

**A FAMILY OF SERIES AND INTEGRALS INVOLVING  
WHITTAKER, BESSEL FUNCTIONS, AND THEIR  
PRODUCTS DERIVABLE FROM THE REPRESENTATION  
OF THE GROUP  $SO(2, 1)$**

JUNESANG CHOI AND I. A. SHILIN

**ABSTRACT.** By mainly using certain properties arising from the semisimple Lie group  $SO(2, 1)$ , we aim to show how a family of some interesting formulas for bilateral series and integrals involving Whittaker, Bessel functions, and their product can be obtained.

**1. Introduction and preliminaries**

It is well known that the irreducible representations of the three-dimensional Lorentz group  $SO(2, 1)$  can be constructed in the space of  $\sigma$ -homogeneous infinitely differentiable functions ( $\sigma \in \mathbb{C} \setminus \mathbb{Z}$ ) defined on the cone

$$\Lambda := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 - x_2^2 - x_3^2 = 0 \right\}.$$

Here and in the following, we denote  $\mathbb{Z}$ ,  $\mathbb{R}$  and  $\mathbb{C}$  by the sets of integers, real and complex numbers, respectively. For the fixed  $\sigma$  with  $-1 < \Re(\sigma) < 0$ , we denote this space by  $\mathfrak{D}$  and the analogous space consisting of  $\varsigma$ -homogeneous functions when  $\varsigma := -\sigma - 1$  by  $\mathfrak{D}^\bullet$ . The linear operator of the space  $\mathfrak{D}$ , arising in this case and corresponding to the Lorentz group element  $g$ , can be written in the form  $T(g)[f(x)] = f(g^{-1}x)$ . Since the functions belonging to  $\mathfrak{D}$  are homogeneous, we can realize the representation  $T_\sigma$  in the space of the restrictions  $\hat{f}$  of functions  $f \in \mathfrak{D}$  to any conic section  $\gamma$ , because  $\hat{f}(\xi) = t^{-\sigma} f(t\xi) = t(x)$  for  $\xi \in \gamma$ . Vilenkin and Shleinikova [15] considered the following three sections of  $\Lambda$ : The circle  $\gamma_1 : x_1 = 1$ ; The hyperbola  $\gamma_2 : x_2 = \pm 1$ ; The parabola  $\gamma_3 : x_1 + x_2 = 1$ . They introduced three bases in  $\mathfrak{D}$ , which consist of the eigenfunctions of the restrictions of the representation  $T$  to the subgroups  $H_i$  acting transitively on the above conic sections, respectively, and obtained formulas for ‘matrix

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elements' of the linear operator, which transforms the 'circle' basis into the 'parabolic' basis, and some 'matrix elements' of the linear operator, which transforms the 'parabolic' basis into the 'hyperbolic' basis.

Here, we denote the 'circle' and 'parabolic' bases by  $B_1$  and  $B_2$ , respectively. Namely,

$$B_1 := \{f_k(x) := x_1^{\sigma-k} (x_2 + \mathbf{i}x_3)^k \mid k \in \mathbb{Z}\}$$

and

$$B_2 := \left\{ f_\lambda^*(x) := (x_1 + x_2)^\sigma \exp\left(\frac{\mathbf{i}\lambda x_3}{x_1 + x_2}\right) \mid \lambda \in \mathbb{R} \right\}.$$

Just as matrix elements of a linear operator can be computed by using a scalar product, the matrix elements of the operators  $B_1^\bullet \leftrightarrow B_2^\bullet$  and  $T(g)$ ,  $g \in SO(2,1)$ , can be derived via the functionals

$$F_i : \mathfrak{D} \times \mathfrak{D}^\bullet \longrightarrow \mathbb{C}, (f, g) \longmapsto \int_{\gamma_i} f(x) g(x) (dx)_i \quad (i \in \{1, 2\}),$$

where  $(dx)_i$  is a  $H_i$ -invariant measure. Shilin and Choi [9] computed the matrix elements of the operator, which acts in the  $SO(3,1)$ -representation space and transforms the 'spherical' basis into the 'hyperbolic' basis, and obtained the corresponding integral formulas for Legendre functions arising around these matrix elements. Shilin and Choi [12] computed the matrix elements of the operator  $T^\bullet(a) \in GL(SO(2,1))$  for some block-diagonal matrices and evaluated some integrals involving a product of two Bessel functions (basic or modified). Shilin and Choi [10, 11] have employed the same idea which was in [12] for  $SO(2,2)$  and  $SO(3,1)$ , respectively. In particular, Shilin and Choi [11] evaluated some integrals involving a product of the Macdonald function and the modified hyper-Bessel function of the first or second kind.

