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(p, q)-EXTENSION OF THE WHITTAKER FUNCTION AND ITS CERTAIN PROPERTIES

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Abstract. In this paper, we obtain a (p, q)-extension of the Whittaker function $M_{k,\mu}(z)$ together with its integral representations, by using the extended confluent hypergeometric function of the first kind $\Phi_{p,q}(b; c; z)$ [recently extended by J. Choi]. Also, we give some of its main properties, namely the summation formula, a transformation formula, a Mellin transform, a differential formula and inequalities. In addition, our extension on Whittaker function finds interesting connection with the Laguerre polynomials.

1. Introduction and preliminaries

The simple Beta function $B(\alpha, \beta)$ is defined by [11, p. 26, eq. (48)]

$$B(\alpha,\beta) = \begin{cases} \displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt, & (\Re(\alpha) > 0; \Re(\beta) > 0), \\ \dfrac{\Gamma(\alpha)(\beta)}{\Gamma(\alpha+\beta)}, & (\Re(\alpha) < 0; \Re(\beta) < 0), \end{cases} \tag{1.1}$$

where $\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-$.

Gauss hypergeometric function (GHF) is defined by [12, p. 124]

$$F\left(\begin{array}{c} a,\ b; \\ c; \end{array} z\right) = \sum_{\ell=0}^{\infty} \frac{(a)_\ell (b)_\ell}{(c)_\ell} \frac{z^\ell}{\ell!}\ , \tag{1.2}$$

where $|z| < 1,\ c \in \mathbb{C}$.

Confluent hypergeometric function (CHF) of the first kind $\Phi(\cdot)$ is defined by [12, p. 124]

$$\Phi\left(\begin{array}{c} b; \\ c; \end{array} z\right) = \sum_{\ell=0}^{\infty} \frac{(b)_\ell}{(c)_\ell} \frac{z^\ell}{\ell!}\ , \tag{1.3}$$

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An integral representations of GHF and CHF are given by

$$F(a,b;c;z)=\frac{1}{B(b,c-b)}\int_0^1\left[t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\right]dt, \tag{1.4}$$

where $|arg(1-z)|<\pi,\ \Re(c)>\Re(b)>0,$
and

$$\Phi(b;c;z)=\frac{1}{B(b,c-b)}\int_0^1\left[t^{b-1}(1-t)^{c-b-1}\exp(zt)\right]dt, \tag{1.5}$$

where $\Re(c)>\Re(b)>0$.

From the equations (1.4) and (1.5), using summation series of $(1-zt)^{-a}$ and $\exp(zt)$, we find

$$F(a,b;c;z)=\sum_{\ell=0}^{\infty}\frac{(a)_\ell B(b+\ell,c-b)}{B(b,c-b)}\frac{z^\ell}{\ell!},\quad |z|<1,\ \Re(c)>\Re(b)>0, \tag{1.6}$$

and

$$\Phi(b;c;z)=\sum_{\ell=0}^{\infty}\frac{B(b+\ell,c-b)}{B(b,c-b)}\frac{z^\ell}{\ell!},\quad \Re(c)>\Re(b)>0. \tag{1.7}$$

The definition of Whittaker function of the first kind $M_{k,\mu}(z)$ in [11, p. 39, eq.(23)] and [13] is given by

$$M_{k,\mu}(z)=z^{\mu+\frac{1}{2}}\exp\left(-\tfrac{z}{2}\right)\Phi\left(\mu-k+\frac{1}{2};\ 2\mu+1;\ z\right), \tag{1.8}$$

where $-\pi<arg(z)\leq\pi,\ k\in\mathbb{C},\ \Re(\mu)>-\frac{1}{2},\ \Re\ (\mu-k)>-\frac{1}{2},\ 2\mu\in\mathbb{C}\setminus\mathbb{Z}^-$ and $\Phi(\cdot)$ is the confluent hypergeometric function of first kind.

In 1997, Chaudhry et al. [3, p.20, Eq. (1.7)] gave a p-extension of the Beta function $B(x,y)$ given by

$$B(x,y;p)=\int_0^1 t^{x-1}(1-t)^{y-1}\exp\left[\frac{-p}{t(1-t)}\right]dt,\ \Re(p)>0 \tag{1.9}$$

and they proved that this extension has connections with the Macdonald, error and Whittaker function. In 2004, Chaudhry et al. [4], given extension of GHF $F(a,b;c;z)$ and CHF $\Phi(b;c;z)$ based on the extended Beta function $B(x,y;p)$,which they were denoted by $F_p(a,b;c;z)$ and $\Phi_p(b;c;z)$.

