Commun. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/CKMS.c170039
pISSN: 1225-1763 / eISSN: 2234-3024

EXTENDED WRIGHT-BESSEL FUNCTION AND ITS PROPERTIES

Muhammad Arshad, Shahid Mubeen, Kottakkaran Sooppy Nisar, and Gauhar Rahman

Abstract. In this present paper, our aim is to introduce an extended Wright-Bessel function $J_{\alpha,q}^{\lambda,\gamma,c}(z)$ which is established with the help of the extended beta function. Also, we investigate certain integral transforms and generalized integration formulas for the newly defined extended Wright-Bessel function $J_{\alpha,q}^{\lambda,\gamma,c}(z)$ and the obtained results are expressed in terms of Fox-Wright function. Some interesting special cases involving an extended Mittag-Leffler functions are deduced.

1. Introduction

The well known Wright–Bessel function (or Bessel-Maitland function, misnamed after the second name Maitland of E. M. Wright) defined in [24] as:

$$J_{\alpha}^{\lambda}(z)=\sum_{n=0}^{\infty}\frac{(-z)^n}{\Gamma(\alpha+\lambda n+1)n!};\lambda>-1,z\in\mathbb{C}. \tag{1.1}$$

Pathak [19] gave the generalization of (1.1) as:

$$J_{\alpha,\beta}^{\lambda}(z)=\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{z}{2})^{\alpha+2\beta+2n}}{\Gamma(\beta+n+1)\Gamma(\alpha+\beta+\lambda n+1)}, \tag{1.2}$$

where $\lambda>0$, $z\in\mathbb{C}\setminus(-\infty,0)$, $\alpha,\beta\in\mathbb{C}$.

Singh et al. [22] defined another generalization of Wright-Bessel function in the following form:

$$J_{\alpha,q}^{\lambda,\gamma}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-z)^n}{\Gamma(\alpha+\lambda n+1)n!}, \tag{1.3}$$

where $\alpha,\lambda,\gamma\in\mathbb{C}$, $\Re(\lambda)>0$, $\Re(\alpha)>-1$, $\Re(\gamma)>0$ and $q\in(0,1)\cup\mathbb{N}$ and $(\gamma)_0=1$ and $(\gamma)_{nq}=\frac{\Gamma(\gamma+nq)}{\Gamma(\gamma)}$ is the generalized Pochhammer symbol (see [13]).

Received February 8, 2017; Accepted March 14, 2017.
2010 *Mathematics Subject Classification.* 33B20, 33C20, 33C45, 33C60, 33B15, 26A33.
Key words and phrases. gamma function, generalized hypergeometric function, Oberhettinger formula, Wright-Bessel function, generalized Wright function, Mittag-Leffler function.

Clearly, if $q=0$, then (1.3) reduces to (1.1).

We recall the generalized hypergeometric function ${}_pF_q(z)$ is defined in [5] as:

$$
\begin{aligned}
{}_pF_q(z) = {}_pF_{q,}\left[\begin{array}{cc} (\alpha_1),(\alpha_2),\cdots(\alpha_p) & \\ & ;z \\ (\beta_1),(\beta_2),\cdots(\beta_q) & \end{array}\right] \\
= \sum_{n=0}^{\infty}\frac{(\alpha_1)_n(\alpha_2)_n\cdots(\alpha_p)_n}{(\beta_1)_n(\beta_2)_n\cdots(\beta_q)_n}\frac{z^n}{n!},
\end{aligned} \tag{1.4}
$$

where $\alpha_i, \beta_j \in \mathbb{C}$; $i=1,2,\ldots,p$, $j=1,2,\ldots,q$ and $b_j \neq 0,-1,-2,\ldots$ and $(z)_n$ is the Pochhammer symbols. The gamma function is defined as:

$$
\Gamma(\mu)=\int_0^\infty t^{\mu-1}e^{-t}dt, \mu\in\mathbb{C}, \tag{1.5}
$$

$$
\Gamma(z+1)=z\Gamma(z), z\in\mathbb{C}, \tag{1.6}
$$

and beta function is defined as:

$$
B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt. \tag{1.7}
$$

For $i=1,2,\ldots,p$ and $j=1,2,\ldots,q$, the Wright type hypergeometric function is defined (see [24]-[26]) by the following series as:

$$
\begin{aligned}
{}_p\Psi_q(z) = {}_p\Psi_q\left[\begin{array}{cc} (\alpha_i,A_i)_{1,p} & \\ & ;z \\ (\beta_j,B_j)_{1,q} & \end{array}\right] \\
= \sum_{n=0}^{\infty}\frac{\Gamma(\alpha_1+A_1n)\cdots\Gamma(\alpha_p+A_pn)}{\Gamma(\beta_1+B_1n)\cdots\Gamma(\beta_q+B_qn)}\frac{z^n}{n!},
\end{aligned} \tag{1.8}
$$

where β_r and μ_s are real positive numbers such that

$$
1+\sum_{s=1}^{q}\beta_s-\sum_{r=1}^{p}\alpha_r>0.
$$

Equation (4.1) differs from the generalized hypergeometric function ${}_pF_q(z)$ defined (3.3) only by a constant multiplier. The generalized hypergeometric function ${}_pF_q(z)$ is a special case of ${}_p\Psi_q(z)$ for $A_i=B_j=1$, where $i=1,2,\ldots,p$ and $j=1,2,\ldots,q$:

$$
\frac{1}{\prod\limits_{j=1}^{q}\Gamma(\beta_j)}\;{}_pF_q\left[\begin{array}{cc} (\alpha_1),\ldots(\alpha_p) & \\ & ;z \\ (\beta_1),\ldots(\beta_q) & \end{array}\right] = \frac{1}{\prod\limits_{i=1}^{p}\Gamma(\alpha_i)}\;{}_p\Psi_q\left[\begin{array}{cc} (\alpha_i,1)_{1,p} & \\ & ;z \\ (\beta_j,1)_{1,q} & \end{array}\right]. \tag{1.9}
$$

