)$ defined by (see, e.g., [14, Section 1.1])

$$B(\alpha,\beta)=\begin{cases}\displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt & (\Re(\alpha)>0;\Re(\beta)>0)\\ \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} & \left(\alpha,\beta\in\mathbb{C}\setminus\mathbb{Z}_0^-\right),\end{cases} \tag{3.6}$$

to (3.5), we get

$$\begin{aligned}\mathscr{B}=&\,x^{\rho-\sigma-1}\prod_{i=1}^{r}\frac{\mathtt{k}_i^{1-\frac{\mu_i}{\mathtt{k}_i}-\frac{b_i+1}{2\mathtt{k}_i}}}{\Gamma\left(\frac{\gamma_i}{\mathtt{k}_i}\right)}\left(\frac{x}{2}\right)^{\mu_i}\left\{\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{\gamma_i}{\mathtt{k}_i}+n\right)}{\Gamma\left(\frac{\mu_i}{\mathtt{k}_i}+\frac{b_i+1}{\mathtt{k}_i}+\frac{q_i n}{\mathtt{k}_i}\right)}\right.\\ &\times\frac{\Gamma(\mu_1+\cdots+\mu_r+\rho+2nr)\Gamma(\mu_1+\cdots+\mu_r+\rho+\vartheta-\sigma+2nr)}{\Gamma(\mu_1+\cdots+\mu_r+\rho-\sigma+2nr)\Gamma(\mu_1+\cdots+\mu_r+\rho+\lambda+\vartheta+2nr)\Gamma(n+1)}\\ &\left.\times\frac{1}{n!}\left(\frac{x^{2r}(-c_i)\mathtt{k}_i^{(1-q_i/\mathtt{k}_i)}}{4}\right)^n\right\}\frac{\Gamma(l+\mu_1+\cdots+\mu_r+2nr)\Gamma(m)}{\Gamma(l+m+\mu_1+\cdots+\mu_r+2nr)},\end{aligned} \tag{3.7}$$

which, upon expressing in terms of the Fox-Wright function (1.12), yields the right-hand side of (3.2). □

**Theorem 6.** *Let $x, \mathtt{k}_i, q_i \in \mathbb{R}^+$ $(i = 1, \ldots, r)$. Also let $\lambda, \sigma, \vartheta, \rho, \mu_i, \gamma_i, b_i, c_i \in \mathbb{C}$ such that $\Re(\lambda) > 0$, $\Re(\rho) < 1 + \min\{\Re(\sigma), \Re(\vartheta)\}$ and $\Re(\mu_i) > 0$ $(i = 1, \ldots, r)$. Further let $l, m \in \mathbb{C}$ with $\Re(m) > 0$ and $\Re\Big(l + \sum_{i=1}^{r} \mu_i\Big) > 0$. Then*

$$
\begin{aligned}
(3.8)\quad & B\left\{\left(J_{x,\infty}^{\lambda,\sigma,\vartheta} t^{\rho-1} \prod_{i=1}^{r} \omega_{\mathtt{k}_i,\mu_i,b_i,c_i}^{\gamma_i,q_i}(z/t)\right)(x) : l, m\right\} \\
&= \Gamma(m) x^{\rho-\sigma-1} \prod_{i=1}^{r} \frac{\mathtt{k}_i^{1-\frac{\mu_i}{\mathtt{k}_i}-\frac{b_i+1}{2\mathtt{k}_i}}}{\Gamma\left(\frac{\gamma_i}{\mathtt{k}_i}\right)} \left(\frac{1}{2x}\right)^{\mu_i} \\
&\times {}_{r+3}\Psi_{r+4}\Bigg[\begin{matrix} (\gamma_1/\mathtt{k}_1, 1), \ldots, (\gamma_r/\mathtt{k}_r, 1), \\ (\mu_1/\mathtt{k}_1 + (b_1+1)/\mathtt{k}_1, q_1/\mathtt{k}_1), \ldots, (\mu_r/\mathtt{k}_r + (b_r+1)/\mathtt{k}_r, q_r/\mathtt{k}_r), \end{matrix} \\
&\qquad \begin{matrix} (1+\sigma+\mathbf{u}-\rho, 2r), (1+\vartheta+\mathbf{u}-\rho, 2r), (l+\mathbf{u}, 2r); \\ (1+\mathbf{u}-\rho, 2r), (1+\lambda+\sigma+\vartheta+\mathbf{u}-\rho, 2r), (l+m+\mathbf{u}, 2r), (1,1); \end{matrix} \\
&\qquad \frac{(-c_1)\mathtt{k}_1^{(1-q_1/\mathtt{k}_1)} \cdots (-c_r)\mathtt{k}_r^{(1-q_r/\mathtt{k}_r)}}{4x^{2r}}\Bigg],
\end{aligned}
$$

*where $\mathbf{u}$ is given as in* (2.10).