An integral representations of $F_p(a,b;c;z)$ and $\Phi_p(b;c;z)$ are given by [4]

$$F_p(a,b;c;z)=\frac{1}{B(b,c-b)}\int_0^1\left[t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\exp\left(-\frac{p}{t(1-t)}\right)\right]dt, \tag{1.10}$$

where $|arg(1-z)|<\pi,\ p>0, p=0\ ,\Re(c)>\Re(b)>0,$
and

$$\Phi_p(b;c;z)=\frac{1}{B(b,c-b)}\int_0^1\left[t^{b-1}(1-t)^{c-b-1}\exp\left(zt-\frac{p}{t(1-t)}\right)\right]dt, \tag{1.11}$$

where $p>0, p=0$, and $\Re(c)>\Re(b)>0$.

The extended Whittaker function $M_{p,k,\mu}(\cdot)$ based on the extended confluent hypergeometric function $\Phi_p(\cdot)$ is defined by [8]

$$M_{p,k,\mu}(z) = z^{\mu+\frac{1}{2}} \exp\left(-\frac{z}{2}\right) \Phi_p\left(\mu - k + \frac{1}{2}; 2\mu+1; z\right), \tag{1.12}$$

where $p \geq 0, -\pi < arg(z) \leq \pi,\ k \in \mathbb{C},\ \Re(\mu) > -\frac{1}{2},\ \Re\ (\mu - k) > -\frac{1}{2},\ 2\mu \in \mathbb{C} \setminus \mathbb{Z}^-$. This definition clearly reduces to the simple Whittaker function, when we set $p = 0$.

Recently, Choi et al. [5] has given (p,q)-extension of the Beta function $B(x,y)$ by adding one more parameter q, denoted and defined by

$$B(x,y;p,q) \equiv B_{p,q}(x,y) = \int_0^1 \left[t^{x-1}(1-t)^{y-1}\ \exp\left(-\frac{p}{t} - \frac{q}{1-t}\right)\right] dt, \tag{1.13}$$

where $\Re(p) > 0,\ \Re(q) > 0$ and also given extension of GHF $F(a,b;c;z)$ and CHF $\Phi(b;c;z)$, based on the extended Beta function $B_{p,q}(x,y)$, which are defined by [5]

$$F_{p,q}(a,b;c;z) = \sum_{\ell=0}^{\infty} \frac{(a)_\ell B_{p,q}(b+\ell, c-b)}{B(b,c-b)} \frac{z^\ell}{\ell!}, \tag{1.14}$$

where $|z| < 1$, $p \geq 0$, $q \geq 0$, $\Re(c) > \Re(b) > 0$, and

$$\Phi_{p,q}(b;c;z) = \sum_{\ell=0}^{\infty} \frac{B_{p,q}(b+\ell, c-b)}{B(b,c-b)} \frac{z^\ell}{\ell!}, \tag{1.15}$$

where $p \geq 0$, $q \geq 0$, $\Re(c) > \Re(b) > 0$.

An integral representation of $F_{p,q}(a,b;c;z)$ and $\Phi_{p,q}(b;c;z)$ are given by [5]

$$F_{p,q}(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 \left[t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} \exp\left(-\frac{p}{t} - \frac{q}{1-t}\right)\right] dt, \tag{1.16}$$

where $p \geq 0, q \geq 0$, $\Re(c) > \Re(b) > 0,\ |arg(1-z)| < \pi$,
and

$$\Phi_{p,q}(b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 \left[t^{b-1}(1-t)^{c-b-1} \exp\left(zt - \frac{p}{t} - \frac{q}{1-t}\right)\right] dt, \tag{1.17}$$

where $p \geq 0$, $q \geq 0$, $\Re(c) > \Re(b) > 0$.

Differentiation formula of Eq. (1.15) with respect to z is given by

$$\frac{d^n}{dz^n}\left\{\Phi_{p,q}(b;c;z)\right\} = \frac{(b)_n}{(c)_n} \Phi_{p,q}(b+n;c+n;z),\ (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}). \tag{1.18}$$

Transformation formula of $\Phi_{p,q}(b;c;z)$ is defined by

$$\Phi_{p,q}(b;c;z) = \exp(z)\ \Phi_{q,p}(c-b;c;-z). \tag{1.19}$$

The Laguerre polynomials $L_n^{\alpha}(z)$ is defined by [10, pp. 201–202]

$$L_n^{\alpha}(z)=\frac{(1+\alpha)_n}{n!}\,{}_1F_1\left(\ -n;1+\alpha;\ z\right)=\sum_{k=0}^{n}\frac{(-1)^k(1+\alpha)_n z^k}{k!(n-k)!(1+\alpha)_k}. \tag{1.20}$$

In view of Eq. (1.20),the definition of simple Laguerre polynomials $L_m(p)=L_m^0(p)(m\in\mathbb{N}_0)$ is given by the generating function [10, p. 202]

$$\exp\left(-\frac{pt}{1-t}\right)=(1-t)^{1+\alpha}\sum_{m=0}^{\infty}t^mL_m^{\alpha}(p)\ ,\ (\alpha\in\mathbb{C},\ |t|<1), \tag{1.21}$$

and for $\alpha\geq 0,\ 0\leq p<\infty$.