Recently Sharma and Devi [21] defined extended Wright type function in the following series:

$$
{}_{p+1}\Psi_{q+1}(z,p) = {}_{p+1}\Psi_{q+1}\left[\begin{array}{cc} (\alpha_i,A_i)_{1,p},(\gamma,1) & \\ & ;(z,p) \\ (\beta_j,B_j)_{1,q},(c,1) & \end{array}\right]
$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + A_1 n) \cdots \Gamma(\alpha_p + A_p n)}{\Gamma(\beta_1 + B_1 n) \cdots \Gamma(\beta_q + B_q n)} \frac{B_p(\gamma + n, c - \gamma) z^n}{\Gamma(c - \gamma) n!}, \tag{1.10}$$

where $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$, $\Re(p) > 0$, $\Re(c) > \Re(\gamma) > 0$ and $p \geq 0$. In this paper, we introduce an extended Wright-Bessel (Bessel-Maitland) function as defined in (1.3) as:

$$J_{\alpha,q}^{\lambda,\gamma,c}(z, p) = \sum_{n=0}^{\infty} \frac{B_p(\gamma + nq, c - \gamma)}{B(\gamma, c - \gamma)} \frac{(c)_{nq} (-z)^n}{\Gamma(\alpha + \lambda n + 1) n!}, \tag{1.11}$$

where $p \geq 0$, $\alpha, \lambda, \gamma, c \in \mathbb{C}$, $\Re(p) > 0$, $\Re(\lambda) \geq 0$, $\Re(\alpha) > -1$, $\Re(\gamma) > 0$, $\Re(c) > 0$ and $B_p(x, y)$ is an extended beta function (see [3], [2]).

Remark 1.1. i) If we set $p = 0$ in (1.11), then extended Wright-Bessel function reduces to the (1.3).

ii) If we set $p, q = 0$ in (1.11), then extended Wright-Bessel function reduces to the (1.1).

iii) If α is replaced by $\alpha - 1$ and z by $-z$, then (1.11) reduces to

$$J_{\alpha-1,q}^{\lambda,\gamma,c}(-z) = E_{\lambda,\alpha}^{\gamma,q,c}(z)$$

which is an extended Mittag-Leffler function (see [14]).

iv) If $q = 1$ and α is replaced by $\alpha - 1$ and z by $-z$, then (1.11) reduces to

$$J_{\alpha-1,1}^{\lambda,\gamma,c}(-z) = E_{\lambda,\alpha}^{\gamma,c}(z)$$

which is an extended Mittag-Leffler function (see [18]).

For this continuation of our study, we recall the following result of Oberhettinger [17]

$$\begin{aligned} &\int_0^\infty z^{\alpha-1} \left(z + b + \sqrt{z^2 + 2bz}\right)^{-\beta} dz \\ &= 2\beta b^{-\beta} \left(\frac{b}{2}\right)^{\alpha} \frac{\Gamma(2\alpha)\Gamma(\beta - \alpha)}{\Gamma(1 + \alpha + \beta)}, \; 0 < \Re(\alpha) < \Re(\beta). \end{aligned} \tag{1.12}$$

For various other investigation containing special function, the reader may refer to the recent work of researchers (see [1], [4], [6], [12], [15], [16]).

In the investigation of various properties and relations of the function $J_{\alpha,q}^{\lambda,\gamma,c}(z)$ as defined in (1.11), we need the following well known facts. Sneddon [23] defined Beta Euler and Laplace transforms in following forms:

$$B(f(z); \alpha, \beta) = \int_0^1 z^{\alpha-1} (1 - z)^{\beta-1} f(z) dz, \tag{1.13}$$

where $\Re(\alpha) > 0$ and $\Re(\beta) > 0$.

$$L\{f(z)\} = \int_0^\infty e^{-sz} f(z) dz, \tag{1.14}$$

respectively.

Meijer [11] defined the following K-transform integral equation as:

$$R_v\{f(x);p\} = \int_0^\infty (px)^{\frac{1}{2}} K_v(px) f(x) dx, \tag{1.15}$$

where the parameter $p \in \mathbb{C}$ and $K_v(z)$ represent a modified Bessel function of third kind defined in [11].

Varma transform (see [20]) is defined by the following integral equation as:

$$V(f,k,m;s) = \int_0^\infty (sx)^{m-\frac{1}{2}} W_{k,m}(sx) f(x) dx, \tag{1.16}$$

where $W_{k,m}(z)$ represent Wittaker function (see [9, p. 55]).

2. Integral transforms of an extended Bessel function

In this section, we investigate some integral transforms such as Beta, Laplace, Varma and K-transforms of an extended Wright-Bessel function as defined in (1.11) and we express the obtained result in terms of an extended Wright-type hypergeometric function (1.10).

Theorem 2.1. *Let* α, λ, γ, c, $\mu, \delta, \nu \in \mathbb{C}$, $\Re(\alpha) \geq -1$, $\Re(\lambda) \geq 0$, $\Re(\gamma) \geq 0$, $\Re(\mu) \geq 0$, $\Re(c) \geq 0$, $\Re(\nu) \geq 0$, $p \geq 0$, $q \in (0,1) \setminus \mathbb{N}$, *then the following integral transforms holds*:

$$\begin{aligned} &\int_0^1 z^{\mu-1}(1-z)^{\nu-1} J_{\alpha,q}^{\lambda,\gamma,c}(xz^\delta, p) dz \\ &= \frac{\Gamma(\nu)}{\Gamma(\gamma)} \cdot {}_3\Psi_3 \left[\begin{matrix} (c,q), (\mu,\delta), (\gamma,1) & ; \\ & & |(-x,p) \\ (\alpha+1,\lambda), (\mu+\nu,\delta), (c,1) & ; \end{matrix} \right]. \end{aligned} \tag{2.1}$$