In this paper, we reconsider the group  $SO(2,1)$  and compute the matrix elements of the operator  $T^\bullet(g_i)$  with respect to the basis  $B_{3-i}^\bullet$ . By using the matrix elements of the transformations  $B_1^\bullet \leftrightarrow B_2^\bullet$  and decompositions  $G = KAN$  (Iwasawa decomposition) and  $G = KAK$  (Cartan decomposition) of Lie groups, we obtain some formulas for series and integrals involving Whittaker or Bessel functions or their product. In particular, in Theorem 1, we consider series involving product of two Whittaker functions  $W_{k, \mathbf{i}\tau}$ , where  $k$  or  $-k$  is the parameter of series. Some similar integrals involving the same product, where the variable of integration is  $\tau$ , are pointed out to be considered in [1].

## 2. Some series involving one or two Whittaker functions of the second order

Certain bilateral series involving a product of two Whittaker functions  $W_{k, \mu}$  and one Whittaker function are evaluated as in Theorem 1. In the following,  $\Gamma(z)$  and  $B(\alpha, \beta)$  are the familiar Gamma and Beta functions (see, *e.g.*, [13, Section 1.1]), respectively,  $J_\nu(z)$  and  $K_\nu(z)$  are Bessel functions of the first kind and modified Bessel functions (Macdonald functions) (see, *e.g.*, [4, Sections 8.4-8.5, p. 900]), respectively,  $W_{\lambda, \mu}(z)$  are the Whittaker functions (see, *e.g.*,

[4, Sections 9.22-9.23, p. 1014]), and  ${}_2F_1$  is the hypergeometric function (see, e.g., [13, Section 1.5]).

**Theorem 1.** *Let  $\theta \neq \pi n$ , where  $n \in \mathbb{Z}$  with  $n \equiv 1 \pmod{2}$ . Then the following formulas hold true:*

- For  $-1 < \Re(\sigma) < 0$ ,

$$\sum_{k \in \mathbb{Z}} \exp(-ik\theta) \{\Gamma(1 + \sigma + k \operatorname{sign} \hat{\lambda}) \Gamma(-\sigma - k \operatorname{sign} \lambda)\}^{-1} \\ \times W_{k \operatorname{sign} \hat{\lambda}, \sigma + \frac{1}{2}}(2|\hat{\lambda}|) W_{-k \operatorname{sign} \lambda, \sigma + \frac{1}{2}}(2|\lambda|) = \Theta(\operatorname{sign}(\lambda \hat{\lambda})),$$

where

$$\Theta(1) = -\frac{1}{\pi} \sec^2 \frac{\theta}{2} (\lambda \hat{\lambda})^{\frac{1}{2}} \sin(\pi\sigma) K_{2\sigma+1} \left( 2 \left| \sec \frac{\theta}{2} \sqrt{\lambda \hat{\lambda}} \right. \right)$$

and

$$\Theta(-1) = \frac{1}{2} \sec^2 \frac{\theta}{2} |\lambda \hat{\lambda}|^{\frac{1}{2}} \cos(\pi\sigma) \\ \times \left[ J_{-2\sigma-1} \left( 2 \left| \sec \frac{\theta}{2} \sqrt{|\lambda \hat{\lambda}|} \right. \right) - J_{2\sigma+1} \left( 2 \left| \sec \frac{\theta}{2} \sqrt{|\lambda \hat{\lambda}|} \right. \right) \right].$$