Replace $t\longrightarrow 1-t$ and $\alpha=0$ in (1.21), which yield

$$\exp\left(-\frac{p}{t}\right)=e^{-p}\left[t\sum_{n=0}^{\infty}L_n(p)(1-t)^n\right],\ (t\in\mathbb{D}_1:=\{t\in\mathbb{R}:|1-t|<1\}), \tag{1.22}$$

Also, replace $t\longrightarrow 1-t$ and $p=q$ in (1.22), which yield

$$\exp\left(-\frac{q}{1-t}\right)=e^{-q}\left[(1-t)\sum_{m=0}^{\infty}L_m(q)\ t^m\right],\ (t\in\mathbb{D}_2:=\{t\in\mathbb{R}:|t|<1\}), \tag{1.23}$$

comparing the equations (1.22) and (1.23), we find

$$\exp\left(-\frac{p}{t}-\frac{q}{1-t}\right)=e^{-p-q}\left[\sum_{n,m=0}^{\infty}L_n(p)L_m(q)(1-t)^{n+1}t^{m+1}\right],\ (t\in\mathbb{D}_1\cap\mathbb{D}_2). \tag{1.24}$$

In this paper, we obtain a (p,q)-extension of the Whittaker function $M_{k,\mu}(\cdot)$ together with its integral representations,which we denoted by $M_{p,q,k,\mu}(\cdot)$. Moreover, our work is also motivated by the following references [1, 2, 6, 9]. The plan of this paper is as follow. The (p,q)-extension of the Whittaker function in (1.12) is given in Section 2 and its integral representations are shown in Section 3. We consider some properties of this function, namely the summation formula, a transformation formula, inequalities, a Mellin transform and a differential formula are given in Sections 4-8. We introduce concluding remarks about the function are given in Section 9.

2. The (p,q)-extended Whittaker function $M_{p,q,k,\mu}(\cdot)$

Here, we consider the following (p,q)-extension of the Whittaker function $M_{k,\mu}(z)$ based on the extended Beta function $B_{p,q}(x,y)$, which we denoted by $M_{p,q,k,\mu}(\cdot)$, and defined by

$$M_{p,q,k,\mu}(z)=z^{\mu+\frac{1}{2}}\ \exp\left(-\frac{z}{2}\right)\Phi_{p,q}\left(\mu-k+\frac{1}{2};\ 2\mu+1;\ z\right), \tag{2.1}$$

where $p\geq 0, q\geq 0,\ -\pi<arg(z)\leq\pi,\ k\in\mathbb{C},\ \Re(\mu)>-\frac{1}{2},\ \Re\ (\mu-k)>-\frac{1}{2},\ 2\mu\in\mathbb{C}\backslash\mathbb{Z}^-$ and $\Phi_{p,q}(\cdot)$ is the extended confluent hypergeometric function of the first kind defined in (1.17). This definition clearly reduces to the simple Whittaker function when $p = 0 = q$.

3. Integral representations for $M_{p,q,k,\mu}(\cdot)$

Theorem 1. *Each of the following integral representations of (p, q)-extended Whittaker functions hold for $\Re(p) > 0, \Re(q) > 0,\ \Re(\mu) > -\frac{1}{2},\ \Re(\mu \pm k) > -\frac{1}{2}$.*

• *The integral representation of $M_{p,q,k,\mu}(\cdot)$ can be obtained by using (1.17) in Eq. (2.1), we find*

$$M_{p,q,k,\mu}(z) = \frac{z^{\mu+\frac{1}{2}}\ \exp\left(-\frac{z}{2}\right)}{B(\mu-k+\frac{1}{2}, \mu+k+\frac{1}{2})} \times \int_0^1 \left[t^{\mu-k-\frac{1}{2}}(1-t)^{\mu+k-\frac{1}{2}}\ \exp\left(zt - \frac{p}{t} - \frac{q}{1-t}\right)\right] dt. \tag{3.1}$$

• *If we set the transformation $t \to \frac{(u-\alpha)}{(\beta-\alpha)}$, where $(\beta - \alpha) > 0$ [when $t = 0, u = \alpha$ and $t = 1, u = \beta$] in Eq. (3.1), we find*

$$M_{p,q,k,\mu}(z) = \frac{(\beta-\alpha)^{-2\mu} z^{\mu+\frac{1}{2}}\ \exp\left(-\frac{z}{2}\right)}{B(\mu-k+\frac{1}{2}, \mu+k+\frac{1}{2})} \int_\alpha^\beta \left[(u-\alpha)^{\mu-k-\frac{1}{2}}(\beta-u)^{\mu+k-\frac{1}{2}} \times \exp\left(\frac{z(u-\alpha)}{\beta-\alpha} - \frac{p(\beta-\alpha)}{u-\alpha} - \frac{q(\beta-\alpha)}{\beta-u}\right)\right] du. \tag{3.2}$$