*Proof.* A similar argument as in the proof of Theorem 5 will establish this theorem. We omit the details. □

We recall the Laplace transform of $f(z)$ (see, e.g., [12, 13]).

$$
(3.9)\qquad L\{f(z)\} = \int_0^\infty e^{-sz} f(z)\,dz.
$$

**Theorem 7.** *Let $x, \mathtt{k}_i, q_i \in \mathbb{R}^+$ $(i = 1, \ldots, r)$. Also let $\lambda, \sigma, \vartheta, \rho, \mu_i, \gamma_i, b_i, c_i \in \mathbb{C}$ such that $\Re(\lambda) > 0$, $\Re(\rho) > \max\{0, \Re(\sigma - \vartheta)\}$, and $\Re(\mu_i) > 0$ $(i = 1, \ldots, r)$. Further let $s, l \in \mathbb{C}$ with $\Re(s) > 0$ and $\Re\Big(l + \sum_{i=1}^{r} \mu_i\Big) > 0$. Then*

$$
\begin{aligned}
(3.10)\quad & L\left\{z^{l-1}\left(I_{0,x}^{\lambda,\sigma,\vartheta} t^{\rho-1} \prod_{i=1}^{r} \omega_{\mathtt{k}_i,\mu_i,b_i,c_i}^{\gamma_i,q_i}(tz)\right)(x)\right\} \\
&= \frac{x^{\rho-\sigma-1}}{s^l} \prod_{i=1}^{r} \frac{\mathtt{k}_i^{1-\frac{\mu_i}{\mathtt{k}_i}-\frac{b_i+1}{2\mathtt{k}_i}}}{\Gamma\left(\frac{\gamma_i}{\mathtt{k}_i}\right)} \left(\frac{x}{2s}\right)^{\mu_i} \\
&\times {}_{r+3}\Psi_{r+3}\Bigg[\begin{matrix} (\gamma_1/\mathtt{k}_1, 1), \ldots, (\gamma_r/\mathtt{k}_r, 1), \\ (\mu_1/\mathtt{k}_1 + (b_1+1)/\mathtt{k}_1, q_1/\mathtt{k}_1), \ldots, (\mu_r/\mathtt{k}_r + (b_r+1)/\mathtt{k}_r, q_r/\mathtt{k}_r), \end{matrix} \\
&\qquad \begin{matrix} (\mathbf{u}+\rho, 2r), (\mathbf{u}+\rho+\vartheta-\sigma, 2r), (l+\mathbf{u}, 2r); \\ (\mathbf{u}+\rho-\sigma, 2r), (\mathbf{u}+\rho+\lambda+\vartheta, 2r), (1,1); \end{matrix} \\
&\qquad \frac{(-c_1)\mathtt{k}_1^{(1-q_1/\mathtt{k}_1)} \cdots (-c_r)\mathtt{k}_r^{(1-q_r/\mathtt{k}_r)}}{4} \left(\frac{x}{s}\right)^{2r}\Bigg],
\end{aligned}
$$

*where $\mathbf{u}$ is given as in* (2.10).

*Proof.* Let $\mathscr{L}$ be the left-hand side of (3.10). Applying the Laplace transform (3.9), using (1.9), and changing the order of integration and summation, which is valid under the conditions of Theorem 7, we have