Proof. From (1.11) and (1.13), we obtain

$$\begin{aligned} &\int_0^1 z^{\mu-1}(1-z)^{\nu-1} J_{\alpha,q}^{\lambda,\gamma,c}(xz^\delta, p) dz \\ &= \int_0^1 z^{\mu-1}(1-z)^{\nu-1} \left(\sum_{n=0}^\infty \frac{B_p(\gamma+nq, c-\gamma)}{B(\gamma, c-\gamma)} \frac{(c)_{nq}(-1)^n (xz^\delta)^n}{\Gamma(\alpha+\lambda n+1) n!} \right) dz. \end{aligned}$$

Interchanging the order of integration and summation, we have

$$\begin{aligned} &\int_0^1 z^{\mu-1}(1-z)^{\nu-1} J_{\alpha,q}^{\lambda,\gamma,c}(xz^\delta, p) dz \\ &= \sum_{n=0}^\infty \frac{B_p(\gamma+nq, c-\gamma)}{B(\gamma, c-\gamma)} \frac{(c)_{nq}(-1)^n (x)^n}{\Gamma(\alpha+\lambda n+1) n!} \int_0^1 z^{\mu+\delta n-1}(1-z)^{\nu-1} dz \\ &= \sum_{n=0}^\infty \frac{B_p(\gamma+nq, c-\gamma)}{B(\gamma, c-\gamma)} \frac{(c)_{nq}(-1)^n (x)^n}{\Gamma(\alpha+\lambda n+1) n!} \frac{\Gamma(\mu+\delta)\Gamma(\nu)}{\Gamma(\mu+\nu+\delta)} \end{aligned}$$

$$= \frac{\Gamma(\nu)}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{B_p(\gamma+nq, c-\gamma)}{\Gamma(c-\gamma)} \frac{\Gamma(c+nq)(-1)^n (x)^n}{\Gamma(\alpha+\lambda n+1)n!} \frac{\Gamma(\mu+\delta)}{\Gamma(\mu+\nu+\delta)},$$

which upon using (1.10), we get the required result. □

From above theorem, we observe the following two cases.
Case (i). If $\lambda = \delta$ and $\mu = \alpha + 1$, then (2.1) reduces to the following result:

$$\int_0^1 z^{\alpha}(1-z)^{\nu-1} J_{\alpha,q}^{\lambda,\gamma,c}(xz^{\lambda}, p)dz = \Gamma(\nu) J_{\alpha+\nu,q}^{\delta,\gamma,c}(x) \tag{2.2}$$

and
Case (ii). If $\lambda = \delta$ and $\mu = \alpha + 1$ and $z = (1-z)$, then (2.1) reduces to the following result:

$$\int_0^1 z^{\mu-1}(1-z)^{\alpha} J_{\alpha,q}^{\lambda,\gamma,c}(x(1-z)^{\lambda}, p)dz = \Gamma(\mu) J_{\alpha+\mu,q}^{\delta,\gamma,c}(x). \tag{2.3}$$

Theorem 2.2. *Let* $\alpha, \lambda, \gamma, c, \mu, \delta \in \mathbb{C}$, $\Re(\alpha) \geq -1$, $\Re(\lambda) \geq 0$, $\Re(\gamma) \geq 0$, $\Re(\mu) \geq 0$, $\Re(c) \geq 0$, $\Re(s) \geq 0$, $p \geq 0$, $q \in (0,1) \setminus \mathbb{N}$, *then the following integral transforms holds*:
(2.4)

$$\int_0^{\infty} z^{\mu-1} e^{-sz} J_{\alpha,q}^{\lambda,\gamma,c}(xz^{\delta}, p)dz = \frac{s^{-\mu}}{\Gamma(\gamma)} \; {}_3\Psi_2 \left[\begin{array}{ll} (c,q),(\mu,\delta),(\gamma,1) & ; \\ & |(-\frac{x}{s^{\delta}}, p) \\ (\alpha+1,\lambda),(c,1) & ; \end{array} \right].$$

Proof. From (1.11) and (1.14), we have

$$\int_0^{\infty} z^{\mu-1} e^{-sz} J_{\alpha,q}^{\lambda,\gamma,c}(xz^{\delta}, p)dz$$
$$= \int_0^{\infty} z^{\mu-1} e^{-sz} \left(\sum_{n=0}^{\infty} \frac{B_p(\gamma+nq, c-\gamma)}{B(\gamma, c-\gamma)} \frac{(c)_{nq}(-1)^n (xz^{\delta})^n}{\Gamma(\alpha+\lambda n+1)n!} \right) dz.$$

Interchanging the order of integration and summation, we have

$$\int_0^{\infty} z^{\mu-1} e^{-sz} J_{\alpha,q}^{\lambda,\gamma,c}(xz^{\delta}, p)dz$$
$$= \sum_{n=0}^{\infty} \frac{B_p(\gamma+nq, c-\gamma)}{B(\gamma, c-\gamma)} \frac{(c)_{nq}(-1)^n (x)^n}{\Gamma(\alpha+\lambda n+1)n!} \int_0^{\infty} z^{\mu+\delta n-1} e^{-sz} dz$$
$$= \sum_{n=0}^{\infty} \frac{B_p(\gamma+nq, c-\gamma)}{B(\gamma, c-\gamma)} \frac{(c)_{nq}(-1)^n (x)^n}{\Gamma(\alpha+\lambda n+1)n!} s^{-\mu-\delta n} \Gamma(\mu+\delta n)$$
$$= \frac{s^{-\mu}}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{B_p(\gamma+nq, c-\gamma)}{\Gamma(c-\gamma)} \frac{\Gamma(c+nq)(-1)^n (x)^n}{\Gamma(\alpha+\lambda n+1)n!} \Gamma(\mu+\delta n),$$

which by using (1.10), we get the required result. □

From above theorem, we observe the following two particular cases:

