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ON SUBSPACE-SUPERCYCLIC SEMIGROUP

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Abstract. A C_0-semigroup $\mathcal{T} = (T_t)_{t\geq 0}$ on a Banach space X is called subspace-supercyclic for a subspace M, if $\mathbb{C}Orb(\mathcal{T}, x) \bigcap M = \{\lambda T_t x : \lambda \in \mathbb{C}, t \geq 0\} \bigcap M$ is dense in M for a vector $x \in M$. In this paper we characterize the notion of subspace-supercyclic C_0-semigroup. At the same time, we also provide a subspace-supercyclicity criterion C_0-semigroup and offer two equivalent conditions of this criterion.

1. Introduction

Let X be a separable infinite dimensional Banach space over the scalar field $\mathbb{C}$ and let $\mathcal{B}(X)$ denote the set of all bounded linear operators on X and we will usually refer to elements of $\mathcal{B}(X)$ as just operators. A bounded linear operator $T : X \to X$ is called hypercyclic (respectively, supercyclic) if there is some vector $x \in X$ such that $Orb(T, x) = \{T^n x : n = 0, 1, 2, \ldots\}$ (respectively, the projective orbit $\mathbb{C}Orb(T, x) = \{\lambda T^n x : \lambda \in \mathbb{C}, n = 0, 1, 2, \ldots\}$) is dense in X. Such a vector x is said hypercyclic (respectively, supercyclic) for T. Refer to $[1, 2, 8, 18]$ for more informations about hypercyclicity and supercyclicity.

The definition and the properties of supercyclicity operators were introduced by Hilden and Wallen [9]. They proved that all unilateral backward weighted shifts on a Hilbert space are supercyclic. The study of supercyclic operators has experienced a great of development in recent years. Salas gave a characterization of supercyclic bilateral backward weighted shifts via the Supercyclicity Criterion in [16]. Montes and Salas [13] refined the Supercyclicity Criterion and proved that it is equivalent to the former given by Salas. Besides, T. Bermúdez, A. Bonilla and A. Peris [3] showed that the equivalence of two supercyclicity criteria given by N. Feldman et al. in [7] to the Supercyclicity Criterion.

In 2011, B. F. Madore and R. A. Martnez-Avendano in [12] introduced and studied the concept of subspace-hypercyclicity for an operator. An operator T is subspace-hypercyclic or M-hypercyclic for a subspace M of X, if there exists $x \in X$ such that $Orb(T, x) \bigcap M$ is dense in M. Such a vector x is called

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a M-hypercyclic vector for T, they showed that there are operators which are M-hypercyclic but not hypercyclic. They introduced analogously the concept of subspace-transitivity. Let $T \in \mathcal{B}(X)$ and M be a closed subspace of X, we say that T is M-transitive, if for any non-empty open sets U, V in M, there exists $n \geq 0$ such that $T^{-n}(U) \bigcap V$ contain a non-empty open subset of M.The authors showed that M-transitivity implies M-hypercyclicity. Note that the converse is not true, this is proven recently by C. M. Le in [11]; for more informations see [10, 15].

Similarly, for subspace-supercyclicity, X. F. Zhao et al. in [21] provided a subspace-supercyclicity criterion and offered two necessary and sufficient conditions for a path of bounded linear operators to have a dense G_δ set of common subspace-hypercyclic vectors and common subspace-supercyclic vectors and they also constructed examples to show that subspace-supercyclic is not a strictly infinite dimensional phenomenon and that some subspace-supercyclic operators are not supercyclic.

Recall that a one-parameter family $(T_t)_{t\geq 0}$ of operators on X is called a strongly continuous semigroup (or C_0-semigroup) of operators, if $T_0 = I$, $T_{t+s} = T_t T_s$ for all $t, s \geq 0$ and $\lim_{t\to s} T_t(x) = T_s(x)$ for all $s \geq 0$ and $x \in X$; see [6, 8, 14].

A C_0-semigroup $\mathcal{T} = (T_t)_{t\geq 0}$ of linear and continuous operators on X is said to be hypercyclic (respectively, supercyclic), if there is some vector $x \in X$ such that the set $Orb(\mathcal{T}, x) = \{T_t x : t \geq 0\}$ (respectively, the projective orbit $\mathbb{C}Orb(\mathcal{T}, x) = \{\lambda T_t x : \lambda \in \mathbb{C}, t \geq 0\}$) is dense in X. Such a vector x is said hypercyclic (respectively, supercyclic) for $\mathcal{T}$.