- For  $-\frac{1}{2} < \Re(\sigma) < 0$  and  $\lambda \neq 0$ ,

$$\sum_{k \in \mathbb{Z}} \exp(-ik\theta) \{\Gamma(-\sigma - k \operatorname{sign} \lambda) B(k - \sigma, -k - \sigma)\}^{-1} W_{-k \operatorname{sign} \lambda, \sigma + \frac{1}{2}}(2|\lambda|) \\ = 2^{-\sigma-1} \pi^{-1} (-2\sigma - 1) |\lambda|^{-\sigma} \left( \cos \frac{\theta}{2} \right)^{4\sigma} \sin(\pi\sigma) \Gamma(2\sigma + 1).$$

- For  $-\frac{1}{2} < \Re(\sigma) < 0$  and  $\hat{\lambda} \neq 0$ ,

$$\sum_{k \in \mathbb{Z}} \exp(-ik\theta) \{\Gamma(1 + \sigma + k \operatorname{sign} \hat{\lambda}) B(k - \sigma, -k - \sigma)\}^{-1} W_{k \operatorname{sign} \lambda, \sigma + \frac{1}{2}}(2|\hat{\lambda}|) \\ = 2^{-\sigma-1} \pi^{-1} (2\sigma + 1) |\hat{\lambda}|^{\sigma+1} \left( \sec \frac{\theta}{2} \right)^{4\sigma+4} \sin(\pi\sigma) \Gamma(-2\sigma - 1).$$

*Proof.* Let us consider the matrix elements  $t_{\lambda \hat{\lambda}}(g)$  of the representation  $T^\bullet$  with respect to the basis  $B_2^\bullet$ . Since

$$(2.1) \quad T^\bullet(g)[f_\lambda^{*\bullet}(x)] = \int_{-\infty}^{+\infty} t_{\lambda \hat{\lambda}}(g) f_{\hat{\lambda}}^{*\bullet}(x) d\hat{\lambda},$$

we have

$$F_i(T^\bullet(g)[f_\lambda^{*\bullet}], f_\tau^*) = \int_{-\infty}^{+\infty} t_{\lambda \mu}(g) F_j(f_{\hat{\lambda}}^{*\bullet}, f_\tau^*) d\hat{\lambda}$$

for any  $i$  and  $j$ . Taking

$$F_3(f_{\hat{\lambda}}^{*\bullet}, f_\tau^*) = \int_{-\infty}^{+\infty} \exp(\mathbf{i}[\mu + \tau]y) dy = \delta(\mu + \tau),$$

where  $\delta(\mu + \tau)$  is the  $(-\tau)$ -delayed Dirac delta function, we obtain

$$F_i(T^\bullet(g)[f_\lambda^{\bullet\bullet}], f_\tau^*) = \int_{-\infty}^{+\infty} t_{\lambda\mu}(g) \delta(\mu + \tau) d\hat{\lambda} = 2\pi t_{\hat{\lambda}, -\tau}(g).$$

Thus we have

$$(2.2) \quad t_{\lambda\hat{\lambda}}(g) = \frac{1}{2\pi} F_i(T^\bullet(g)[f_\lambda^{\bullet\bullet}], f_{-\hat{\lambda}}^*).$$

By using the connection  $f_\lambda^{\bullet\bullet}(x) = \sum_{k \in \mathbb{Z}} c_{\lambda k} f_k^\bullet(x)$  between the bases  $B_1^\bullet$  and  $B_3^\bullet$ , we have

$$t_{\lambda\hat{\lambda}}(g) = \frac{1}{2\pi} F_i \left( \sum_{k \in \mathbb{Z}} c_{\lambda k} T^\bullet(g)[f_k^\bullet], \sum_{\bar{k} \in \mathbb{Z}} c_{-\bar{k}\hat{\lambda}} f_{\bar{k}} \right).$$