(i) If $p=0$, $q=1$, $\mu=\alpha+1$, $\lambda=\delta$, then (2.4) reduces to

$$\int_0^\infty z^\alpha e^{-sz} J_{\alpha,1}^{\lambda,\gamma}(xz^\lambda)dz = s^{-\alpha-1}(1+xs^{-\lambda})^{-\gamma}, \tag{2.5}$$

by using the relation $\sum\limits_{n=0}^{\infty} \frac{(a)_n z^n}{n!} = (1-z)^{-a}$ (see [20]), and

(ii) If $p=0$, $\gamma=q=1$, $\mu=\alpha+1$, $\lambda=\delta$ and $x=\pm t$, then (2.4) reduces to

$$\begin{aligned}\int_0^\infty z^\alpha e^{-sz} J_{\alpha,1}^{\lambda,1}(\mp z^\lambda)dz &= s^{-\alpha-1}\sum_{n=0}^{\infty}\left(\frac{\mp t}{s^\lambda}\right)^n \\ &= \frac{s^{-\alpha-1}}{1-\frac{\mp t}{s^\lambda}}; \left|\frac{\mp t}{s^\lambda}\right| < 1 \\ &= \frac{s^{\lambda-\alpha-1}}{s^\lambda \pm t}. \end{aligned} \tag{2.6}$$

Theorem 2.3. *Let α, λ, δ, ρ, $\gamma, c \in \mathbb{C}$ and $\Re(\alpha) \geq -1$, $\Re(\gamma) \geq 0$, $\Re(\delta) \geq 0$, $\Re(\rho) \geq 0$, $\Re(c) \geq 0$, $p \geq 0$ and $q \in \mathbb{N}$, then the following result holds:*

$$\begin{aligned}&\int_0^\infty t^{\delta-1} K_\mu(st) J_{\alpha,q}^{\lambda,\gamma,c}(\omega t^\rho, p)dt \\ &= \frac{2^{\delta-2}}{s^\delta \Gamma(\gamma)} {}_4\Psi_2 \left[\begin{matrix} (c,q),(\delta \pm \mu, \rho),(\gamma,1) & ; & \\ & & |(-\omega(\frac{2}{s})^\rho, p) \\ (\alpha+1,\lambda),(c,1) & ; & \end{matrix} \right]. \end{aligned} \tag{2.7}$$

Proof. From (1.11) and (1.15), we have

$$\begin{aligned}&\int_0^\infty t^{\delta-1} K_\mu(st) J_{\alpha,q}^{\lambda,\gamma,c}(\omega t^\rho)dt \\ &= \int_0^\infty t^{\delta-1} K_\mu(st) \left(\sum_{n=0}^{\infty} \tfrac{B_p(\gamma+nq,c-\gamma)}{B(\gamma,c-\gamma)} \tfrac{(c)_{nq}(-\omega t^\rho)^n}{\Gamma(\alpha+\lambda n+1)n!} \right) dt.\end{aligned}$$

By changing the variable $t = \frac{z}{s}$ and interchanging of sum and integration, we have

$$\begin{aligned}&\int_0^\infty (\frac{z}{s})^{\delta-1} K_\mu(z) \left(\sum_{n=0}^{\infty} \frac{B_p(\gamma+nq,c-\gamma)}{B(\gamma,c-\gamma)} \frac{(c)_{nq}(-\omega)^n(\frac{z}{s^\rho})^n}{\Gamma(\alpha+\lambda n+1)n!} \right) \frac{1}{s}dz \\ &= s^{-\delta} \sum_{n=0}^{\infty} \frac{B_p(\gamma+nq,c-\gamma)}{B(\gamma,c-\gamma)} \frac{(c)_{nq}(-\frac{\omega}{s^\rho})^n}{\Gamma(\alpha+\lambda n+1)n!} \int_0^\infty (\frac{z}{s})^{\delta+\rho n-1} K_\mu(z)dz.\end{aligned}$$

Now, by using the following Mathai and Saxena ([10, p. 78]) formula

$$\int_0^\infty x^{\rho-1} K_\mu(x)dx = 2^{\rho-2}\Gamma(\frac{\rho \pm \mu}{2}), \tag{2.8}$$

in the above equation, we obtain

$$\int_0^\infty (\frac{z}{s})^{\delta-1} K_\mu(z) \left(\sum_{n=0}^{\infty} \frac{B_p(\gamma+nq,c-\gamma)}{B(\gamma,c-\gamma)} \frac{(c)_{nq}(-\omega)^n(\frac{z}{s^\rho})^n}{\Gamma(\alpha+\lambda n+1)n!} \right) \frac{1}{s}dz$$

$$= \frac{1}{\Gamma(\gamma)} s^{-\delta} \sum_{n=0}^{\infty} \frac{B_p(\gamma+nq, c-\gamma)}{\Gamma(c-\gamma)} \frac{\Gamma(c+nq)(-\frac{\omega}{s^\rho})^n}{\Gamma(\alpha+\lambda n+1)n!} 2^{\delta-\rho n-2}\Gamma(\delta+\rho n \pm \mu)$$

which by using (1.10), we get the required result. □

Theorem 2.4. *Let $\alpha,\ \lambda,\ \delta,\ \rho,\ \gamma, \mu, \nu c \in \mathbb{C}$ and $\Re(\alpha) \geq -1$, $\Re(\gamma) \geq 0$, $\Re(\delta) \geq 0$, $\Re(\rho) \geq 0$, $\Re(c) \geq 0$, $p \geq 0$ and $q \in \mathbb{N}$, then the following result holds*:

$$\int_0^\infty e^{-\frac{st}{2}} t^{\delta-1} W_{\mu,\nu}(st) J_{\alpha,q}^{\lambda,\gamma,c}(\omega t^\rho, p) dt$$

$$= \frac{1}{s^\delta \Gamma(\gamma)} {}_4\Psi_3 \left[\begin{array}{ccc} (c,q), (\frac{1}{2}\pm\nu+\delta,\rho), (\gamma,1) & ; & \\ & & |(-\frac{\omega}{s^\rho}, p) \\ (\alpha+1,\lambda), (1-\mu+\delta,\rho), (c,1) & ; & \end{array} \right]. \tag{2.9}$$