In [5] J. A. Conjero and A. Peris showed that different transitivity criteria for strongly continuous semigroups of operators are equivalent. They also gave new results concerning the equivalence of transitivity criteria in the case of iterations of a single operator; for more informations see [4].

Recently, in [17] A. Tajmouati et al. introduced and studied the M-hypercyclicity of C_0-semigroup $\mathcal{T} = (T_t)_{t\geq 0}$ on an infinite-dimensional separable complex Banach space X and gave sufficient conditions of being M-hypercyclic for this semigroup. Moreover, some proprieties and analogous results for the notion of M-transitive; see also [19], [20].

In this present paper, we will partially characterize the notion of subspace-supercyclic C_0-semigroup. At the same time, we also provide a subspace-supercyclicity criterion C_0-semigroup and offer two equivalent conditions of this criterion.

2. Main results

We will assume that the subspace $\mathcal{M} \subset X$ is topologically closed. We start with our main definitions.

Definition 2.1. Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup and M be a nonzero subspace of X. We say that $\mathcal{T}$ is M-supercyclic if there exists a vector $x \in X$

such that $\mathbb{C}Orb(\mathcal{T}, x) \bigcap M = \{\lambda T_t x : \lambda \in \mathbb{C}, t \geq 0\} \bigcap M$ is dense in M. We call x a M-supercyclic vector.

Remark 1. The definition above reduces to the classical definition of supercyclicity C_0-semigroup if M = X and we may assume that the subspace-supercyclic vector $x \in M$, if needed.

Example 1 ([21])**.** Let T be a supercyclic operator on X with supercyclic vector x and let I be the identity operator on X. Then the operator $T \oplus I : X \oplus X \to X \oplus X$ is subspace-supercyclic for the subspace $M := X \oplus \{0\}$ with the subspace-supercyclic vector $x \oplus \{0\}$, but $T \oplus I$ is not supercyclic on the space $X \oplus X$.

Theorem 2.1. *Let $\mathcal{T} = (T_t)_{t \geq 0}$ be a C_0-semigroup and M be a nonzero subspace of X. Then*

$$SC(\mathcal{T}, M) = \bigcap_{k \geq 0} \bigcup_{\lambda \in \mathbb{C} \backslash \{0\}, t} ((\lambda T_t)^{-1})(B_k),$$

where $(B_k)_{k \geq 0}$ is a countable open basis for the relative topology of M as a subspace of X.

Proof. Let $(B_k)_{k \geq 0}$ is a countable open basis for the relative topology of M as a subspace of X. We have $x \in SC(\mathcal{T}, M)$ if and only if $\{\lambda T_t x : \lambda \in \mathbb{C}, t \geq 0\} \bigcap M$ is dense in M if and only if for each $k \geq 0$, there are $\lambda \in \mathbb{C} \backslash \{0\}$, and $t \geq 0$ such that $\lambda T_t x \in B_k$ if and only if

$$\bigcap_{k \geq 0} \bigcup_{\lambda} \bigcup_{t \geq 0} ((\lambda T_t)^{-1})(B_k) = \bigcap_{k \geq 0} \bigcup_{\lambda, t} ((\lambda T_t)^{-1})(B_k).$$

□

Theorem 2.2. *Let $\mathcal{T} = (T_t)_{t \geq 0}$ be a C_0-semigroup and M be a nonzero subspace of X. Then the following conditions are equivalent*:

(1) *For every non-empty open U and V of M, there exists $t_0 \geq 0$ and $\lambda \in \mathbb{C} \backslash \{0\}$ such that $(\lambda T_{t_0})^{-1}(U) \bigcap V$ contains a non-empty open subset of M.*
(2) *For every non-empty open U and V of M, there exists $t_0 \geq 0$ and $\lambda \in \mathbb{C} \backslash \{0\}$ such that $(\lambda T_{t_0})^{-1}(U) \bigcap V$ is non-empty and $T_t(M) \subset M$.*
(3) *For every non-empty open U and V of M, there exists $t_0 \geq 0$ and $\lambda \in \mathbb{C} \backslash \{0\}$ such that $(\lambda T_{t_0})^{-1}(U) \bigcap V$ is non-empty open subset of M.*

Proof. (1) $\Rightarrow$ (2). Let U and V be tow nonempty open subsets of M. By (1) there exists $t_0 \geq 0$ and $\lambda \in \mathbb{C} \backslash \{0\}$ such that $(\lambda T_{t_0})^{-1}(U) \bigcap V$ contains a non-empty open W of M, it follows that $W \subset (\lambda T_{t_0})^{-1}(U) \cap V$ and $(\lambda T_{t_0})^{-1}(U) \cap V \neq \emptyset$.

