

## VARIOUS CENTROIDS AND SOME CHARACTERIZATIONS OF CATENARY CURVES

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ABSTRACT. For every interval  $[a, b]$ , we denote by  $(\bar{x}_A, \bar{y}_A)$  and  $(\bar{x}_L, \bar{y}_L)$  the geometric centroid of the area under a catenary curve  $y = k \cosh((x - c)/k)$  defined on this interval and the centroid of the curve itself, respectively. Then, it is well-known that  $\bar{x}_L = \bar{x}_A$  and  $\bar{y}_L = 2\bar{y}_A$ .

In this paper, we fix an end point, say 0, and we show that one of  $\bar{x}_L = \bar{x}_A$  and  $\bar{y}_L = 2\bar{y}_A$  for every interval with an end point 0 characterizes the family of catenaries among nonconstant  $C^2$  functions.

### 1. Introduction

For the catenary given by  $f(x) = k \cosh((x - c)/k)$  with a positive constant  $k$ , it is well-known that the ratio of the area under the curve to the arc length of the curve is independent of the interval over which these quantities are concurrently measured. For a positive  $C^1$  function  $f(x)$  defined on an interval  $I$  and an interval  $[a, b] \subset I$ , we consider the area  $A(a, b)$  over the interval  $[a, b]$  and the arc length  $L(a, b)$  of the graph of  $f(x)$ . Then, the function  $f(x) = k \cosh((x - c)/k)$ ,  $k > 0$  satisfies for every interval  $[a, b] \subset I$ ,  $A(a, b) = kL(a, b)$ . This property characterizes the family of catenaries among nonconstant  $C^2$  functions ([13]). Thus, we have the following.

**Proposition 1.1.** *For a nonconstant positive  $C^2$  function  $f(x)$  defined on an interval  $I$ , the followings are equivalent.*

- (1) *There exists a positive constant  $k$  such that for every interval  $[a, b] \subset I$ ,  $A(a, b) = kL(a, b)$ .*
- (2) *The function  $f(x)$  satisfies  $f(x) = k\sqrt{1 + f'(x)^2}$ , where  $k$  is a positive constant.*

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- (3) For some  $k > 0$  and  $c \in \mathbb{R}$ ,

$$f(x) = k \cosh\left(\frac{x-c}{k}\right).$$

For a positive  $C^1$  function  $f(x)$  defined on an interval  $I$  and an interval  $[a, b] \subset I$ , we denote by  $(\bar{x}_A, \bar{y}_A) = (\bar{x}_A(a, b), \bar{y}_A(a, b))$  and  $(\bar{x}_L, \bar{y}_L) = (\bar{x}_L(a, b), \bar{y}_L(a, b))$  the geometric centroid of the area under the graph of  $f(x)$  defined on this interval and the centroid of the graph itself, respectively. Then, for a catenary curve we have the following ([13]).

**Proposition 1.2.** *A catenary curve  $f(x) = k \cosh((x-c)/k)$  satisfies the following properties.*

- (1) For every interval  $[a, b] \subset I$ ,  $\bar{x}_L(a, b) = \bar{x}_A(a, b)$ .
- (2) For every interval  $[a, b] \subset I$ ,  $\bar{y}_L(a, b) = 2\bar{y}_A(a, b)$ .

Conversely, in a recent paper [10], it was shown that one of  $\bar{x}_L = \bar{x}_A$  and  $\bar{y}_L = 2\bar{y}_A$  for all interval  $[a, b]$  characterizes the family of catenaries among nonconstant  $C^2$  functions as follows.