• *If we set $\alpha = -1$ and $\beta = 1$ in Eq. (3.2), we get following another integral representation*

$$M_{p,q,k,\mu}(z) = \frac{(2)^{-2\mu} z^{\mu+\frac{1}{2}}\ \exp\left(-\frac{z}{2}\right)}{B(\mu-k+\frac{1}{2}, \mu+k+\frac{1}{2})} \int_{-1}^1 \left[(1+u)^{\mu-k-\frac{1}{2}}(1-u)^{\mu+k-\frac{1}{2}} \times \exp\left(\frac{z(u+1)}{2} - \frac{2p}{u+1} - \frac{2q}{1-u}\right)\right] du. \tag{3.3}$$

• *Upon setting $t \longrightarrow \frac{u}{1+u}$, $(u \neq -1)$ [when $t \to 0, u \to 0$, and $t \to 1, u \to \infty$] in Eq. (3.1), we obtain*

$$M_{p,q,k,\mu}(z) = \frac{z^{\mu+\frac{1}{2}}\ \exp\left(-\frac{z}{2}\right)}{B(\mu-k+\frac{1}{2}, \mu+k+\frac{1}{2})} \times \int_0^\infty \left[\frac{u^{\mu-k-\frac{1}{2}}}{(1+u)^{2\mu+1}} \exp\left(\frac{zu}{1+u} - \frac{p(1+u)}{u} - q(1+u)\right)\right] du. \tag{3.4}$$

• *Clearly, if we consider $p = 0 = q$ in (3.1)-(3.4), we get another following integral representations of the simple Whittaker function:*

$$\begin{aligned} M_{k,\mu}(z) &= \frac{z^{\mu+\frac{1}{2}}\ \exp\left(-\frac{z}{2}\right)}{B(\mu-k+\frac{1}{2}, \mu+k+\frac{1}{2})} \int_0^1 \left[t^{\mu-k-\frac{1}{2}}(1-t)^{\mu+k-\frac{1}{2}} \exp(zt)\right] dt, \\ &= \frac{(\beta-\alpha)^{-2\mu}\ z^{\mu+\frac{1}{2}}\ \exp\left(-\frac{z}{2}\right)}{B(\mu-k+\frac{1}{2}, \mu+k+\frac{1}{2})} \end{aligned}$$

$$\times \int_{\alpha}^{\beta} \left[(u-\alpha)^{\mu-k-\frac{1}{2}}(\beta-u)^{\mu+k-\frac{1}{2}} \exp\left(\frac{z(u-\alpha)}{\beta-\alpha}\right)\right] du, \tag{3.5}$$

where $\beta \neq \alpha$ *and* α, β *are two scalers.*

$$\begin{aligned} M_{k,\mu}(z) &= \frac{(2)^{-2\mu} z^{\mu+\frac{1}{2}} \ \exp\left(-\frac{z}{2}\right)}{B(\mu-k+\frac{1}{2}, \mu+k+\frac{1}{2})} \\ &\quad \times \int_{-1}^{1} \left[(1+u)^{\mu-k-\frac{1}{2}}(1-u)^{\mu+k-\frac{1}{2}} \exp\left(\frac{z(u+1)}{2}\right)\right] du \\ &= \frac{z^{\mu+\frac{1}{2}} \ \exp\left(-\frac{z}{2}\right)}{B(\mu-k+\frac{1}{2}, \mu+k+\frac{1}{2})} \\ &\quad \times \int_{0}^{\infty} \left[\frac{(u)^{\mu-k-\frac{1}{2}}}{(1+u)^{2\mu+1}} \exp\left(\frac{zu}{1+u}\right)\right] du, \ u \neq -1. \end{aligned} \tag{3.6}$$

4. Summation formula for $M_{p,q,k,\mu}(\cdot)$

We obtain the following summation formula of the (p, q)-extended Whittaker function $M_{p,q,k,\mu}(\cdot)$.

Theorem 2. *The following summation formula hold true*:

$$\begin{aligned} M_{p,q,k,\mu}(z) &= \frac{\exp(-p-q) \ z^{\mu+\frac{1}{2}} \ \exp\left(-\frac{z}{2}\right)}{B(\mu-k+\frac{1}{2}, \mu+k+\frac{1}{2})} \\ &\quad \times \sum_{n,m=0}^{\infty} \Bigg[L_n(p)L_m(q)B(\mu-k+m+\frac{3}{2}, \mu+k+n+\frac{3}{2}) \\ &\quad \times \Phi(\mu-k+m+\frac{3}{2}; \ 2\mu+n+m+3; \ z)\Bigg], \end{aligned} \tag{4.1}$$

where $\Re(\mu) > -\frac{1}{2}, \ \Re(\mu-k+m) > -\frac{3}{2}, \Re(\mu \pm k) > -\frac{1}{2}$.