Next, we prove that $T_{t_0}(M) \subset M$.
Let $x \in M$, we have $W \subset (\lambda T_{t_0})^{-1}(U) \cap V$, this implies that $T_{t_0}(W) \subset U \subset M$. Let $x_0 \in W$, since W is open of M then for all r enough small we have $x_0 + rx \in W$, therefore $\lambda T_{t_0}(x_0 + rx) = (\lambda T_{t_0} x_0 + \lambda r T_{t_0} x) \in M$. Since $\lambda T_{t_0} x_0 \in M$, it follows that $\lambda r T_{t_0} x \in M$. We then conclude that $T_{t_0}(M) \subset M$.

(2) $\Rightarrow$ (3). Let U and V be nonempty open subsets of M, by (2) there exists $t_0 \geq 0$ and $\lambda \in \mathbb{C}\backslash\{0\}$ such that $(\lambda T_{t_0})^{-1}(U) \bigcap V$ is non-empty and $T_{t_0}(M) \subset M$.

Since $(\lambda T_{t_0})|M : M \longrightarrow M$ is continuous, then $(\lambda T_{t_0})^{-1}(U)$ is open in M, therefore $(\lambda T_{t_0})^{-1}(U) \cap V$ is nonempty open of M.

At last, we see that the implication (3) $\Rightarrow$ (1) is obvious and this completes the whole proof of the theorem. □

Corollairy 2.1. *Let $\mathcal{T} = (T_t)_{t\geq 0}$ be a C_0-semigroup and M be a nonzero subspace of X. If any of the conditions in theorem 2.2 is satisfied, then $SC(\mathcal{T}, M)$ is a dense subset of M.*

Proof. We may assume the condition (3) in Theorem 2.2 is satisfied, then for each $i, j \geq 0$, there exist $t = t_{i,j} \geq 0$ and $\lambda = \lambda_{i,j} \in \mathbb{C}\backslash\{0\}$ such that the set $(\lambda_{i,j} T_{t_{i,j}})^{-1}(B_i) \bigcap B_j$ is nonempty and open. Hence the set

$$A_i = \bigcup_j (\lambda_{i,j} T_{t_{i,j}})^{-1}(B_i) \bigcap B_j$$

is nonempty and open. Furthermore, each A_i is dense in M. By the Baire category theorem, we have

$$\bigcap_i A_i = \bigcap_i \bigcup_j (\lambda_{i,j} T_{t_{i,j}})^{-1}(B_i) \bigcap B_j$$

is a dense set in M. By Theorem 2.1, we know that

$$SC(\mathcal{T}, M) = \bigcap_{i\geq 0} \bigcup_{\lambda, t} ((\lambda T_t)^{-1})(B_i)$$

and the result is obtained. □

Let M_1 and M_2 be subspaces of a Banach space X, then the direct sum of M_1 and M_2 is defined as follows:

$$M_1 \oplus M_2 = \{(x, y) : x \in M_1, y \in M_2\}.$$

Theorem 2.3. *Let $\mathcal{T} = (T_t)_{t\geq 0}$ and $\mathcal{S} = (S_t)_{t\geq 0}$ be a C_0-semigroups and M_1, M_2 be a nonzero closed subspaces of X and $(T_t \oplus S_t)_{t\geq 0}$ is $(M_1 \oplus M_2)$-supercyclic C_0-semigroup, then $\mathcal{T}$ and $\mathcal{S}$ are M_1-supercyclic and M_2-supercyclic C_0-semigroups, respectively.*

Proof. Let $x_1 \in M_1$ and $x_2 \in M_2$, and let $(x, y) \in SC(T_t \oplus S_t, M_1 \oplus M_2)$, then there exist an $\varepsilon > 0$ and an increasing sequence of positive real numbers $(t_n)_{n\in\mathbb{N}}$ and $\lambda \in \mathbb{C}$ such that

$$||\lambda(T_{t_n} \oplus S_{t_n})(x, y) - (x_1, x_2)||_{M_1\oplus M_2} \leq \varepsilon.$$

It follows

$$||\lambda T_{t_n} x - x_1||_{M_1} + ||\lambda S_{t_n} y - x_2||_{M_2} \leq \varepsilon.$$

Then

$$||\lambda T_{t_n}x - x_1||_{M_1} \leq \varepsilon \text{ and } ||\lambda S_{t_n}y - x_2||_{M_2} \leq \varepsilon.$$