**Proposition 1.3.** *For a nonconstant positive  $C^2$  function  $f(x)$  defined on an interval  $I$ , the followings are equivalent.*

- (1) For every interval  $[a, b] \subset I$ ,  $\bar{x}_L(a, b) = \bar{x}_A(a, b)$ .
- (2) For every interval  $[a, b] \subset I$ ,  $\bar{y}_L(a, b) = 2\bar{y}_A(a, b)$ .
- (3) For some  $k > 0$  and  $c \in \mathbb{R}$ ,

$$f(x) = k \cosh\left(\frac{x-c}{k}\right).$$

In this paper, we consider intervals with a fixed end point, say 0. For a nonzero real number  $x$ , we denote by  $I_x$  the interval defined by

$$(1.1) \quad I_x = \begin{cases} [0, x], & \text{if } x > 0, \\ [x, 0], & \text{if } x < 0. \end{cases}$$

We also denote by  $A(x)$ ,  $L(x)$ ,  $(\bar{x}_A(x), \bar{y}_A(x))$  and  $(\bar{x}_L(x), \bar{y}_L(x))$  the area under the graph of  $f(x)$ , the arc length of the graph of  $f(x)$ , the geometric centroid of the area under the graph of  $f(x)$  and the centroid of the graph itself over the interval  $I_x$ , respectively. Then, it is trivial to show the following.

**Proposition 1.4.** *For a nonconstant positive  $C^2$  function  $f(x)$  defined on an interval  $I$  containing 0, the followings are equivalent.*

- (1) There exists a positive constant  $k$  such that for every nonzero real number  $x$ ,  $A(x) = kL(x)$ .
- (2) The function  $f(x)$  satisfies  $f(x) = k\sqrt{1 + f'(x)^2}$ , where  $k$  is a positive constant.

(3) For some  $k > 0$  and  $c \in \mathbb{R}$ ,

$$f(x) = k \cosh\left(\frac{x-c}{k}\right).$$

As a result, in Section 3 we establish the following characterization theorem for catenary curves.

**Theorem 1.5.** *For a nonconstant positive  $C^2$  function  $f(x)$  defined on an interval  $I$  containing 0, the followings are equivalent.*

- (1) For every nonzero real number  $x \in I$ ,  $\bar{x}_L(x) = \bar{x}_A(x)$ .
- (2) For every nonzero real number  $x \in I$ ,  $\bar{y}_L(x) = 2\bar{y}_A(x)$ .
- (3) For some  $k > 0$  and  $c \in \mathbb{R}$ ,

$$f(x) = k \cosh\left(\frac{x-c}{k}\right).$$

In order to prove the above mentioned main theorem, first of all in Section 2 we prove some lemmas. In particular, Lemma 2.1 establishes a sufficient and necessary condition for two positive  $C^2$  functions to be proportional.

To find the centroid of polygons, see [3]. For the perimeter centroid of a polygon, we refer [2]. In [11], mathematical definitions of centroid of planar bounded domains were given. For various centroids of higher dimensional simplexes, see [12]. The relationships between various centroids of a quadrangle were given in [5, 9]

Archimedes proved the area properties of parabolic sections and then formulated the centroid of parabolic sections ([14]). Some characterizations of parabolas using these properties were given in [4, 6, 7].

Two higher dimensional generalizations of Proposition 1.1 were established in [1]. In [8], it was shown that among nonconstant  $C^2$  functions  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  with isolated singularities,  $S = kV$ ,  $k \in \mathbb{R}$  characterizes the family of catenary rotation surfaces  $f(x, y) = k \cosh(r/k)$ ,  $r = |(x, y)|$ , where  $V$  and  $S$  denote the volume and the surface area of the graph of  $z = f(x, y)$  over a rectangular domain.

## 2. Some lemmas

In this section, first of all we prove the following lemma which is useful in the proof of Theorems stated in Section 1.