Let  $g_1$  be the rotation through an angle  $\theta$  in the plane generated by axes  $Ox_2$  and  $Ox_3$ . The matrix representation of  $g_1$  is

$$g_1 \longleftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix}.$$

$g_1 \in H_1$  and  $f_k$  is the eigenfunction of the operator  $T^\bullet(g_1)$  associated with the eigenvalue  $\exp(-ik\theta)$ . The matrix elements of the linear operator that transforms  $B_1^\bullet$  into  $B_2^\bullet$  were computed for  $\lambda \neq 0$  in [15] and for  $\lambda = 0$  in [12]:

$$c_{\lambda k} \equiv c_{\lambda k}(\sigma) = \begin{cases} |\lambda|^{-\sigma-1} \{\Gamma(-\sigma - k \operatorname{sign} \lambda)\}^{-1} W_{-k \operatorname{sign} \lambda, \sigma + \frac{1}{2}}(2|\lambda|), & \text{if } \lambda \neq 0, \\ 2^{\sigma+1} \{(-2\sigma - 1) B(k - \sigma, -k - \sigma)\}^{-1}, & \text{if } \lambda = 0. \end{cases}$$

Thus, choosing  $i = 1$  in (2.2) and taking the basis  $B_1^\bullet$  which is orthonormal with respect to the bilinear functional  $\frac{1}{2\pi} F_1$ , we obtain

$$t_{\lambda\hat{\lambda}}(g_1) = |\hat{\lambda}|^\sigma |\lambda|^{-\sigma-1} \sum_{k \in \mathbb{Z}} \exp(-ik\theta) \{\Gamma(1 + \sigma + k \operatorname{sign} \hat{\lambda})\}^{-1} \\ \times \{\Gamma(-\sigma - k \operatorname{sign} \lambda)\}^{-1} W_{k \operatorname{sign} \hat{\lambda}, \sigma + \frac{1}{2}}(2|\hat{\lambda}|) W_{-k \operatorname{sign} \lambda, \sigma + \frac{1}{2}}(2|\lambda|).$$

Let us represent  $g_1$  as a product

$$g_1 = h_1 \operatorname{diag}(1, -1, -1) h_2 h_1,$$

where  $h_1$  belongs to the maximal nilpotent subgroup  $H_3$  and depends on the only parameter  $\theta^*$ , and  $h_2$  is the hyperbolic rotation through an angle  $\theta^{**}$  in the plane generated by axes  $Ox_1$  and  $Ox_2$ . Taking the matrix representations

$$h_1 \longleftrightarrow \begin{pmatrix} 1 + \frac{\theta^{*2}}{2} & \frac{\theta^{*2}}{2} & \theta^* \\ -\frac{\theta^{*2}}{2} & 1 - \frac{\theta^{*2}}{2} & -\theta^* \\ \theta^* & \theta^* & 1 \end{pmatrix}, \quad h_2 \longleftrightarrow \begin{pmatrix} \cosh \theta^{**} & \sinh \theta^{**} & 0 \\ \sinh \theta^{**} & \cosh \theta^{**} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we have

$$\begin{cases} \frac{1}{2}\theta^{*4} \exp(-\theta^{**}) + \theta^{*2} \exp(-\theta^{**}) + \cosh \theta^{**} - \theta^{*2} = 1, \\ \frac{1}{2}\theta^{*4} \exp(-\theta^{**}) + \sinh \theta^{**} - \theta^{*2} = 0, \\ \theta^{*3} \exp(-\theta^{**}) + \theta^* \exp(-\theta^{**}) - \theta^* = 0, \\ \theta^{*2} + \theta^{*2} \exp(-\theta^{**}) - \frac{1}{2}\theta^{*4} \exp(-\theta^{**}) - \cosh \theta^{**} = \cos \theta, \\ \theta^{*3} \exp(-\theta^{**}) - \theta^* \exp(-\theta^{**}) - \theta^* = -\sin \theta, \end{cases}$$

that is,  $\frac{2\theta^*}{\theta^{*2}+1} = \sin \theta$  and, therefore,  $\theta^* = \tan \frac{\theta}{2}$  and  $\exp(-\theta^{**}) = (\theta^{*2}+1)^{-1} = \cos^2 \frac{\theta}{2}$ . Since  $T^\bullet$  is a group homomorphism and for any  $g \in SO(2,1)$  the functional  $F_3$  is invariant with respect to the pair  $(T(g), T^\bullet(g))$  of operators arising in the corresponding spaces [12], we derive