Thus, there exist an increasing sequence of positive real numbers $(t_n)_{n\in\mathbb{N}}$ and $\lambda \in \mathbb{C}$ such that $\{\lambda T_{t_n} : \lambda \in \mathbb{C}, t_n \geq 0\}$ and $\{S_{t_n} : \lambda \in \mathbb{C}, t_n \geq 0\}$ are dense in M_1 and M_2, respectively. Therefore $\mathbb{C}Orb(\mathcal{T}, x)$ and $\mathbb{C}Orb(\mathcal{S}, y)$ are dense in M_1 and M_2, respectively. □

Next we get the following theorem, which is the subspace-supercyclicity criterion C_0-semigroup, which is similar to the supercyclictiy criterion that was stated in [3]; see also [8].

Theorem 2.4 (Subspace-supercyclicity criterion C_0-semigroup)**.** *Let*

$$\mathcal{T} = (T_t)_{t\geq 0}$$

be a C_0-semigroup and M be a nonzero subspace of X. Assume that there exist M_0 and M_1, dense subsets of M, an increasing sequence $(t_n)_{n\geq 0}$ in $\mathbb{R}_+$ with $t_n \to \infty$, a sequence $(\lambda_{t_n})_{n\geq 0} \subseteq \mathbb{C}\backslash\{0\}$ and a sequence of mappings $S_{t_n} : M_1 \to M$, $n \in \mathbb{N}$ such that

(1) *for each $x \in M_0$, $\lambda_{t_n} T_{t_n} x \to 0$,*
(2) *for each $y \in M_1$, $\frac{1}{\lambda_{t_n}} S_{\lambda_{t_n}} y \to 0$,*
(3) *for each $y \in M_1$, $(T_{t_n} \circ S_{t_n})y \to y$,*
(4) *M is an invariant subspace for T_{t_n} for all $n \geq 0$.*

Then $\mathcal{T} = (T_t)_{t\geq 0}$ is subspace-supercyclic C_0-semigroup for M

Proof. Let U and V be non-empty open subsets of M. By Theorem 2.2, it is enough to prove that there exist $t_0 \geq 0$ and $\lambda \in \mathbb{C}\backslash\{0\}$ such that

$$(\lambda T_{t_0})^{-1}(U) \bigcap V \text{ is non-empty and } T_{t_0}(M) \subset M.$$

Since M_0 and M_1 are dense in M, there exist $x \in M_0 \cap V$, $y \in M_1 \cap U$. And since U and V are nonempty open subsets, there exists $\varepsilon > 0$ such that $B_M(x, \varepsilon) \subseteq V$ and $B_M(y, \varepsilon) \subseteq U$. By assumption, there exist $(t_n)_{n\geq 0}$ in $\mathbb{R}_+$ and $(\lambda_{t_n})_{n\geq 0} \subseteq \mathbb{C}\backslash\{0\}$ such that

$$||\lambda_{t_n} T_{t_n} x|| \leq \tfrac{\varepsilon}{2}, ||\lambda_{t_n}^{-1} S_{t_n} y|| \leq \tfrac{\varepsilon}{2} \text{ and } ||T_{t_n} S_{t_n} y - y|| \leq \tfrac{\varepsilon}{2}.$$

Define $u = x + \lambda_{t_n}^{-1} S_{t_n} y$. We know that $u \in M$ and $u \in V$, since $||u - x|| = ||\lambda_{t_n}^{-1} S_{t_n} y|| \leq \frac{\varepsilon}{2}$. Observe that $\lambda_{t_n} T_{t_n} u = \lambda_{t_n} T_{t_n} x + T_{t_n} S_{t_n} y, so \lambda_{t_n} T_{t_n} u \in M$. Since

$$||\lambda_{t_n} T_{t_n} u - y|| = ||\lambda_{t_n} T_{t_n} x|| + ||T_{t_n} S_{t_n} y - y|| < \varepsilon,$$

we have that $\lambda_{t_n} T_{t_n} u \in U$. Then $(\lambda_{t_n} T_{t_n})^{-1}(U) \bigcap V \neq \emptyset$ and $\mathcal{T}$ is a subspace-supercyclic C_0-semigroup for M. □

T. Bermúdez, A. Bonilla and A. Peris in [3] showed that the equivalence of two supercyclicity criteria given by N. Feldman, V. Miller and L. Miller in [7] to the supercyclicity criterion; see also [8]. Now, we will show that they hold for subspace M.