**Lemma 2.1.** *We denote by  $f(x)$  and  $g(x)$  two positive  $C^2$  functions defined on an interval  $I$  containing  $0 \in \mathbb{R}$ . Suppose that  $f(x)$  and  $g(x)$  satisfy the following property:*

$$(2.1) \quad \frac{\int_0^x tf(t)dt}{\int_0^x f(t)dt} = \frac{\int_0^x tg(t)dt}{\int_0^x g(t)dt}, \quad x \in I, x \neq 0.$$

*Then the ratio of  $f(x)$  and  $g(x)$  is constant, that is, for some constant  $k \in \mathbb{R}$  we have  $f(x) = kg(x)$ .*

*Proof.* Suppose that  $f(x)$  and  $g(x)$  satisfy (2.1). Then for all  $x \in I$  we get

$$(2.2) \quad \int_0^x g(t)dt \int_0^x tf(t)dt = \int_0^x f(t)dt \int_0^x tg(t)dt.$$

By differentiating (2.2) with respect to the variable  $x$ , we obtain

$$(2.3) \quad g(x) \int_0^x tf(t)dt + xf(x) \int_0^x g(t)dt = f(x) \int_0^x tg(t)dt + xg(x) \int_0^x f(t)dt.$$

We denote by  $k(x)$  the ratio of  $f(x)$  and  $g(x)$ , that is, we let

$$(2.4) \quad k(x) = \frac{f(x)}{g(x)}.$$

Then the function  $k(x)$  is a  $C^2$  function on the interval  $I$ . It follows from (2.3) that

$$(2.5) \quad \int_0^x tf(t)dt + xk(x) \int_0^x g(t)dt = k(x) \int_0^x tg(t)dt + x \int_0^x f(t)dt.$$

Differentiating (2.5) with respect to  $x$  gives

$$(2.6) \quad (xk(x))' \int_0^x g(t)dt = k'(x) \int_0^x tg(t)dt + \int_0^x f(t)dt.$$

By differentiating (2.6) with respect to  $x$  once more, we obtain

$$(2.7) \quad (xk(x))'' \int_0^x g(t)dt = k''(x) \int_0^x tg(t)dt.$$

First, suppose that the open set  $I_1$  defined by  $I_1 = \{x \in I \mid k''(x) \neq 0, x \neq 0\}$  is nonempty. Then, on the open set  $I_1$ , from (2.7) we get

$$(2.8) \quad \int_0^x tg(t)dt = h(x) \int_0^x g(t)dt,$$

where

$$(2.9) \quad h(x) = \frac{(xk(x))''}{k''(x)}.$$

It follows from (2.8) that the function  $h(x)$  is a  $C^2$  function on the open set  $I_1$ . Differentiating (2.8) with respect to  $x$  shows that

$$(2.10) \quad xg(x) - h(x)g(x) = h'(x) \int_0^x g(t)dt.$$

Second, suppose that the open set  $I_2$  defined by  $I_2 = \{x \in I_1 \mid h'(x) \neq 0, x \neq 0\}$  is nonempty. Then, on the open set  $I_2$ , from (2.10) we get

$$(2.11) \quad (j(x)g(x))' = g(x),$$

where

$$(2.12) \quad j(x) = \frac{x - h(x)}{h'(x)}.$$

It also follows from (2.10) that the function  $j(x)$  is a  $C^2$  function on the open set  $I_2$ .

On the other hand, together with (2.8), (2.1) implies

$$(2.13) \quad \int_0^x tf(t)dt = h(x) \int_0^x f(t)dt, \quad x \in I_1,$$

where  $h(x)$  is given in (2.9). Differentiating (2.13) with respect to  $x$  gives

$$(2.14) \quad xf(x) - h(x)f(x) = h'(x) \int_0^x f(t)dt.$$

Hence as in the discussions above, on the open set  $I_2$ , from (2.14) we obtain

$$(2.15) \quad (j(x)f(x))' = f(x),$$

where  $j(x)$  is defined in (2.12).