$$\begin{aligned} t_{\lambda\hat{\lambda}}(g_1) &= \frac{1}{2\pi} F_3(T^\bullet(h_2 h_1) [f_{\lambda^*}^\bullet], T(\text{diag}(1, -1, -1) h_1^{-1}) [f_{-\hat{\lambda}}^*]) \\ &= \frac{1}{2\pi} \sec^2 \frac{\theta}{2} \int_{-\infty}^{+\infty} y^{2\sigma} \exp\left(\mathbf{i} \sec^2 \frac{\theta}{2} \left[\lambda y - \frac{\hat{\lambda}}{y}\right]\right) dy \end{aligned}$$

and use the following formulas:

- $\lambda\hat{\lambda} > 0$  (see, *e.g.*, [7, Entry 2.5.24.4]),

$$\int_0^{+\infty} x^{\alpha-1} \cos\left(ax - \frac{b}{x}\right) dx = 2 \left(\frac{b}{a}\right)^{\frac{\alpha}{2}} \cos\left(\frac{\alpha\pi}{2}\right) K_\alpha(2\sqrt{ab})$$

$(a, b > 0; |\Re(\alpha)| < 1).$

- $\lambda\hat{\lambda} < 0$  (see, *e.g.*, [7, Entry 2.5.24.7]).

$$\int_0^{+\infty} x^{\alpha-1} \cos\left(ax + \frac{b}{x}\right) dx = \frac{\pi}{2} \left(\frac{b}{a}\right)^{\frac{\alpha}{2}} \sin\left(\frac{\alpha\pi}{2}\right) [J_{-\alpha}(2\sqrt{ab}) - J_\alpha(2\sqrt{ab})]$$

$(a, b > 0; |\Re(\alpha)| < 1).$

- $\lambda \neq 0$  and  $\hat{\lambda} = 0$  (see, *e.g.*, [7, Entry 2.5.3.10]).

$$(2.3) \quad \int_0^{+\infty} x^{\alpha-1} \cos(bx) dx = b^{-\alpha} \cos\left(\frac{\alpha\pi}{2}\right) \Gamma(\alpha)$$

$(b > 0; 0 < \Re(\alpha) < 1).$

- $\lambda = 0$  and  $\hat{\lambda} \neq 0$ .

Substitute  $x = z^{-1}$  and use the formula (2.3).

By comparing the results for  $t_{\lambda\hat{\lambda}}(g_1)$  and  $t_{\lambda\hat{\lambda}}(h_1 \text{diag}(1, -1, -1) h_2 h_1)$ , we completes the proof.  $\square$

### 3. Some integrals involving a product of Bessel and Whittaker functions

Throughout this section, we assume that  $\theta \neq \pi n$  ( $n \in \mathbb{Z}$  with  $n \equiv 1 \pmod{2}$ ). Here we establish a formula for certain integrals involving Bessel functions of the first kind, Macdonald functions, and Whittaker functions, which is asserted by Theorem 2.

**Theorem 2.** *For  $-1 < \Re(\sigma) < 0$  and  $\lambda > 0$ , the following formula holds true:*

$$\begin{aligned} & \frac{1}{2} \cos(\pi\sigma) \{\Gamma(k - \sigma)\}^{-1} \int_0^{+\infty} \left[ J_{-2\sigma-1} \left( 2 \left| \sec \frac{\theta}{2} \right| \sqrt{\lambda \hat{\lambda}} \right) \right. \\ & \quad \left. - J_{2\sigma+1} \left( 2 \left| \sec \frac{\theta}{2} \right| \sqrt{\lambda \hat{\lambda}} \right) \right] W_{k, \sigma + \frac{1}{2}}(2\hat{\lambda}) d\hat{\lambda} \\ & \quad - \frac{1}{\pi} \sin(\pi\sigma) \Gamma(-k - \sigma) \int_0^{+\infty} K_{2\sigma+1} \left( 2 \left| \sec \frac{\theta}{2} \right| \sqrt{\lambda \hat{\lambda}} \right) W_{-k, \sigma + \frac{1}{2}}(2\hat{\lambda}) d\hat{\lambda} \\ & = \lambda^{-\frac{1}{2}} \left( \cos \frac{\theta}{2} \right)^2 \exp(-ik\theta) \{\Gamma(-k - \sigma)\}^{-1} W_{-k, \sigma + \frac{1}{2}}(2\lambda). \end{aligned}$$