$$\begin{aligned}
\mathscr{L} &= x^{\rho-\sigma-1}\prod_{i=1}^{r}\frac{\mathtt{k}_i^{1-\frac{\mu_i}{\mathtt{k}_i}-\frac{b_i+1}{2\mathtt{k}_i}}}{\Gamma\left(\frac{\gamma_i}{\mathtt{k}_i}\right)}\left(\frac{x}{2}\right)^{\mu_i}\left\{\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{\gamma_i}{\mathtt{k}_i}+n\right)}{\Gamma\left(\frac{\mu_i}{\mathtt{k}_i}+\frac{b_i+1}{\mathtt{k}_i}+\frac{q_i n}{\mathtt{k}_i}\right)}\right.\\
&\quad\times\frac{\Gamma(\mu_1+\cdots+\mu_r+\rho+2nr)\Gamma(\mu_1+\cdots+\mu_r+\rho+\vartheta-\sigma+2nr)}{\Gamma(\mu_1+\cdots+\mu_r+\rho-\sigma+2nr)\Gamma(\mu_1+\cdots+\mu_r+\rho+\lambda+\vartheta+2nr)\Gamma(n+1)}\\
&\quad\left.\times\frac{1}{n!}\left(\frac{x^{2r}(-c_i)\mathtt{k}_i^{(1-q_i/\mathtt{k}_i)}}{4}\right)^n\right\}\int_0^{\infty}e^{-sz}z^{l+\mu_1+\cdots+\mu_r+2nr-1}dz\\
&= x^{\rho-\sigma-1}\prod_{i=1}^{r}\frac{\mathtt{k}_i^{1-\frac{\mu_i}{\mathtt{k}_i}-\frac{b_i+1}{2\mathtt{k}_i}}}{\Gamma\left(\frac{\gamma_i}{\mathtt{k}_i}\right)}\left(\frac{x}{2}\right)^{\mu_i}\left\{\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{\gamma_i}{\mathtt{k}_i}+n\right)}{\Gamma\left(\frac{\mu_i}{\mathtt{k}_i}+\frac{b_i+1}{\mathtt{k}_i}+\frac{q_i n}{\mathtt{k}_i}\right)}\right.\\
&\quad\times\frac{\Gamma(\mu_1+\cdots+\mu_r+\rho+2nr)\Gamma(\mu_1+\cdots+\mu_r+\rho+\vartheta-\sigma+2nr)}{\Gamma(\mu_1+\cdots+\mu_r+\rho-\sigma+2nr)\Gamma(\mu_1+\cdots+\mu_r+\rho+\lambda+\vartheta+2nr)\Gamma(n+1)}\\
&\quad\left.\times\frac{1}{n!}\left(\frac{x^{2r}(-c_i)\mathtt{k}_i^{(1-q_i/\mathtt{k}_i)}}{4}\right)^n\right\}\frac{\Gamma(l+\mu_1+\cdots+\mu_r+2nr)}{s^{l+\mu_1+\cdots+\mu_r+2nr}},
\end{aligned}\tag{3.11}$$

which, upon using (1.12), leads to the right-hand side of (3.10). □

**Theorem 8.** *Let* $x, \mathtt{k}_i, q_i \in \mathbb{R}^+$ $(i = 1, \ldots, r)$. *Also let* $\lambda, \sigma, \vartheta, \rho, \mu_i, \gamma_i, b_i, c_i \in \mathbb{C}$ *such that* $\Re(\lambda) > 0$, $\Re(\rho) < 1 + \min[\Re(\sigma), \Re(\vartheta)]$, *and* $\Re(\mu_i) > 0$ $(i = 1, \ldots, r)$. *Further let* $s, l \in \mathbb{C}$ *with* $\Re(s) > 0$ *and* $\Re\Big(l + \sum_{i=1}^{r}\mu_i\Big) > 0$. *Then*