It follows from (2.11) and (2.15) that

$$(2.16) \quad j(x)g'(x) = g(x) - g(x)j'(x)$$

and

$$(2.17) \quad j(x)f'(x) = f(x) - f(x)j'(x),$$

respectively. For the function  $k(x) = f(x)/g(x)$ , (2.16) and (2.17) show that on the open set  $I_2$ ,

$$(2.18) \quad j(x)k'(x) = 0.$$

Now, we claim that the derivative  $k'(x)$  of the function  $k(x)$  vanishes on  $I_2$ . Otherwise, on a nonempty open set  $I_3$  contained in  $I_2$ , the function  $j(x)$  vanishes. Then, it follows from (2.12) that on  $I_3$ ,  $h(x) = x$ . Hence (2.10) shows that on  $I_3$ ,  $\int_0^x g(t)dt = 0$ , which is a contradiction. Thus, together with (2.18), this contradiction yields that on the open set  $I_2$ ,  $k'(x)$  must vanish, which completes the proof of the above claim.

But, the vanishing of the derivative  $k'(x)$  of  $k(x)$  on  $I_2$  contradicts to the hypothesis on  $I_1$ . Therefore, this contradiction implies that  $I_2$  is empty, that is, on the open set  $I_1$ ,  $h'(x)$  must vanish. Hence, it follows from (2.10) that  $h(x) = x$  on  $I_1$ , which leads to a contradiction. This contradiction shows that  $I_1$  is empty, that is, on the whole domain  $I$ ,  $k''(x)$  must vanish.

Since  $k''(x) = 0$  on  $I$ , (2.7) implies that  $(xk(x))'' = 2k'(x) + xk''(x)$  also vanishes on  $I$ . Combining them, we see that  $k'(x) = 0$  on  $I$ . Thus the function  $k(x) = f(x)/g(x)$  is a constant  $k$ . This completes the proof of Lemma 2.1.  $\square$

Now, we prove the following lemma which is crucial in the proof of Theorem 1.5.

**Lemma 2.2.** *We consider two positive  $C^2$  functions  $f(x)$  and  $g(x)$  defined on an interval  $I$  containing  $0 \in \mathbb{R}$  and denote by  $k(x)$  the ratio  $k(x) = f(x)/g(x)$*

of  $f(x)$  and  $g(x)$ . Suppose that  $f(x)$  and  $g(x)$  satisfy the following:

$$(2.19) \quad \frac{\int_0^x f(t)g(t)dt}{\int_0^x f(t)dt} = \frac{\int_0^x g(t)^2 dt}{\int_0^x g(t)dt}, \quad x \in I, x \neq 0.$$

Then, on the open set  $I_1 = \{x \in I \mid k'(x) \neq 0\}$ , we have  $g'(x) = 0$ .

*Proof.* Suppose that  $f(x)$  and  $g(x)$  satisfy (2.19). Then for all  $x \in I$  we get

$$(2.20) \quad \int_0^x g(t)dt \int_0^x f(t)g(t)dt = \int_0^x f(t)dt \int_0^x g(t)^2 dt.$$

By differentiating (2.20) with respect to the variable  $x$ , we obtain

$$(2.21) \quad g(x) \int_0^x f(t)g(t)dt + f(x)g(x) \int_0^x g(t)dt = f(x) \int_0^x g(t)^2 dt + g(x)^2 \int_0^x f(t)dt.$$

We denote by  $k(x)$  the ratio of  $f(x)$  and  $g(x)$ , that is, we put

$$(2.22) \quad k(x) = \frac{f(x)}{g(x)}.$$

Then the function  $k(x)$  is a  $C^2$  function on the interval  $I$ . It follows from (2.21) that

$$(2.23) \quad \int_0^x f(t)g(t)dt + f(x) \int_0^x g(t)dt = k(x) \int_0^x g(t)^2 dt + g(x) \int_0^x f(t)dt.$$

Differentiating (2.23) with respect to  $x$  gives

$$(2.24) \quad f'(x) \int_0^x g(t)dt = k'(x) \int_0^x g(t)^2 dt + g'(x) \int_0^x f(t)dt.$$