*Proof.* Taking the connection between the bases  $B_1^\bullet$  and  $B_2^\bullet$ , we rewrite (2.1) for the case  $g = g_1$  in the following manner:

$$(3.1) \quad T^\bullet(g_1)[f_\lambda^{\bullet\bullet}(x)] = \sum_{k \in \mathbb{Z}} \left( \int_{-\infty}^{+\infty} t_{\lambda \hat{\lambda}}(g_1) c_{\hat{\lambda}k} d\hat{\lambda} \right) f_k^\bullet(x).$$

Since  $f_k^\bullet$  is an eigenfunction of the operator  $T^\bullet(g_1)$  associated with the eigenvalue  $\exp(-ik\theta)$ , we have

$$(3.2) \quad T^\bullet(g_1)[f_\lambda^{\bullet\bullet}(x)] = \sum_{k \in \mathbb{Z}} c_{\lambda k} \exp(-ik\theta) f_k^\bullet(x).$$

By using formulas for  $c_{\lambda k}$  and  $t_{\lambda \hat{\lambda}}(g_1)$  given in the proof of Theorem 1, from (3.1) and (3.2) we derive the desired result.  $\square$

A similar integral formula can be established for the case  $\lambda < 0$ . For the case  $\lambda = 0$ , we obtain an integral formula asserted by Theorem 3.

**Theorem 3.** *For  $-\frac{1}{2} < \Re(\sigma) < 0$ , the following formula holds true:*

$$(3.3) \quad \begin{aligned} & \int_{-\infty}^{+\infty} \hat{\lambda}^{2\sigma+1} |\hat{\lambda}|^{-\sigma-1} \{\Gamma(-\sigma - k \operatorname{sign} \hat{\lambda})\}^{-1} W_{-k \operatorname{sign} \hat{\lambda}, \sigma + \frac{1}{2}}(2|\hat{\lambda}|) d\hat{\lambda} \\ & = 2^{\sigma+1} \frac{\pi(2\sigma+1)}{\Gamma(-2\sigma-1) \Gamma(k-\sigma) \Gamma(-k-\sigma) \cos(\sigma\pi)}. \end{aligned}$$

*Proof.* The formula (3.3) follows from (3.1) and (3.2), where  $g_1$  is the identity matrix, that is,  $\theta = 0$ .  $\square$

Setting here  $k = 0$  and using a known formula (see, *e.g.*, [4, Entry 8.476.5]):

$$(3.4) \quad K_\nu(e^{m\pi i} z) = e^{-m\nu\pi i} K_\nu(z) - i\pi \frac{\sin(m\nu\pi)}{\sin(\nu\pi)} I_\nu(z),$$

we obtain

$$\begin{aligned} & \int_0^{+\infty} \hat{\lambda}^{\sigma+\frac{1}{2}} \left( K_{\sigma+\frac{1}{2}}(\hat{\lambda}) + i\pi \cos(\sigma\pi) I_{\sigma+\frac{1}{2}}(\hat{\lambda}) \right) d\hat{\lambda} \\ &= 2^\sigma \frac{\pi^{\frac{3}{2}} (2\sigma + 1)}{\Gamma(-2\sigma - 1) \Gamma(-\sigma) \cos(\sigma\pi)}. \end{aligned}$$

#### 4. The cosine-Fourier transform of squared Macdonald function

Here we present a formula for the cosine-Fourier transform of squared Macdonald function given in Theorem 4.