Proof. From (1.11) and (1.16), we have

$$\int_0^\infty e^{-\frac{st}{2}} t^{\delta-1} W_{\mu,\nu}(st) J_{\alpha,q}^{\lambda,\gamma,c}(\omega t^\rho) dt$$
$$= \int_0^\infty e^{-\frac{st}{2}} t^{\delta-1} W_{\mu,\nu}(st) \left(\sum_{n=0}^{\infty} \frac{B_p(\gamma+nq, c-\gamma)}{B(\gamma, c-\gamma)} \frac{(c)_{nq}(-\omega t^\rho)^n}{\Gamma(\alpha+\lambda n+1)n!} \right) dt.$$

By changing the variable $st = z$ and interchanging of sum and integration in above equation yields

$$\int_0^\infty e^{-\frac{z}{2}} (\frac{z}{s})^{\delta-1} W_{\mu,\nu}(z) J_{\alpha,q}^{\lambda,\gamma,c}(t^\rho) \frac{1}{s} dz$$
$$= s^{-\delta} \sum_{n=0}^{\infty} \frac{B_p(\gamma+nq, c-\gamma)}{B(\gamma, c-\gamma)} \frac{(c)_{nq}(-\frac{\omega}{s^\rho})^n}{\Gamma(\alpha+\lambda n+1)n!} \left(\int_0^\infty e^{-\frac{z}{2}} (\frac{z}{s})^{\delta+\rho n-1} W_{\mu,\nu}(z) dz \right).$$

Now, by using the formula

$$\int_0^\infty e^{-\frac{s}{2}} s^{\beta-1} W_{\mu,\nu}(s) ds = \frac{\Gamma(\frac{1}{2}+\nu+\beta)\Gamma(\frac{1}{2}-\nu+\beta)}{\Gamma(1-\mu+\beta)}, \tag{2.10}$$

(where $\Re(\beta \pm \nu) > -\frac{1}{2}$) in above equation (see Mathai and Saxena [10, p. 79]), we get

$$\int_0^\infty e^{-\frac{z}{2}} (\frac{z}{s})^{\delta-1} W_{\mu,\nu}(z) J_{\alpha,q}^{\lambda,\gamma,c}(t^\rho) \frac{1}{s} dz$$
$$= \frac{s^{-\delta}}{\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{B_p(\gamma+nq, c-\gamma)}{\Gamma(c-\gamma)} \frac{\Gamma(c+nq)(-\frac{\omega}{s^\rho})^n}{\Gamma(\alpha+\lambda n+1)n!} \frac{\Gamma(\frac{1}{2}\pm\nu+\delta+\rho n)}{\Gamma(1-\mu+\delta+\rho n)}$$

which by using (1.10), we get the required result. □

3. Unified integral formulas of an extended Bessel function

In this section, we establish two generalized integral formulas containing extended Wright-Bessel function as defined in (1.11) which is represented in terms of Wright-type function defined in (1.10) by inserting with the suitable argument defined in (1.12).

Theorem 3.1. *Let* α, μ, δ, v, $\lambda, \gamma, c \in \mathbb{C}$ *with* $\Re(\delta) > 0$, $\Re(\alpha) > -1$, $\Re(\lambda) > 0$, $\Re(\gamma) > 0$, $\Re(c) > 0$, $\Re(\mu) > \Re(\delta) > 0$, $p \geq 0$, $q \in (0,1) \cup \mathbb{N}$ *and* $z > 0$, *then the following result holds*:

$$\int_0^\infty z^{\delta-1}\left(z+b+\sqrt{z^2+2bz}\right)^{-\mu} J_{\alpha,q}^{\lambda,\gamma,c}\left(\frac{y}{z+b+\sqrt{z^2+2bz}}\right)dz$$

$$= \frac{\Gamma(2\delta)b^{\delta-\mu}}{2^{\delta-1}\Gamma(\gamma)}\ {}_4\Psi_4\left[\begin{matrix}(\mu+1;1),(\mu-\delta,1),(c,q),(\gamma,1); \\ (\alpha+1,\lambda),(\mu,1),(\delta+\mu+1,1),(c,1)\end{matrix}\ \Big|(-\tfrac{y}{b},p)\right]. \tag{3.1}$$

Proof. Let $\mathfrak{L}_1$ be the left hand side of Theorem (3.1) and applying (1.11) to the integrand of (3.1), we have

$$\mathfrak{L}_1 = \int_0^\infty z^{\delta-1}\left(\ z+b+\sqrt{z^2+2bz}\ \right)^{-\mu}$$
$$\times \sum_{n=0}^\infty \frac{B_p(\gamma+nq,c-\gamma)(c)_{nq}}{B(\gamma,c-\gamma)\Gamma(\lambda n+\alpha+1)n!}\left(\ -\frac{y}{(z+b+\sqrt{z^2+2bz})}\ \right)^n dz.$$

By interchanging the order of integration and summation, which is verified by the uniform convergence of the series under the given assumption of Theorem 3.1, we have

(3.2)
$$\mathfrak{L}_1 = \sum_{n=0}^\infty \frac{B_p(\gamma+nq,c-\gamma)(c)_{nq}(-1)^n y^n}{B(\gamma,c-\gamma)\Gamma(\lambda n+\alpha+1)n!}\int_0^\infty z^{\delta-1}\left(z+b+\sqrt{z^2+2bz}\right)^{-(\mu+n)}dz.$$

By considering the assumption given in theorem 3.1, since $\Re(\mu+n) > \Re(\delta) > 0$, $\Re(\alpha) > -1$ and applying (1.12) to (3.2), we obtain

$$\mathfrak{L}_1 = \sum_{n=0}^\infty \frac{B_p(\gamma+nq,c-\gamma)(c)_{nq}(-1)^n y^n}{B(\gamma,c-\gamma)\Gamma(\lambda n+\alpha+1)n!}2(\mu+n)b^{-(\mu+n)}(\frac{b}{2})^\delta\frac{\Gamma(2\delta)\Gamma(\mu+n-\alpha)}{\Gamma(1+\mu+\delta+n)}.$$