First, suppose that the open set  $I_1$  defined by  $I_1 = \{x \in I \mid k'(x) \neq 0\}$  is nonempty. Then, on the open set  $I_1$ , from (2.24) we get

$$(2.25) \quad \phi(x) \int_0^x g(t)dt = \int_0^x g(t)^2 dt + \psi(x) \int_0^x f(t)dt,$$

where we put

$$(2.26) \quad \phi(x) = \frac{f'(x)}{k'(x)}, \quad \psi(x) = \frac{g'(x)}{k'(x)}.$$

By differentiating (2.25) with respect to  $x$ , we obtain

$$(2.27) \quad \phi'(x) \int_0^x g(t)dt = \psi'(x) \int_0^x f(t)dt.$$

Next, suppose that the open set  $I_2$  defined by  $I_2 = \{x \in I_1 \mid \psi'(x) \neq 0, x \neq 0\}$  is nonempty. Then, on the open set  $I_2$ , from (2.27) we get

$$(2.28) \quad \int_0^x g(t)dt = h(x) \int_0^x f(t)dt,$$

where

$$(2.29) \quad h(x) = \frac{\psi'(x)}{\phi'(x)}.$$

It follows from (2.28) that the function  $h(x)$  is a  $C^2$  function on the open set  $I_2$ . Differentiating (2.28) with respect to  $x$  shows that

$$(2.30) \quad g(x) - h(x)f(x) = h'(x) \int_0^x f(t)dt.$$

Finally, suppose that the open set  $I_3$  defined by  $I_3 = \{x \in I_2 \mid h'(x) \neq 0, x \neq 0\}$  is nonempty. Then, on the open set  $I_3$ , from (2.30) we get

$$(2.31) \quad (j(x)g(x))' = f(x),$$

where

$$(2.32) \quad j(x) = \frac{1 - k(x)h(x)}{h'(x)}.$$

It follows from (2.30) that the function  $j(x)$  is a  $C^2$  function on the open set  $I_3$ .

On the other hand, together with (2.28), (2.19) implies

$$(2.33) \quad \int_0^x g(t)^2 dt = h(x) \int_0^x f(t)g(t)dt, \quad x \in I_2,$$

where  $h(x)$  is given in (2.29). Differentiating (2.33) with respect to  $x$  gives

$$(2.34) \quad g(x)^2 - h(x)f(x)g(x) = h'(x) \int_0^x f(t)g(t)dt.$$

Hence as in the discussions above, on the open set  $I_3$ , from (2.34) we obtain

$$(2.35) \quad (j(x)g(x)^2)' = f(x)g(x),$$

where  $j(x)$  is defined in (2.32) and we use (2.22).

On the open set  $I_3$ , it follows from (2.31) and (2.35) that

$$(2.36) \quad j(x)g'(x) = 0,$$

and hence

$$(2.37) \quad f(x) = (j(x)g(x))' = j'(x)g(x).$$

Together with  $f(x) = k(x)g(x)$ , this shows that

$$(2.38) \quad j'(x) = k(x), \quad x \in I_3.$$

Since  $k(x) > 0$ , (2.36) and (2.38) imply that  $g'(x)$  vanishes on  $I_3$ . This shows that on  $I_3$ ,  $\psi(x)$  (and hence  $h(x)$ ) vanishes, which contradicts to the hypothesis on  $I_2$ . This contradiction yields that  $I_3$  is empty, that is, on the open set  $I_2$ ,  $h(x)$  is a constant  $h_0$ .

It follows from (2.30) that on  $I_2$ ,  $k(x) = f(x)/g(x) = 1/h_0$  is constant, which contradicts to the hypothesis on  $I_1$ . This contradiction shows that  $I_2$  is empty, that is, on the open set  $I_1$ ,  $\psi(x)$  is a constant  $b$ .