Proof. We begin with left hand side of (4.1) and involving integral representation (3.1), which yield

$$\begin{aligned} M_{p,q,k,\mu}(z) &= \frac{z^{\mu+\frac{1}{2}} \ \exp\left(-\frac{z}{2}\right)}{B(\mu-k+\frac{1}{2}, \mu+k+\frac{1}{2})} \\ &\quad \times \int_{0}^{1} \left[t^{\mu-k-\frac{1}{2}}(1-t)^{\mu+k-\frac{1}{2}} dt \ \exp(zt) \ \exp\left(-\frac{p}{t}-\frac{q}{1-t}\right)\right] dt, \end{aligned} \tag{4.2}$$

we find (4.3) upon using the result (1.24) in Eq. (4.2)

$$\begin{aligned} M_{p,q,k,\mu}(z) &= \frac{\exp(-p-q) \ z^{\mu+\frac{1}{2}} \ \exp\left(-\frac{z}{2}\right)}{B(\mu-k+\frac{1}{2}, \mu+k+\frac{1}{2})} \sum_{n,m=0}^{\infty} \Bigg[L_n(p)L_m(q) \\ &\quad \times \int_{0}^{1} t^{\mu-k+m+\frac{3}{2}-1}(1-t)^{\mu+k+n+\frac{3}{2}-1} \ \exp(zt)\Bigg] dt, \end{aligned} \tag{4.3}$$

Finally, applying the confluent hypergeometric function (1.5) in the above Eq. (4.3), we get right hand side of (4.1) which is our required result. □

5. Transformation formula for $M_{p,q,k,\mu}(\cdot)$

Theorem 3. *The following transformation formula hold true*:

$$M_{p,q,k,\mu}(-z) = (-1)^{\mu+\frac{1}{2}} M_{p,q,-k,\mu}(z), \tag{5.1}$$

where $p \geq 0,\ q \geq 0,\ \Re(\mu) > -\frac{1}{2},\ 2\mu \neq -1, -2, -3, \ldots$.

Proof. We take left hand side of (5.1), and applying the definition of Whittaker function given in (2.1). Then we find

$$M_{p,q,k,\mu}(-z) = (-1)^{\mu+\frac{1}{2}} (z)^{\mu+\frac{1}{2}}\ \exp\left(\frac{z}{2}\right) \Phi_{p,q}(\mu - k + \frac{1}{2};\ 2\mu + 1; -z). \tag{5.2}$$

Using transformation formula of the extended confluent hypergeometric function (1.19) in the above Eq. (5.2), we obtain

$$M_{p,q,k,\mu}(-z) = (-1)^{\mu+\frac{1}{2}} (z)^{\mu+\frac{1}{2}}\ \exp\left(-\frac{z}{2}\right) \Phi_{q,p}(\mu + k + \frac{1}{2};\ 2\mu + 1;\ z). \tag{5.3}$$

Lastly, applying (2.1) in the above Eq. (5.3), we get right hand side of (5.1). □

6. Inequalities for $M_{p,q,k,\mu}(\cdot)$

Theorem 4. *The following inequalities hold true.*

$$M_{p,q,k,\mu}(z) \leq \exp(-2p - 2q) M_{k,\mu}(z), \tag{6.1}$$

where $p > 0,\ q > 0,\ \Re(\mu) > -\frac{1}{2},\ \Re(\mu \pm k) > -\frac{1}{2}$.

Proof. We start with left hand side of (6.1) and involving the integral representation (3.4), we get

$$\begin{aligned} M_{p,q,k,\mu}(z) = & \frac{z^{\mu+\frac{1}{2}}\ \exp\left(-\frac{z}{2}\right)}{B(\mu - k + \frac{1}{2}, \mu + k + \frac{1}{2})} \\ & \times \int_0^\infty \left[\frac{(u)^{\mu-k-\frac{1}{2}}}{(1+u)^{2\mu+1}}\ \exp\left(\frac{zu}{1+u}\right) \exp\left\{-p(1+u^{-1})\right\} \right. \\ & \left. \times \exp\left\{-q(1+u)\right\}\right] du. \end{aligned} \tag{6.2}$$