**Theorem 4.** *For  $-1 < \Re(\sigma) < 0$ , the following formula holds true:*

$$\begin{aligned} & \int_0^{+\infty} \left\{ K_{\sigma+\frac{1}{2}}(\lambda) \right\}^2 \cos(b\lambda) d\lambda \\ &= \frac{\pi^2}{4} \operatorname{cosec}(\pi\sigma) \exp(-ib_1 k) \exp(\sigma b_2) {}_2F_1 \left( -\sigma, \frac{1}{2}; 1; 1 - \exp(-2b_2) \right), \end{aligned}$$

where

$$(4.1) \quad b_1 = b_3 - \pi = \arccos \frac{b}{\sqrt{b^2 + 4}} \quad \text{and} \quad \cosh b_2 = \frac{b^2}{2} + 1.$$

*Proof.* Let  $g_2(b) \in H_3$ , that is, there is  $b \in \mathbb{R}$  such that

$$g_2 = \begin{pmatrix} 1 + \frac{b^2}{2} & \frac{b^2}{2} & b \\ -\frac{b^2}{2} & 1 - \frac{b^2}{2} & -b \\ b & b & 1 \end{pmatrix}.$$

Let

$$f_k^\bullet(x) = \int_{-\infty}^{+\infty} \tilde{c}_{k\lambda} f_\lambda^{\bullet\bullet}(x) d\lambda \quad \text{and} \quad T^\bullet(g)[f_k^\bullet(x)] = \sum_{k \in \mathbb{Z}} t_{k\hat{k}}(g) f_{\hat{k}}^\bullet(x).$$

We readily obtain

$$\tilde{c}_{k\lambda}(\sigma) = \frac{1}{2\pi} F_i(f_k^\bullet, f_{-\lambda}^*) = c_{-\lambda, -k}(-\sigma - 1)$$

and, therefore,

$$(4.2) \quad \begin{aligned} t_{k\hat{k}}(g_2) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{c}_{k\lambda} \tilde{c}_{\hat{k}, -\lambda} \exp(-ib\lambda) d\lambda \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} c_{-\lambda, -k}(-\sigma - 1) c_{\lambda, -\hat{k}}(\sigma) \exp(-ib\lambda) d\lambda. \end{aligned}$$

Let us represent  $g_2$  as a product  $g_2 = \tilde{h}_1(b_1)\tilde{h}_2(b_2)\tilde{h}_3(b_3)$ , where  $\tilde{h}_1, \tilde{h}_3 \in H_1$  and  $\tilde{h}_2$  belongs to the subgroup of hyperbolic rotations in the plane generated by axes  $Ox_1$  and  $Ox_2$ . By solving the system of equations

$$\begin{pmatrix} 1 + \frac{b^2}{2} & \frac{b^2}{2} & b \\ -\frac{b^2}{2} & 1 - \frac{b^2}{2} & -b \\ b & b & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos b_1 & -\sin b_1 \\ 0 & \sin b_1 & \cos b_1 \end{pmatrix} \begin{pmatrix} \cosh b_2 & \sinh b_2 & 0 \\ \sinh b_2 & \cosh b_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos b_3 & -\sin b_3 \\ 0 & \sin b_3 & \cos b_3 \end{pmatrix},$$

we obtain the relations (4.1), that is,

$$g_2 = \tilde{h}_1(b_1)\tilde{h}_2(b_2) \text{diag}(1, -1, -1) \tilde{h}_1(b_1).$$

Since  $T^\bullet$  is a group homomorphism,  $(\tilde{h}_i(b_i))^{-1} = \tilde{h}_i(-b_i)$  for any  $i$ , and  $F_1$  is invariant with respect to the pair  $(T^\bullet, T)$ , we have

$$\begin{aligned} t_{k\hat{k}}(g_2) &= \frac{1}{2\pi} F_1(T^\bullet(h_1)T^\bullet(h_2)[f_k^\bullet], T(h_3^{-1})T(\text{diag}(1, -1, -1))[f_{-\hat{k}}]) \\ &= \frac{1}{2\pi \exp(\mathbf{i}(\pi + b_1)\hat{k})} \int_{-\pi}^{\pi} \exp(\mathbf{i}(k + \hat{k})\varphi) (\cosh b_2 - \sinh b_2 \cos \varphi)^{\sigma - k} \\ &\quad \times (\cosh b_2 \cos \varphi - \sinh b_2 + \mathbf{i} \sin \varphi)^k d\varphi. \end{aligned}$$