Since  $\psi'(x) = 0$ , it follows from (2.27) that on the open set  $I_1$ ,  $\phi(x)$  is a constant  $a$ . Hence, (2.26) yields that on the open set  $I_1$  for some constants  $c$  and  $d$

$$(2.39) \quad f(x) = ak(x) + c, \quad g(x) = bk(x) + d.$$

Thus, it follows from the definition of  $k(x)$  that  $k(x)$  is a root of the quadratic polynomial  $q(t) = bt^2 + (d - a)t - c = 0$ . If the quadratic polynomial  $q(t) = bt^2 + (d - a)t - c = 0$  is nontrivial, then the ratio  $k(x)$  must be constant on the open set  $I_1$ . This contradiction shows that  $q(t)$  is trivial. That is, we have  $b = 0$ ,  $d = a$  and  $c = 0$ , and hence from (2.39) we get on the open set  $I_1$

$$(2.40) \quad f(x) = ak(x), \quad g(x) = a.$$

This completes the proof of Lemma 2.2.  $\square$

### 3. Proof of Theorem 1.5

In this section, with the help of Lemmas in Section 2, we prove Theorem 1.5 stated in Section 1.

First, suppose that a positive nonconstant  $C^2$  function  $f(x)$  defined on an interval  $I$  containing  $0 \in \mathbb{R}$  satisfies  $\bar{x}_L(x) = \bar{x}_A(x)$  for every nonzero real number  $x$ . Then for all  $x \in I$  with  $x \neq 0$ , we have

$$(3.1) \quad \frac{\int_0^x tw(t)dt}{\int_0^x w(t)dt} = \frac{\int_0^x tf(t)dt}{\int_0^x f(t)dt},$$

where we put

$$(3.2) \quad w(x) = \sqrt{1 + f'(x)^2}.$$

It follows from Lemma 2.1 that for some constant  $k \in \mathbb{R}$  we have  $f(x) = kw(x)$ . Therefore Proposition 1.1 implies that for some  $c \in \mathbb{R}$ ,

$$f(x) = k \cosh\left(\frac{x - c}{k}\right).$$

This completes the proof of (1)  $\Rightarrow$  (3).

Next, suppose that a nonconstant positive  $C^2$  function  $f(x)$  defined on an interval  $I$  containing  $0 \in \mathbb{R}$  satisfies  $\bar{y}_L(x) = 2\bar{y}_A(x)$  for every nonzero real number  $x$ . Then for all  $x \in I$  with  $x \neq 0$ , we have

$$(3.3) \quad \frac{\int_0^x f(t)w(t)dt}{\int_0^x w(t)dt} = \frac{\int_0^x f(t)^2 dt}{\int_0^x f(t)dt},$$

where  $w(x)$  is given in (3.2). For the ratio  $k(x) = w(x)/f(x)$ , we consider the open set  $I_1 = \{x \in I \mid k'(x) \neq 0\}$ . Then, it follows from Lemma 2.2 that on the open set  $I_1$ , we have  $f'(x) = 0$ .

Suppose that the open set  $I_1$  is not empty. Then on a fixed connected component  $I_0$  of  $I_1$ ,  $f(x)$  is a constant  $a$ . Therefore, we have  $w(x) = 1$ , and hence on  $I_0$ ,  $k(x) = 1/a$ . This contradiction shows that the open set  $I_1$  must



be empty. That is, the ratio  $k(x)$  is a constant. Hence, for some constant  $k \in \mathbb{R}$  we have  $f(x) = kw(x)$ . Therefore Proposition 1.1 implies that for some  $c \in \mathbb{R}$ ,

$$f(x) = k \cosh\left(\frac{x-c}{k}\right).$$

This completes the proof of (2)  $\Rightarrow$  (3).

Conversely, it follows from Proposition 1.2 that (3)  $\Rightarrow$  (1) and (2). This completes the proof of Theorem 1.5.

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