$$\begin{aligned}
&L\left\{z^{l-1}\left(J_{x,\infty}^{\lambda,\sigma,\vartheta}t^{\rho-1}\prod_{i=1}^{r}\omega_{\mathtt{k}_i,\mu_i,b_i,c_i}^{\gamma_i,q_i}(z/t)\right)(x)\right\}\\
&= \frac{x^{\rho-\sigma-1}}{s^l}\prod_{i=1}^{r}\frac{\mathtt{k}_i^{1-\frac{\mu_i}{\mathtt{k}_i}-\frac{b_i+1}{2\mathtt{k}_i}}}{\Gamma\left(\frac{\gamma_i}{\mathtt{k}_i}\right)}\left(\frac{1}{2xs}\right)^{\mu_i}\\
&\quad\times {}_{r+3}\Psi_{r+3}\left[\begin{matrix}(\gamma_1/\mathtt{k}_1, 1), \ldots, (\gamma_r/\mathtt{k}_r, 1),\\ (\mu_1/\mathtt{k}_1 + (b_1+1)/\mathtt{k}_1, q_1/\mathtt{k}_1), \ldots, (\mu_r/\mathtt{k}_r + (b_r+1)/\mathtt{k}_r, q_r/\mathtt{k}_r),\end{matrix}\right.\\
&\qquad\begin{matrix}(1+\sigma+\mathbf{u}-\rho, 2r), (1+\vartheta+\mathbf{u}-\rho, 2r), (l+\mathbf{u}, 2r);\\ (1+\mathbf{u}-\rho, 2r), (1+\lambda+\sigma+\vartheta+\mathbf{u}-\rho, 2r), (1,1);\end{matrix}\\
&\qquad\left.\frac{(-c_1)\mathtt{k}_1^{(1-q_1/\mathtt{k}_1)}\cdots(-c_r)\mathtt{k}_r^{(1-q_r/\mathtt{k}_r)}}{4(xs)^{2r}}\right],
\end{aligned}\tag{3.12}$$

*where* $\mathbf{u}$ *is given as in* (2.10).

*Proof.* The proof would run parallel to that of Theorem 7. We omit the details. □

We recall the following formula (see, e.g., [5, p. 816, Entry 11])

$$\int_0^\infty z^{\nu-1}e^{-\frac{1}{2}z}\mathcal{W}_{\eta,\theta}(z)dz = \frac{\Gamma\left(\frac{1}{2}+\theta+\nu\right)\Gamma\left(\frac{1}{2}-\theta+\nu\right)}{\Gamma(1-\eta+\nu)} \quad \left(\Re(\nu\pm\theta) > -\tfrac{1}{2}\right), \tag{3.13}$$

where $\mathcal{W}_{\eta,\theta}(\cdot)$ denote the Whittaker functions (see, e.g., [5, Sections 9.22-9.23]).

**Theorem 9.** *Let $x, \mathtt{k}_i, q_i \in \mathbb{R}^+$ $(i = 1, \ldots, r)$. Also, let $\lambda, \sigma, \vartheta, \rho, \mu_i, \gamma_i, b_i, c_i \in \mathbb{C}$ such that $\Re(\lambda) > 0$, $\Re(\rho) > \max\{0, \Re(\sigma-\vartheta)\}$, and $\Re(\mu_i) > 0$ $(i = 1, \ldots, r)$. Further, let $\xi, \eta, \tau, \omega, \in \mathbb{C}$ with $\Re(\eta) > 0$ and $\Re\Big(\xi+\omega\Big) > -\frac{1}{2}$. Then*

$$\begin{aligned}
&\int_0^\infty z^{\xi-1}e^{-\eta z/2}W_{\tau,\omega}(\eta z)\left\{\left(I_{0,x}^{\lambda,\sigma,\vartheta}t^{\rho-1}\prod_{i=1}^{r}\omega_{\mathtt{k}_i,\mu_i,b_i,c_i}^{\gamma_i,q_i}(tz)\right)(x)\right\}dz \\
&= \frac{x^{\rho-\sigma-1}}{\eta^{\xi}}\prod_{i=1}^{r}\frac{\mathtt{k}_i^{1-\frac{\mu_i}{\mathtt{k}_i}-\frac{b_i+1}{2\mathtt{k}_i}}}{\Gamma\left(\frac{\gamma_i}{\mathtt{k}_i}\right)}\left(\frac{x}{2\eta}\right)^{\mu_i} \\
&\times {}_{r+4}\Psi_{r+4}\left[\begin{matrix}(\gamma_1/\mathtt{k}_1,1),\ldots,(\gamma_r/\mathtt{k}_r,1), \\ (\mu_1/\mathtt{k}_1+(b_1+1)/\mathtt{k}_1, q_1/\mathtt{k}_1),\ldots,(\mu_r/\mathtt{k}_r+(b_r+1)/\mathtt{k}_r, q_r/\mathtt{k}_r),\end{matrix}\right. \\
&\qquad \begin{matrix}(\mathbf{u}+\rho,2r),(\mathbf{u}+\rho+\vartheta-\sigma,2r),(1/2+\omega+\xi+\mathbf{u},2r),(1/2-\omega+\xi+\mathbf{u},2r); \\ (\mathbf{u}+\rho-\sigma,2r),(\mathbf{u}+\rho+\lambda+\vartheta,2r),(1-\tau+\xi+\mathbf{u},2r),(1,1);\end{matrix} \\
&\qquad \left.\frac{(-c_1)\mathtt{k}_1^{(1-q_1/\mathtt{k}_1)}\cdots(-c_r)\mathtt{k}_r^{(1-q_r/\mathtt{k}_r)}}{4}\left(\frac{x}{\eta}\right)^{2r}\right],
\end{aligned} \tag{3.14}$$

*where $\mathbf{u}$ is given as in* (2.10).