Theorem 2.5. *Let $\mathcal{T}=(T_t)_{t\geq 0}$ be a C_0-semigroup and M be a nonzero subspace of X. Then the following $(1),(2)$ and (3) are equivalent:*

(1) *$\mathcal{T}$ satisfies subspace-supercyclicity criterion C_0-semigroup.*

(2) *(Outer subspace-supercyclicity criterion C_0-semigroup) There exist an increasing sequence $(t_n)_{n\geq 0}$ in $\mathbb{R}_+$ with $t_n\to\infty$, a sequence $(\lambda_{t_n})_{n\geq 0}\subseteq \mathbb{C}\backslash\{0\}$, a dense linear subspace $Y_0\subseteq M$ and, for each $y\in Y_0$, a dense linear subspace $X_y\in M$ such that*
   (a) *there exists a sequence of mappings $S_{t_n}: Y_0\to M$, $n\in\mathbb{N}$ such that $(T_{t_n}\circ S_{t_n})y \ \to y$ for each $y\in Y_0$ and*
   (b) *$||T_{t_n}x||\ ||S_{t_n}y|| \ \to 0$ for each $y\in Y_0$ and $x\in X_y$,*
   (c) *M is an invariant subspace for T_{t_n} for all $n\geq 0$.*

(3) *(Inner subspace-supercyclicity criterion C_0-semigroup) There exist an increasing sequence $(t_n)_{n\geq 0}$ in $\mathbb{R}_+$ with $t_n\to\infty$, a sequence $(\lambda_{t_n})_{n\geq 0}\subseteq \mathbb{C}\backslash\{0\}$, a dense linear subspace $Y_0\subseteq M$ and, for each $y\in Y_0$, a dense linear subspace $X_y\in M$ such that*
   (a) *there exists a sequence of mappings $S_{y,t_n}: X_y\to M$ such that $(T_{t_n}\circ S_{y,t_n})x \ \to x$ for each $x\in X_y$ and*
   (b) *$||T_{t_n}y||\ ||S_{t_n}x|| \ \to 0$ for each $y\in Y_0$ and $x\in X_y$,*
   (c) *M is an invariant subspace for T_{t_n} for all $n\in\mathbb{N}$.*

Proof. It is obvious that any operator satisfying the subspace-supercyclicity criterion C_0-semigroup also satisfies the criteria of (2) and (3). It suffices to show that (2) implies (1), since the other case is analogous. Let $U_i,V_i\subseteq M$ be non-empty open sets with $i=1,2$. The same argument as in the proof of Theorem 3.2 in [3] can be used to show that there exist $(t_n)_{n\geq 0}$ in $\mathbb{R}_+$, a sequence $(\lambda_{t_n})_{n\geq 0}\subseteq\mathbb{C}\backslash\{0\}$ such that

$$(\lambda_{t_n}T_{t_n})^{-1}(U_i)\bigcap V_i\neq\emptyset \text{ for } i=1,2.$$

Then we can know that $(T_t\oplus T_t)_{t\geq 0}$ is subspace-supercyclic C_0-semigroup for $M\oplus M$ and (x,y) is subspace-supercyclic C_0-semigroup vector for $(T_t\oplus T_t)_{t\geq 0}$. In particular, x is subspace-supercyclic C_0-semigroup vector for $\mathcal{T}$ and $SC(\mathcal{T},M)\bigcap M$ is a dense subset G_δ of M. Let (U_k) be a base of 0-neighborhoods in M. Then there exist $(t_k)_{k\geq 0}\in\mathbb{R}_+$, and $(\lambda_{t_k})_{k\geq 0}\subseteq\mathbb{C}\backslash\{0\}$ such that

$$\lambda_{t_k}T_{t_k}x\in U_k \text{ and } \lambda_{t_k}T_{t_k}y\in x+U_k \text{ for all } k\geq 0.$$

This implies that $\lambda_{t_k}T_{t_k}x\to 0$ and $\lambda_{t_k}T_{t_k}y\to x$. Let

$$M_0=M_1=\mathbb{C}orb(\mathcal{T},x)\bigcap M,$$

which is dense G_δ in M. Also for all $k\geq 0,\lambda\in\mathbb{C}$ and $t\in\mathbb{R}_+$ define

$$S_{t_k}(\lambda T_t x)=\lambda\lambda_{t_k}T_t y.$$

Note that

$$T_{t_k}S_{t_k}(\lambda T_t x)=T_{t_k}(\lambda\lambda_{t_k}T_t y)=\lambda T_t(\lambda_{t_k}T_{t_k}y)\to\lambda T_t x.$$

Hence (1) holds. We complete the proof. □

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