Applying (1.6), we get

$$\mathfrak{L}_1 = \frac{\Gamma(2\delta)b^{\delta-\mu}}{2^{\delta-1}\Gamma(\gamma)}\sum_{n=0}^\infty \frac{(-1)^n}{\Gamma(c-\gamma)n!}\frac{y^n}{b^n}\frac{B_p(\gamma+nq,c-\gamma)\Gamma(c+nq)\Gamma(\mu+n+1)\Gamma(\mu+n-\delta)}{\Gamma(\alpha+\lambda n+1)\Gamma(\mu+n)\Gamma(\delta+\mu+n+1)}$$

which upon using (1.10), we get the required result. □

Theorem 3.2. *Let* α, μ, δ, v, $\lambda, \gamma, c \in \mathbb{C}$ *with* $\Re(\delta) > 0$, $\Re(\alpha) > -1$, $\Re(\lambda) > 0$, $\Re(\gamma) > 0$, $\Re(c) > 0$, $\Re(\mu) > \Re(\delta) > 0$, $p \geq 0$, $q \in (0,1) \cup \mathbb{N}$ *and* $z > 0$. *Then* *the following result holds*:

$$\int_0^\infty z^{\delta-1}\left(z+b+\sqrt{z^2+2bz} \right)^{-\mu} J_{\alpha,q}^{\lambda,\gamma,c}\left(\tfrac{yz}{z+b+\sqrt{z^2+2bz}} \right) dz$$

$$= \frac{\Gamma(\mu-\delta)b^{\delta-\mu}}{2^{\delta-1}\Gamma(\gamma)}\, {}_4\Psi_4 \left[\begin{matrix} (\mu+1;1),(2\delta,2),(c,q),(\gamma,1); \\ (\mu,1),(\alpha+1,\lambda),(\mu+\delta+1,2),(c,1) \end{matrix} \mid -(\tfrac{y}{2},p) \right]. \tag{3.3}$$

Proof. Let $\mathfrak{L}_2$ be the left hand side of (3.2) and applying (1.11) to the integrand of (3.3), we have

$$\mathfrak{L}_2 = \int_0^\infty z^{\delta-1}\left(z+b+\sqrt{z^2+2bz} \right)^{-\mu} \times \sum_{n=0}^{\infty} \frac{B_p(\gamma+nq,c-\gamma)(c)_{nq}}{B(\gamma,c-\gamma)\Gamma(\alpha+\lambda n+1)n!}\left(-\tfrac{yz}{z+b+\sqrt{z^2+2bz}} \right)^n dz.$$

By interchanging the order of integration and summation, which is verified by the uniform convergence of the series under the given assumption of Theorem 3.2, we have

$$\mathfrak{L}_2 = \sum_{n=0}^{\infty} \frac{B_p(\gamma+nq,c-\gamma)(c)_{nq}(-1)^n y^n}{B(\gamma,c-\gamma)\Gamma(\alpha+\lambda n+1)n!} \times \int_0^\infty z^{\delta+n-1}\left(z+b+\sqrt{z^2+2bz}\right)^{-(\mu+n)} dz. \tag{3.4}$$

By considering the assumption given in Theorem 3.2, since $\Re(\beta) > \Re(\alpha) > 0$, $\Re(\alpha) > -1$ and applying (1.12) to (3.4), we obtain

$$\mathfrak{L}_2 = \sum_{n=0}^{\infty} \frac{B_p(\gamma+nq,c-\gamma)(c)_{nq}(-1)^n y^n}{B(\gamma,c-\gamma)\Gamma(\alpha+\lambda n+1)n!} 2(\mu+n)b^{\delta-\mu}\left(\frac{1}{2^{\delta+n}}\right)\frac{\Gamma(2\delta+2n)\Gamma(\mu-\delta)}{\Gamma(1+\mu+\delta+2n)}$$

$$= \frac{\Gamma(\mu-\delta)b^{\delta-\mu}}{2^{\delta-1}\Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{B_p(\gamma+nq,c-\gamma)(-1)^n y^n}{2^n\Gamma(c-\gamma)} \frac{\Gamma(c+nq)\Gamma(2\delta+2n)\Gamma(\mu+n+1)}{\Gamma(\alpha+\lambda n+1)\Gamma(\mu+n)\Gamma(1+\mu+\delta+2n)}$$

which upon using (1.10), we get the required result. □

4. Special cases

In this section, we present the extended form of generalized Mittag-Leffler functions which are the special cases of extended Wright-Bessel function defined in (1.11). Also, we prove four corollaries which are the special cases of obtained Theorems 3.1 and 3.2.

Case 1. If α is replaced by $\alpha-1$ and z by $-z$, then (1.11) reduces to an extended Mittag-Leffler function defined as:

$$J_{\alpha-1,q}^{\lambda,\gamma,c}(-z) = E_{\lambda,\alpha}^{\gamma,q,c}(z) = \sum_{n=0}^{\infty} \frac{B_p(\gamma+nq,c-\gamma)(c)_{nq}}{B(\gamma,c-\gamma)\Gamma(\lambda n+\alpha)n!}. \tag{4.1}$$

Case 2. If $q=1$ and α is replaced by $\alpha-1$ and z by $-z$, then (1.11) reduces to an extended Mittag-Leffler function

$$J^{\lambda,\gamma,c}_{\alpha-1,1}(-z)=E^{\gamma,c}_{\lambda,\alpha}(z)=\sum_{n=0}^{\infty}\frac{B_p(\gamma+n,c-\gamma)(c)_n}{B(\gamma,c-\gamma)\Gamma(\lambda n+\alpha)n!}. \tag{4.2}$$