*Proof.* Let $\mathscr{W}$ be the left-hand side of the result (3.14). By using (1.9) and changing the order of integration and summation, which is valid under the conditions of Theorem 9, we have

$$\begin{aligned}
\mathscr{W} &= x^{\rho-\sigma-1}\prod_{i=1}^{r}\frac{\mathtt{k}_i^{1-\frac{\mu_i}{\mathtt{k}_i}-\frac{b_i+1}{2\mathtt{k}_i}}}{\Gamma\left(\frac{\gamma_i}{\mathtt{k}_i}\right)}\left(\frac{x}{2}\right)^{\mu_i}\left\{\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{\gamma_i}{\mathtt{k}_i}+n\right)}{\Gamma\left(\frac{\mu_i}{\mathtt{k}_i}+\frac{b_i+1}{\mathtt{k}_i}+\frac{q_i n}{\mathtt{k}_i}\right)}\right. \\
&\times \frac{\Gamma(\mu_1+\cdots+\mu_r+\rho+2nr)\Gamma(\mu_1+\cdots+\mu_r+\rho+\vartheta-\sigma+2nr)}{\Gamma(\mu_1+\cdots+\mu_r+\rho-\sigma+2nr)\Gamma(\mu_1+\cdots+\mu_r+\rho+\lambda+\vartheta+2nr)\Gamma(n+1)} \\
&\times \left.\frac{1}{n!}\left(\frac{x^{2r}(-c_i)\mathtt{k}_i^{(1-q_i/\mathtt{k}_i)}}{4}\right)^{n}\right\}\int_0^\infty z^{\xi+\mu_1+\cdots+\mu_r+2nr-1}e^{-\eta z/2}W_{\tau,\omega}(\eta z)dz,
\end{aligned}$$

which, upon replacing $\eta z$ by $\varsigma$, yields

$$\begin{aligned}
\mathscr{W} &= x^{\rho-\sigma-1}\prod_{i=1}^{r}\frac{\mathtt{k}_i^{1-\frac{\mu_i}{\mathtt{k}_i}-\frac{b_i+1}{2\mathtt{k}_i}}}{\Gamma\left(\frac{\gamma_i}{\mathtt{k}_i}\right)}\left(\frac{x}{2}\right)^{\mu_i}\left\{\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{\gamma_i}{\mathtt{k}_i}+n\right)}{\Gamma\left(\frac{\mu_i}{\mathtt{k}_i}+\frac{b_i+1}{\mathtt{k}_i}+\frac{q_i n}{\mathtt{k}_i}\right)}\frac{\Gamma(\mu_1+\cdots+\mu_r+\rho+2nr)}{\Gamma(\mu_1+\cdots+\mu_r+\rho-\sigma+2nr)}\right. \\
&\times \left.\frac{\Gamma(\mu_1+\cdots+\mu_r+\rho+\vartheta-\sigma+2nr)}{\Gamma(\mu_1+\cdots+\mu_r+\rho+\lambda+\vartheta+2nr)\Gamma(n+1)}\frac{1}{n!}\left(\frac{x^{2r}(-c_i)\mathtt{k}_i^{(1-q_i/\mathtt{k}_i)}}{4}\right)^{n}\right\}
\end{aligned} \tag{3.15}$$

$$\times \frac{1}{\eta^{\xi+\mu_1+\cdots+\mu_r+2nr}} \int_0^{\infty} \varsigma^{\xi+\mu_1+\cdots+\mu_r+2nr-1} e^{-\varsigma/2} W_{\tau,\omega}(\varsigma) d\varsigma.$$