For $p > 0,\ q > 0$, we have

$$p(1 + u^{-1}) \geq 2p;\ u \in (0, 1], \tag{6.3}$$

$$\Rightarrow \quad \exp\{-p(1 + u^{-1})\} \leq \exp(-2p), \tag{6.4}$$

also

$$q(1 + u) \geq 2q;\ u \in [1, \infty), \tag{6.5}$$

$$\Rightarrow \quad \exp\{-q(1 + u)\} \leq \exp(-2q). \tag{6.6}$$

Using inequalities (6.4) and (6.6) in the above Eq. (6.2), which yield

$$M_{p,q,k,\mu}(z) \leq \frac{\exp(-2p-2q)\ z^{\mu+\frac{1}{2}}\ \exp\left(-\frac{z}{2}\right)}{B(\mu-k+\frac{1}{2}.\mu+k+\frac{1}{2})} \times \int_0^\infty \left[\frac{u^{\mu-k-\frac{1}{2}}}{(1+u)^{2\mu+1}}\ \exp\left(\frac{zu}{1+u}\right)\right]du. \tag{6.7}$$

Finally, using (3.6) in the above Eq. (6.7), we get right hand side of (6.1), which is our required result.

Here, we obtain an another interesting result by using the following inequality:

$$\exp(x) \geq 1 + \frac{x^n}{n!},\ n>0,\ x>0, \tag{6.8}$$

in the above Eq. (3.1), we find

$$M_{p,q,k,\mu}(z) \geq \frac{z^{\mu+\frac{1}{2}}\ \exp\left(-\frac{z}{2}\right)}{B(\mu-k+\frac{1}{2},\mu+k+\frac{1}{2})} \times \left[\int_0^1 t^{\mu-k-\frac{1}{2}}(1-t)^{\mu+k-\frac{1}{2}}\ \exp\left(-\frac{p}{t}-\frac{q}{1-t}\right)dt + \frac{z^n}{n!}\int_0^1 t^{\mu-k+n-\frac{1}{2}}(1-t)^{\mu+k-\frac{1}{2}}\ \exp\left(-\frac{p}{t}-\frac{q}{1-t}\right)\right]dt. \tag{6.9}$$

Using the extended Beta function (1.13) in the above Eq. (6.9), we get

$$M_{p,q,k,\mu}(z) \geq \frac{z^{\mu+\frac{1}{2}}\ \exp\left(-\frac{z}{2}\right)}{B(\mu-k+\frac{1}{2},\mu+k+\frac{1}{2})}\left[B\left(\mu-k+\frac{1}{2},\mu+k+\frac{1}{2};\ p,q\right) + \frac{z^n}{n!}B\left(\mu-k+n+\frac{1}{2},\mu+k+\frac{1}{2};\ p,q\right)\right]. \tag{6.10}$$

□

7. Mellin transforms for $M_{p,q,k,\mu}(\cdot)$

We obtain the Mellin transform of $M_{p,q,k,\mu}(\cdot)$ by multiply the both sides of Eq. (2.1) with p^{r-1} and q^{s-1}, then integrate with respect to p and q over $(0,\infty)$, we get the following result.

Theorem 5. *The following Mellin transformation hold true*

$$\int_0^\infty\int_0^\infty p^{r-1}q^{s-1}M_{p,q,k,\mu}(z)dpdq = \frac{\Gamma(r)\Gamma(s)B\left(\mu-k+r+s+\frac{1}{2},\mu+k+r+s+\frac{1}{2}\right)z^{-r-s}}{B\left(\mu-k+\frac{1}{2},\mu+k+\frac{1}{2}\right)}M_{p,q,k,\mu+r+s}(z), \tag{7.1}$$

where $\Re(r)>0,\ \Re(s)>0,\ \Re(\mu\pm k+r+s)>-\frac{1}{2},\ \Re(\mu\pm k)>-\frac{1}{2}$.

Proof. We take left hand side of (7.1) upon using (2.1), we get

$$\int_0^\infty\int_0^\infty p^{r-1}q^{s-1}M_{p,q,k,\mu}(z)dpdq \tag{7.2}$$
$$=\int_0^\infty\int_0^\infty \left[p^{r-1}q^{s-1}z^{\mu+\frac{1}{2}}\exp\left(-\frac{z}{2}\right)\Phi_{p,q}(\mu-k+\frac{1}{2};\ 2\mu+1;z)\right]dpdq.$$

Applying (1.17) in Eq. (7.2) and change the order of integration, we find

$$\int_0^\infty\int_0^\infty p^{r-1}q^{s-1}M_{p,q,k,\mu}(z)dpdq \tag{7.3}$$
$$=\frac{z^{\mu+\frac{1}{2}}\ \exp\left(-\frac{z}{2}\right)}{B(\mu-k+\frac{1}{2},\mu+k+\frac{1}{2})}\int_0^1\Bigg[t^{\mu-k-\frac{1}{2}}(1-t)^{\mu+k-\frac{1}{2}}dt\ \exp(zt)$$
$$\times\left\{\int_0^\infty p^{r-1}\exp\left(-\frac{p}{t}\right)dp\int_0^\infty q^{s-1}\exp\left(-\frac{q}{1-t}\right)dq\right\}\Bigg]dt.$$