Corollary 4.1. *Let the conditions of Theorem 2.1, then the following integral transforms holds*:

$$\begin{aligned}&\int_0^1 z^{\mu-1}(1-z)^{\nu-1}E^{\gamma,q,c}_{\lambda,\alpha}(xz^{\delta},p)dz\\ &=\frac{\Gamma(\nu)}{\Gamma(\gamma)}\cdot{}_3\Psi_3\left[\begin{array}{cc}(c,q),(\mu,\delta),(\gamma,1) & ;\\ & |(x,p)\\ (\alpha,\lambda),(\mu+\nu,\delta),(c,1) & ;\end{array}\right].\end{aligned} \tag{4.3}$$

Corollary 4.2. *Let the conditions of Theorem 2.1, then the following integral transforms holds*:

$$\begin{aligned}&\int_0^1 z^{\mu-1}(1-z)^{\nu-1}E^{\gamma,c}_{\lambda,\alpha}(xz^{\delta},p)dz\\ &=\frac{\Gamma(\nu)}{\Gamma(\gamma)}\cdot{}_3\Psi_3\left[\begin{array}{cc}(c,1),(\mu,\delta),(\gamma,1) & ;\\ & |(x,p)\\ (\alpha,\lambda),(\mu+\nu,\delta),(c,1) & ;\end{array}\right].\end{aligned} \tag{4.4}$$

Corollary 4.3. *Let the conditions of Theorem 2.2 are satisfied, then the following integral transforms holds*:

$$\begin{aligned}&\int_0^\infty z^{\mu-1}e^{-sz}E^{\gamma,q,c}_{\lambda,\alpha}(xz^{\delta},p)dz\\ &=\frac{s^{-\mu}}{\Gamma(\gamma)}\cdot{}_3\Psi_2\left[\begin{array}{cc}(c,q),(\mu,\delta),(\gamma,1) & ;\\ & |(\frac{x}{s^{\delta}},p)\\ (\alpha,\lambda),(c,1) & ;\end{array}\right].\end{aligned} \tag{4.5}$$

Corollary 4.4. *Let the conditions of Theorem 2.2 are satisfied, then the following integral transforms holds*:

$$\begin{aligned}&\int_0^\infty z^{\mu-1}e^{-sz}E^{\gamma,c}_{\lambda,\alpha}(xz^{\delta},p)dz\\ &=\frac{s^{-\mu}}{\Gamma(\gamma)}\cdot{}_3\Psi_2\left[\begin{array}{cc}(c,1),(\mu,\delta),(\gamma,1) & ;\\ & |(\frac{x}{s^{\delta}},p)\\ (\alpha,\lambda),(c,1) & ;\end{array}\right].\end{aligned} \tag{4.6}$$

Similarly, we can derives some corollaries by using the conditions of Theorems 2.3 and 2.4.

Corollary 4.5. *Assume that the conditions of Theorem 3.1 are satisfied. Then the following integral formula holds*:

$$\int_0^\infty z^{\delta-1}\left(z+b+\sqrt{z^2+2bz}\right)^{-\mu}E^{\gamma,q,c}_{\lambda,\alpha}\left(\frac{y}{z+b+\sqrt{z^2+2bz}}\right)dz$$

$$(4.7) \quad = \frac{\Gamma(2\delta)b^{\delta-\mu}}{2^{\delta-1}\Gamma(\gamma)} \; {}_4\Psi_4 \left[\begin{array}{c} (\mu+1;1), (\mu-\delta,1), (c,q), (\gamma,1); \\ \\ (\alpha,\lambda), (\mu,1), (\delta+\mu+1,1), (c,1) \end{array} \Big| (-\tfrac{y}{b}, p) \right].$$

Corollary 4.6. *Assume that the conditions of Theorem 3.1 are satisfied. Then the following integral formula holds*:

$$\int_0^\infty z^{\delta-1} \left(z+b+\sqrt{z^2+2bz}\right)^{-\mu} E^{\gamma,c}_{\lambda,\alpha} \left(\frac{y}{z+b+\sqrt{z^2+2bz}}\right) dz$$

$$(4.8) \quad = \frac{\Gamma(2\delta)b^{\delta-\mu}}{2^{\delta-1}\Gamma(\gamma)} \; {}_4\Psi_4 \left[\begin{array}{c} (\mu+1;1), (\mu-\delta,1), (c,1), (\gamma,1); \\ \\ (\alpha,\lambda), (\mu,1), (\delta+\mu+1,1), (c,1) \end{array} \Big| (-\tfrac{y}{b}, p) \right].$$

Corollary 4.7. *Assume that the conditions of Theorem 3.2 are satisfied. Then the following integral formula holds*:

$$\int_0^\infty z^{\delta-1} \left(z+b+\sqrt{z^2+2bz}\right)^{-\mu} E^{\gamma,q,c}_{\lambda,\alpha} \left(\frac{yz}{z+b+\sqrt{z^2+2bz}}\right) dz$$

$$(4.9) \quad = \frac{\Gamma(\mu-\delta)b^{\delta-\mu}}{2^{\delta-1}\Gamma(\gamma)} \; {}_4\Psi_4 \left[\begin{array}{c} (\mu+1;1), (2\delta,2), (c,q), (\gamma,1); \\ \\ (\mu,1), (\alpha,\lambda), (\mu+\delta+1,2), (c,1) \end{array} \Big| -(\tfrac{y}{2}, p) \right].$$

Corollary 4.8. *Assume that the conditions of Theorem 3.2 are satisfied. Then the following integral formula holds*:

$$\int_0^\infty z^{\delta-1} \left(z+b+\sqrt{z^2+2bz}\right)^{-\mu} E^{\gamma,c}_{\lambda,\alpha} \left(\frac{yz}{z+b+\sqrt{z^2+2bz}}\right) dz$$

$$(4.10) \quad = \frac{\Gamma(\mu-\delta)b^{\delta-\mu}}{2^{\delta-1}\Gamma(\gamma)} \; {}_4\Psi_4 \left[\begin{array}{c} (\mu+1;1), (2\delta,2), (c,1), (\gamma,1); \\ \\ (\mu,1), (\alpha,\lambda), (\mu+\delta+1,2), (c,1) \end{array} \Big| -(\tfrac{y}{2}, p) \right].$$

Concluding Remark. In this paper, we introduced certain integral transforms and representation formulas for newly defined extended Wright-Bessel function and express the obtained results in terms of an extended Wright-type hypergeometric function. If letting $p=0$, then we have the results of the generalized Wright-Bessel function and Mittag-Leffler functions proved in ([7], [22]).