Upon applying (3.13) to the involved integral in (3.15), we obtain

$$\begin{aligned}
\mathscr{W} = & \frac{x^{\rho-\sigma-1}}{\eta^{\xi}} \prod_{i=1}^{r} \frac{\mathtt{k}_i^{1-\frac{\mu_i}{\mathtt{k}_i}-\frac{b_i+1}{2\mathtt{k}_i}}}{\Gamma\left(\frac{\gamma_i}{\mathtt{k}_i}\right)} \left(\frac{x}{2\eta}\right)^{\mu_i} \left\{ \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{\gamma_i}{\mathtt{k}_i}+n\right)}{\Gamma\left(\frac{\mu_i}{\mathtt{k}_i}+\frac{b_i+1}{\mathtt{k}_i}+\frac{q_i n}{\mathtt{k}_i}\right)} \right. \\
& \times \frac{\Gamma(\mu_1+\cdots+\mu_r+\rho+2nr)\Gamma(\mu_1+\cdots+\mu_r+\rho+\vartheta-\sigma+2nr)}{\Gamma(\mu_1+\cdots+\mu_r+\rho-\sigma+2nr)\Gamma(\mu_1+\cdots+\mu_r+\rho+\lambda+\vartheta+2nr)\Gamma(n+1)} \\
& \times \frac{\Gamma(1/2+\omega+\xi+\mu_1+\cdots+\mu_r+2nr)\Gamma(1/2-\omega+\xi+\mu_1+\cdots+\mu_r+2nr)}{\Gamma(1-\tau+\xi+\mu_1+\cdots+\mu_r+2nr)} \\
& \left. \times \frac{1}{n!} \left( \frac{x^{2r}(-c_i)\mathtt{k}_i^{(1-q_i/\mathtt{k}_i)}}{4\eta^{2r}} \right)^n \right\},
\end{aligned}$$

which, upon expressing in terms of (1.12), leads to the right-hand side of (3.14). □

**Theorem 10.** *Let* $x, \mathtt{k}_i, q_i \in \mathbb{R}^+$ $(i = 1, \ldots, r)$*. Also, let* $\lambda, \sigma, \vartheta, \rho, \mu_i, \gamma_i, b_i, c_i \in \mathbb{C}$ *such that* $\Re(\lambda) > 0$*,* $\Re(\rho) < 1+\min[\Re(\sigma), \Re(\vartheta)]$*, and* $\Re(\mu_i) > 0$ $(i = 1, \ldots, r)$*. Further, let* $\xi, \eta, \tau, \omega, \in \mathbb{C}$ *with* $\Re(\eta) > 0$ *and* $\Re\Big(\xi+\omega\Big) > -\frac{1}{2}$*. Then*

(3.16)

$$\begin{aligned}
& \int_0^{\infty} z^{\xi-1} e^{-\eta z/2} W_{\tau,\omega}(\eta z) \left\{ \left( J_{x,\infty}^{\lambda,\sigma,\vartheta} t^{\rho-1} \prod_{i=1}^{r} \omega_{\mathtt{k}_i,\mu_i,b_i,c_i}^{\gamma_i,q_i}(z/t) \right)(x) \right\} dz \\
= & \frac{x^{\rho-\sigma-1}}{\eta^{\xi}} \prod_{i=1}^{r} \frac{\mathtt{k}_i^{1-\frac{\mu_i}{\mathtt{k}_i}-\frac{b_i+1}{2\mathtt{k}_i}}}{\Gamma\left(\frac{\gamma_i}{\mathtt{k}_i}\right)} \left(\frac{1}{2x\eta}\right)^{\mu_i} \\
& \times {}_{r+4}\Psi_{r+4} \left[ \begin{array}{l} (\gamma_1/\mathtt{k}_1, 1), \ldots, (\gamma_r/\mathtt{k}_r, 1), \\ (\mu_1/\mathtt{k}_1 + (b_1+1)/\mathtt{k}_1, q_1/\mathtt{k}_1), \ldots, (\mu_r/\mathtt{k}_r + (b_r+1)/\mathtt{k}_r, q_r/\mathtt{k}_r), \end{array} \right. \\
& \qquad \begin{array}{l} (1+\sigma+\mathbf{u}-\rho, 2r), (1+\vartheta+\mathbf{u}-\rho, 2r), (1/2+\omega+\xi+\mathbf{u}, 2r), (1/2-\omega+\xi+\mathbf{u}, 2r); \\ (1+\mathbf{u}-\rho, 2r), (1+\lambda+\sigma+\vartheta+\mathbf{u}-\rho, 2r), (1-\tau+\xi+\mathbf{u}, 2r), (1, 1); \end{array} \\
& \qquad \left. \frac{(-c_1)\mathtt{k}_1^{(1-q_1/\mathtt{k}_1)} \cdots (-c_r)\mathtt{k}_r^{(1-q_r/\mathtt{k}_r)}}{4(x\eta)^{2r}} \right],
\end{aligned}$$