If we set $\frac{p}{t}\to z$ [when $p=0,z=0$; $p\to\infty,z\to\infty$] and $\frac{q}{1-t}\to u$ [when $q=0,u=0$; $q\to\infty,u\to\infty$] in the above Eq. (7.3), we find

$$\int_0^\infty\int_0^\infty p^{r-1}q^{s-1}M_{p,q,k,\mu}(z)dpdq \tag{7.4}$$
$$=\frac{\Gamma(r)\Gamma(s)z^{\mu+\frac{1}{2}}\ \exp\left(-\frac{z}{2}\right)}{B(\mu-k+\frac{1}{2},\mu+k+\frac{1}{2})}\int_0^1\left[t^{\mu-k+r-\frac{1}{2}}(1-t)^{\mu+k+s-\frac{1}{2}}\exp(zt)\right]dt.$$

Upon using confluent hypergeometric function (1.5) in Eq. (7.4), we obtain

$$\int_0^\infty\int_0^\infty p^{r-1}q^{s-1}M_{p,q,k,\mu}(z)dpdq \tag{7.5}$$
$$=\frac{z^{\mu+r+s+\frac{1}{2}}\ \exp\left(-\frac{z}{2}\right)\Gamma(r)\Gamma(s)\ z^{-r-s}}{B\left(\mu-k+\frac{1}{2},\mu+k+\frac{1}{2}\right)}\Phi\left(\mu-k+r+\tfrac{1}{2};2\mu+r+s+1;z\right)$$
$$\times B\left(\mu-k+r+\tfrac{1}{2},\mu+k+s+\tfrac{1}{2}\right).$$

Applying the classical Whittaker function (1.8) in Eq. (7.5), which gives

$$\int_0^\infty\int_0^\infty p^{r-1}q^{s-1}M_{p,q,k,\mu}(z)dpdq \tag{7.6}$$
$$=\frac{\Gamma(r)\Gamma(s)B\left(\mu-k+r+\frac{1}{2},\mu+k+s+\frac{1}{2}\right)z^{-r-s}}{B\left(\mu-k+\frac{1}{2},\mu+k+\frac{1}{2}\right)}M_{k,\mu+r+s}(z).$$

If we take $r=1=s$ in (7.6), we get the following another interesting result

$$\int_0^\infty\int_0^\infty M_{p,q,k,\mu}(z)dpdq=\frac{B\left(\mu-k+\frac{3}{2},\mu+k+\frac{3}{2}\right)z^{-2}}{B\left(\mu-k+\frac{1}{2},\mu+k+\frac{1}{2}\right)}M_{k,\mu+2}(z). \tag{7.7}$$ □

Theorem 6. *If* $p\geq 0,\ q\geq 0,\ (2\alpha-\beta)>0$ *and* $\Re(\mu\pm\alpha)>-\frac{1}{2}$, *then*

$$\int_0^\infty \exp(-\alpha x)\ x^{\alpha-1}M_{p,q,k,\mu}(\beta x)dx \tag{7.8}$$

$$= \frac{\Gamma(\alpha+\mu+\frac{1}{2})}{(\alpha+\frac{\beta}{2})^{\alpha+\mu+\frac{1}{2}}} F_{p,q}\left(\mu+k+\frac{1}{2}, \mu-k+\frac{1}{2}; 2\mu+1; \frac{2\alpha}{2\alpha+\beta}\right),$$

where $|arg(1-2\beta/2\alpha+\beta)| < \pi$.

Proof. We begin with left hand side of (7.8) and using the definition of extended Whittaker function (2.1), which yields

$$\int_0^\infty \exp(-\alpha x)\ x^{\alpha-1} M_{p,q,k,\mu}(\beta x) dx \tag{7.9}$$
$$= \int_0^\infty \Big[\exp(-\alpha x) x^{\alpha-1}(\beta x)^{\mu+\frac{1}{2}} \exp\left(-\tfrac{\beta x}{2}\right) \Phi_{p,q}(\mu-k+\tfrac{1}{2}; 2\mu+1; \beta x)\Big] dx.$$

Applying the extended confluent hypergeometric function (1.17) in the above Eq. (7.9) and change the order of integration (with uniform convergence), which gives

$$M_{p,q,k,\mu}(z) = \frac{\beta^{\mu+\frac{1}{2}}}{B(\mu-k+\frac{1}{2}, \mu+k+\frac{1}{2})} \int_0^1 \Big[t^{\mu-k-\frac{1}{2}}(1-t)^{\mu+k-\frac{1}{2}}$$
$$\times \exp\left(-\tfrac{p}{t}-\tfrac{q}{1-t}\right) \int_0^\infty \exp\left(-\left(\alpha+\tfrac{\beta}{2}-\beta t\right) x\right)\ x^{\alpha+\mu-\frac{1}{2}} dx\Big] dt. \tag{7.10}$$