By using ‘universal’ substitution  $t = \tan \frac{\varphi}{2}$ , we obtain

$$(4.3) \quad t_{00}(g_2) = \frac{\exp(\sigma b_2)}{2\pi^2 \exp(\mathbf{i}(\pi + b_1)\hat{k})} \int_0^{+\infty} (1 + t^2)^{-\sigma - 1} (\exp(-2b_2) + t^2)^\sigma dt.$$

The integral (4.3) can be evaluated by using the following known formula (see [4, Entry 3.259.3]):

$$\begin{aligned} &\int_0^{+\infty} x^{\lambda - 1} (1 + \alpha x^p)^{-\mu} (1 + \beta x^p)^{-\nu} dx \\ &= \frac{1}{p} \alpha^{-\frac{\lambda}{p}} \mathbf{B}\left(\frac{\lambda}{p}, \mu + \nu - \frac{\lambda}{p}\right) {}_2F_1\left(\nu, \frac{\lambda}{p}; \mu + \nu; 1 - \frac{\beta}{\alpha}\right) \end{aligned}$$

$$(|\arg(\alpha)|, |\arg(\beta)| < \pi, p > 0, 0 < \Re(\lambda) < 2\Re(\mu + \nu)).$$

Finally comparing formulas (4.2) for  $k = \hat{k} = 0$  and (4.3), we obtain the desired result.  $\square$

## 5. Concluding remarks

Theorem 2 is a generalization of [12, Theorem 1] which was derived by using the restriction of the representation  $T^\bullet$  to the diagonal matrix  $\text{diag}(1, -1, -1)$ .

Theorem 4 can be considered as a new approach that differs from the formula (see [8, Entry 2.16.36.2]):

$$(5.1) \quad \int_0^{+\infty} \cos(ax) K_\nu(bx) K_\nu(cx) dx = \frac{\pi^2}{4\sqrt{bc} \cos(\pi\nu)} P_{\nu-\frac{1}{2}} \left( \frac{a^2 + b^2 + c^2}{2bc} \right) \\ \left( \Re(b+c) > |\Im(a)|, \Re(\nu) < \frac{1}{2} \right),$$

which does not use functions  $a_1(a)$  and  $a_2(a)$  in the right side.

The approach employed in this paper looks a good starting point for further research: By using various decompositions of the group  $SO(2, 1)$ , formulas of the transformations  $B_1 \leftrightarrow$  ‘hyperbolic basis’  $B_3$  and  $B_2 \leftrightarrow B_3$ , formulas for the matrix elements of the operator  $T^\bullet(h)$  with respect to the  $B_1^\bullet$  or  $B_2^\bullet$ , where  $h$  belongs to the transitive subgroup of a hyperbolic section of  $\Lambda$ , and formulas for the matrix elements of the  $T^\bullet(g_1)$  and  $T^\bullet(g_2)$  with respect to the  $B_3^\bullet$ , it seems possible to obtain new formulas for the special functions related to the group  $SO(2, 1)$ .

Bessel functions play an important role, in particular, in investigating solutions of various differential equations, and are associated with a wide range of problems in many research areas of mathematical physics. So, Bessel functions, and their extensions and variants have been investigated by many authors (see, e.g., [2, 3, 5, 6, 14]).

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JUNESANG CHOI  
DEPARTMENT OF MATHEMATICS  
DONGGUK UNIVERSITY  
GYEONGJU 38066, KOREA  
*E-mail address:* junesang@mail.dongguk.ac.kr

I. A. SHILIN  
DEPARTMENT OF HIGHER MATHEMATICS  
NATIONAL RESEARCH UNIVERSITY “MPEI”  
KRASNOKAZARMENNAYA 14, MOSCOW 111250, RUSSIA  
AND  
DEPARTMENT OF ALGEBRA  
MOSCOW STATE PEDAGOGICAL UNIVERSITY  
MALAYA PIROGOVSKAYA 1, MOSCOW 119991, RUSSIA  
*E-mail address:* ilyashilin@li.ru; shilinia@mpei.ru