References

[1] M. S. Abouzaid, A. H. Abusufian, and K. S. Nisar, *Some unified integrals associated with generalized Bessel-Maitland function*, Int. Bull. Math. Research **3** (2016), 18–23.

[2] M. A. Chaudhry, A. Qadir, H. M. Srivastava, and R. B. Paris, *Extended hypergeometric and confluent hypergeometric functions*, Appl. Math. Comput. **159** (2004), no. 2, 589–602.

[3] M. A. Chaudhry and S. M. Zubair, *On a Class of Incomplete Gamma Functions with Applications*, CRC Press, Boca Raton, 2002.

[4] J. Choi, P. Agarwal, S. Mathur, and S. D. Purohit, *Certain new integral formulas involving the generalized Bessel functions*, Bull. Korean Math. Soc. **51** (2014), no. 4, 995–1003.

[5] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions*, Vol. 1, McGraw-Hill, NewYork, Toronto, London, 1953.

[6] S. Jain and P. Agarwal, *A new class of integral relation involving general class of Polynomials and I-function*, Walailak J. Sci. Tech. **12** (2015), 1009–1018.
[7] N. U. Khan and T. Kashmin, *Some integrals for the generalized Bessel-Maitland function*, Electron. J. Math. Analy. Appl. **4** (2016), no. 2, 139–149.
[8] V. S. Kiryakova, *Generalized Fractional Calculus and Applications*, Longman & J. Wiley, Harlow and New York, 1994.
[9] A. M. Mathai and R. K. Saxena, *On linear combinations of stochastic variables*, Metrika **20** (1973), no. 3, 160–169.
[10] A. M. Mathai, R. K. Saxena, and H. J. Haubold, *The H-Function Theory and Applications*, Springer New York Dordrecht Heidelberg London, 2010.
[11] C. S. Meijer, *Über eine Erweiterung der Laplace-Transform*, Neder Wetensch Proc. **2** (1940), 269–278.
[12] N. Menaria, S. D. Purohit, and R. K. Parmar, *On a new class of integrals involving generalized Mittag-Leffler function*, Sur. Math. Appl. **11** (2016), 1–9.
[13] G. M. Mitagg-Leffler, *Sur la nouvelle function* $E_\alpha(x)$, C. R. Acad. Soc. Paris **137** (1903), 554–558.
[14] E. Mittal, R. M. Pandey, and S. Joshi, *On extension of Mittag-Leffler function*, Appl. Appl. Math. **11** (2016), no. 1, 307–316.
[15] K. S. Nisar and S. R. Mondal, *Certain unified integral formulas involving the generalized modified k-Bessel function of first kind*, Commun. Korean Math. Soc. **32** (2017), no. 1, 47–53.
[16] K. S. Nisar, R. K. Parmar, and A. H. Abusufian, *Certain new unified integrals associated with the generalized K-Bessel function*, Far East J. Math. Sci. **9** (2016), 1533–1544.
[17] F. Oberhettinger, *Tables of Mellin Transforms*, Springer, New York, 1974.
[18] M. A. Özarslan and B. Yilmaz, *The extended Mittag-Leffler function and its properties*, J. Inequal. Appl. **2014** (2014), 85, 10 pp.
[19] R. S. Pathak, *Certain convergence theorems and asymptotic properties of a generalization of Lommel and Maitland transformations*, Proc. Nat. Acad. Sci. India Sect. A **36** (1966), 81–86.
[20] E. D. Rainville, *Special Functions*, The Macmillan Company, New York, 1960.
[21] S. C. Sharma and M. Devi, *Certain Properties of Extended Wright Generalized Hypergeometric Function*, Ann. Pure Appl. Math. **9** (2015), 45–51.
[22] M. Singh, M. A. Khan, and A. H. Khan, *On some properties of a generalization of Bessel-Maitland function*, Inter. J. Math. Trend Tech. **14** (2014), 46–47.
[23] I. N. Sneddon, *The use of Integral Transform*, Tata Mc Graw-Hill, New Delhi, 1979.
[24] E. M. Wright, *The asymptotic expansion of the generalized Bessel function*, Proc. Lond. Math. Soc. **38** (1935), no. 2, 257–270.
[25] ______, *The asymptotic expansion of integral functions defined by Taylor Series*, Philos. Trans. Roy. Soc. London Ser. A **238** (1940), 423–451.
[26] ______, *The asymptotic expansion of the generalized hypergeometric function II*, Proc. London. Math. Soc. **46** (1935), no. 2, 389–408.

Muhammad Arshad
Department of Mathematics
International Islamic University
Islamabad, Pakistan
E-mail address: marshad_@yahoo.com

Shahid Mubeen
Department of Mathematics
University of Sargodha
Sargodha, Pakistan
E-mail address: smjhanda@gmail.com

Kottakkaran Sooppy Nisar
Department of Mathematics
College of Arts and Science-Wadi Aldawaser
11991, Prince Sattam bin Abdulaziz University
Alkharj, Kingdom of Saudi Arabia
E-mail address: ksnisar1@gmail.com, n.sooppy@psau.edu.sa

Gauhar Rahman
Department of Mathematics
International Islamic University
Islamabad, Pakistan
E-mail address: gauhar55uom@gmail.com