Applying the Euler's integral formula [7, eq. (5)],

$$\int_0^\infty e^{-st} t^{z-1} dt = \frac{\Gamma(z)}{s^z},\ \Re(z) > 0,\ \Re(s) > 0. \tag{7.11}$$

in the above Eq. (7.10), we find

$$M_{p,q,k,\mu}(z) = \frac{\beta^{\mu+\frac{1}{2}}\Gamma(\mu+\alpha+\frac{1}{2})}{(\alpha+\frac{\beta}{2})^{\mu+\alpha+\frac{1}{2}} B(\mu-k+\frac{1}{2}, \mu+k+\frac{1}{2})}$$
$$\times \int_0^1 \Big[t^{\mu-k-\frac{1}{2}}(1-t)^{\mu+k-\frac{1}{2}} \left(1-\frac{2\beta t}{2\alpha+\beta}\right)^{-(\mu+\alpha+\frac{1}{2})} \tag{7.12}$$
$$\times \exp\left(-\frac{p}{t}-\frac{q}{1-t}\right)\Big] dt,$$

where $|arg(1-2\beta/2\alpha+\beta)| < \pi$.

Finally, applying the integral representation (1.16) in Eq. (7.12), we get right hand side of (7.8). □

Corollary. *If we take* $\beta = 2\alpha$ *in the above Eq. (7.8), we find*

$$\int_0^\infty \exp(-\alpha x)\ x^{\alpha-1} M_{p,q,k,\mu}(2\alpha x) dx = \frac{\Gamma(\mu+\alpha+\frac{1}{2}) B\left(\mu-k+\frac{1}{2}, k-\alpha; p, q\right)}{B\left(\mu-k+\frac{1}{2}, \mu+k+\frac{1}{2}\right)(2\alpha)^\alpha}, \tag{7.13}$$

where $\alpha > 0$; $q,\ p > 0$, *and* $\Re(\alpha+\mu) > -\frac{1}{2}$.

8. Differentiation formula for $M_{p,q,k,\mu}(\cdot)$

Differentiation formula of (p,q)-extended Whittaker function $M_{p,q,k,\mu}(\cdot)$ upon using the following formula:

$$(\lambda)_{1+K} = (\lambda)_1(\lambda+1)_K. \tag{8.1}$$

Theorem 7. *The following differential formula hold true:*

$$\begin{aligned}&\frac{d^n}{dz^n}\left\{z^{-\mu-\frac{1}{2}}\exp\left(\frac{z}{2}\right)M_{p,q,k,\mu}(z)\right\}\\ &=\frac{(\mu-k+\frac{1}{2})_n}{(2\mu+1)_n}\exp\left(\frac{z}{2}\right)z^{-\mu-\frac{1}{2}-\frac{n}{2}}M_{p,q,k-\frac{n}{2},\mu+\frac{n}{2}}(z),\end{aligned} \tag{8.2}$$

where $n=0,1,2,3,\ldots$.

Proof. Using (2.1) in the above Eq. (8.2), which yield

$$z^{-\mu-\frac{1}{2}}\exp\left(\frac{z}{2}\right)M_{p,q,k,\mu}(z)=\Phi_{p,q}(\mu-k+\frac{1}{2};\ 2\mu+1;\ z). \tag{8.3}$$

From (8.3) we call n^{th} derivative on both sides and using result (8.1), which yield

$$\begin{aligned}&\frac{d^n}{dz^n}\left\{z^{-\mu-\frac{1}{2}}\exp\left(\frac{z}{2}\right)M_{p,q,k,\mu}(z)\right\}\\ &=\frac{(\mu-k+\frac{1}{2})_n}{(2\mu+1)_n}\Phi_{p,q}(\mu-k+\frac{1}{2}+n;\ 2\mu+1+n;\ z).\end{aligned} \tag{8.4}$$

Finally, applying (2.1) in the above Eq. (8.4), we get right hand side of (8.2). □

9. Conclusion remark

The Whittaker function has various applications in the field of Mathematical physics and engineering sciences. Here in this paper, we have obtained (p,q)-extension of the Whittaker function together with its integral representation. We consider some properties for this function, namely the summation formula, a transformation formula, a Mellin transform, a differential formula and inequalities. Therefore, these evaluated results may be useful in the field of Engineering sciences, Mathematical physics, and so on. Also, this extended Whittaker function have some relations with Laguerre polynomials, Macdonald function, error function, Modified Bessel function and G-function.

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