*where* $\mathbf{u}$ *is given as in* (2.10)*.*

*Proof.* Here, using (2.2), a similar argument as in the proof of Theorem 9 will establish (3.16). So its detailed account is omitted. □

## References

[1] P. Agarwal, M. Chand, and G. Singh, *Certain fractional kinetic equations involving the product of generalized k-Bessel function*, Alexandria Engrg. J. **55** (2016), no. 4, 3053–3059.

[2] R. A. Cerutti, *On the k-Bessel functions*, Int. Math. Forum **7** (2012), no. 37-40, 1851–1857.
[3] M. Chand, J. C. Prajapati, and E. Bonyah, *Fractional integrals and solution of fractional kinetic equations involving k-Mittag-Leffler function*, Trans. A. Razmadze Math. Inst. **171** (2017), no. 2, 144–166.
[4] R. Díaz and E. Pariguan, *On hypergeometric functions and Pochhammer k-symbol*, Divulg. Mat. **15** (2007), no. 2, 179–192.
[5] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, 2000.
[6] A. A. Kilbas and N. Sebastian, *Generalized fractional integration of Bessel function of the first kind*, Integral Transforms Spec. Funct. **19** (2008), no. 11-12, 869–883.
[7] V. Kiryakova, *All the special functions are fractional differintegrals of elementary functions*, J. Phys. A **30** (1997), no. 14, 5085–5103.
[8] K. S. Miller and B. Ross, *An Introduction to Fractional Calculus and Fractional Differential Equations*, A Wiley-Interscience Publication, John Wiley and Sons, New York, Chichester, Brisbane, Toronto and Singapore, 1993.
[9] L. G. Romero and R. A. Cerutti, *Fractional Fourier transform and special k-functions*, Int. J. Contemp. Math. Sci. **7** (2012), no. 13-16, 693–704.
[10] L. G. Romero, R. A. Cerutti, and L. L. Luque, *A new fractional Fourier transform and convolution products*, Int. J. Pure Appl. Math. **66** (2011), no. 4, 397–408.
[11] L. G. Romero, G. A. Dorrego, and R. A. Cerutti, *The k-Bessel function of the first kind*, Int. Math. Forum **7** (2012), no. 37-40, 1859–1864.
[12] I. N. Sneddon, *The Use of Integral Transforms*, Tata McGraw-Hill, New Delhi, 1979.
[13] M. R. Spiegel, *Theory and Problems of Laplace Transforms*, Schaum Publishing Co., New York, 1965.
[14] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier, Inc., Amsterdam, 2012.
[15] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Ellis Horwood Series: Mathematics and its Applications, Ellis Horwood Ltd., Chichester, 1985.
[16] H. M. Srivastava, S.-D. Lin, and P.-Y. Wang, *Some fractional-calculus results for the $\overline{H}$-function associated with a class of Feynman integrals*, Russ. J. Math. Phys. **13** (2006), no. 1, 94–100.

Praveen Agarwal
Department of Mathematics
Anand International College of Engineering
Jaipur-303012, India
*Email address*: praveen.agarwal@anandice.ac.in; goyal.praveen2011@gmail.com

Mehar Chand
Department of Applied Sciences
Guru Kashi University
Bathinda-1513002, India
*Email address*: mehar.jallandhra@gmail.com

Junesang Choi
Department of Mathematics
Dongguk University
Gyeongju 38066, Korea
*Email address*: junesang@mail.dongguk.ac.kr

Gurmej Singh
Department of Mathematics
Mata Sahib Kaur Girls College
Bathinda-151103, India
and
Research Scholar
Department of Mathematics
Singhania University
Pacheri Bari, Jhunjhunu, India
*Email address*: gurmejsandhu11@gmail.com