

SOLUTION TO $\bar{\partial}$ -PROBLEM WITH SUPPORT CONDITIONS IN WEAKLY q -CONVEX DOMAINS

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ABSTRACT. Let X be a complex manifold of dimension $n \geq 2$ and let $\Omega \Subset X$ be a weakly q -convex domain with smooth boundary. Assume that E is a holomorphic line bundle over X and $E^{\otimes m}$ is the m -times tensor product of E for positive integer m . If there exists a strongly plurisubharmonic function on a neighborhood of $b\Omega$, then we solve the $\bar{\partial}$ -problem with support condition in Ω for forms of type (r, s) , $s \geq q$ with values in $E^{\otimes m}$. Moreover, the solvability of the $\bar{\partial}_b$ -problem on boundaries of weakly q -convex domains with smooth boundary in Kähler manifolds are given. Furthermore, we shall establish an extension theorem for the $\bar{\partial}_b$ -closed forms.

1. Introduction

In [3], Derridj considered the $\bar{\partial}$ -problem with exact support by using Carleman type estimates for smooth domains with plurisubharmonic defining functions. In [14], Shaw has obtained a solution to this problem in a pseudo-convex domain in \mathbb{C}^n with C^1 smooth boundary. Cao-Shaw-Wang [2] have obtained a solution to this problem in a locally Stein domain of the complex projective space. On strongly q -convex (or concave) domains, this problem has been studied by Sambou in [13]. In [10], the author studied this problem on a weakly q -pseudoconvex domain with C^1 -smooth boundary in \mathbb{C}^n and extended this result to a Stein manifold in [11]. Also Saber in [12], studies this problem on a weakly pseudoconvex domain with smooth boundary for forms in $E^{\otimes m}$ under the positivity condition on E . The purpose of this paper is to extend this result to a weakly q -convex domain for forms of type (r, s) , $s \geq q$ with values in $E^{\otimes m}$ and under a different condition. More precisely, we prove the following result:

Theorem 1. *Let X be a complex manifold of dimension $n \geq 2$ and let $\Omega \Subset X$ be a weakly q -convex domain with smooth boundary in X . Assume that E is a holomorphic line bundle over X and $E^{\otimes m}$ is the m -times tensor product of E for positive integer m . Suppose that there exists a strongly plurisubharmonic*

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function on a neighborhood of the boundary of Ω . Then, for $\alpha \in L_{r,s}^2(X, E^{\otimes m})$, $\text{supp } \alpha \subset \bar{\Omega}$, with $s \geq q$, satisfying $\bar{\partial}\alpha = 0$ in the distribution sense in X , there exists $u \in L_{r,s-1}^2(X, E^{\otimes m})$, $\text{supp } u \subset \bar{\Omega}$ such that $\bar{\partial}u = \alpha$ in the distribution sense in X .

Applications to the solvability of the $\bar{\partial}_b$ -problem on boundaries of weakly q -convex domains with smooth boundary in Kähler manifolds are given. Furthermore, we shall establish an extension theorem for the $\bar{\partial}_b$ -closed forms.

2. Notation and preliminaries

Let X be an n -dimensional complex manifold. Let Ω be an open subset of X and ρ be its defining function. Let E be a holomorphic line bundle over X and let E^* be its dual. Let $\{U_j\}_{j \in J}$ be an open covering of X such that $E|_{U_j}$ is trivial, namely $\pi^{-1}(U_j) = U_j \times \mathbb{C}$, and $(z_j^1, z_j^2, \dots, z_j^n)$ be local coordinates on U_j . Let $\{e_{jk}\}$ be a system of transition functions of E with respect to a covering $\{U_j\}_{j \in J}$. An (r, s) forms $\alpha = \{\alpha_j\}$ on X can be expressed as follows:

$$\alpha_j = \sum'_{C_r, D_s} \alpha_{j, C_r, \bar{D}_s} dz_j^{C_r} \wedge d\bar{z}_j^{\bar{D}_s},$$

where $C_r = (c_1, \dots, c_r)$ and $D_s = (d_1, \dots, d_s)$ are multiindices and so on. The notation \sum' means the summation over strictly increasing multiindices. Let

$$ds^2 = \sum_{\alpha, \beta=1}^n g_{j, \alpha \bar{\beta}}(z) dz_j^\alpha d\bar{z}_j^\beta$$

be a hermitian metric on X . We associate to ds^2 , the $(1, 1)$ differential form $\omega = \frac{\sqrt{-1}}{2} \sum_{\alpha, \beta=1}^n g_{j, \alpha \bar{\beta}}(z) dz_j^\alpha \wedge d\bar{z}_j^\beta$. If $d\omega = 0$, the metric ds^2 is called Kähler metric and ω is called the Kähler form associated to the metric ds^2 . A complex manifold X is called Kähler manifold if we can define Kähler metric on it. Let $h = \{h_j\}$ be a hermitian metric of $E = \{e_{jk}\}$ with respect to the covering $\{U_j\}_{j \in J}$ satisfies $h_j = |e_{jk}|^2 h_k$ on $U_j \cap U_k$. For integers $r, s \geq 0$, $m \geq 1$, we define the following notations:

- $C_{r,s}^\infty(\Omega, E^{\otimes m})$: the complex vector space of $E^{\otimes m}$ -valued differential forms of class C^∞ and of type (r, s) on Ω .
- $C_{r,s}^\infty(\bar{\Omega}, E^{\otimes m})$: the subspace of $C_{r,s}^\infty(\Omega, E^{\otimes m})$ whose elements can be extended smoothly up to $b\Omega$.
- $\mathcal{D}_{r,s}(\Omega, E^{\otimes m})$: the space of $E^{\otimes m}$ -valued differential forms of type (r, s) with compact support in Ω .
- The operator $\star : C_{r,s}^\infty(X, E^{\otimes m}) \rightarrow C_{n-s, n-r}^\infty(X, E^{\otimes m})$ is the Hodge star operator.
- The operator $\#_{E^{\otimes m}} : C_{r,s}^\infty(X, E^{\otimes m}) \rightarrow C_{s,r}^\infty(X, E^{*\otimes m})$ is defined by $\#_{E^{\otimes m}} \alpha = h^m \bar{\alpha}$, which commutes with the Hodge star operator, and

the corresponding operator $\#_{E^* \otimes m} : C_{r,s}^\infty(X, E^{* \otimes m}) \longrightarrow C_{s,r}^\infty(X, E^{\otimes m})$ is defined by

$$\#_{E^* \otimes m} \alpha = \overline{(h^m)^* \bar{\alpha}} = \overline{i(h^m)^{-1} \bar{\alpha}} = h^{-m} \bar{\alpha} = \#_{E^{\otimes m}}^{-1} \alpha.$$

Thus $\#_{E^* \otimes m} \alpha = \#_{E^{\otimes m}}^{-1} \alpha$.

- $\mathcal{B}_{r,s}(\bar{\Omega}, E^{\otimes m}) = \{\alpha \in C_{r,s}^\infty(\bar{\Omega}, E^{\otimes m}) : \star \#_{E^{\otimes m}} \alpha|_{b\Omega} = 0\}$.
- dV is the volume element with respect to ds^2 .
- $\bar{\partial} : C_{r,s-1}^\infty(\Omega, E^{\otimes m}) \longrightarrow C_{r,s}^\infty(\Omega, E^{\otimes m})$ is the Cauchy-Riemann operator and ϑ_m its formal adjoint.
- $\text{dom}(\bar{\partial}, E^{\otimes m})$, $\text{range}(\bar{\partial}, E^{\otimes m})$ and $\ker(\bar{\partial}, E^{\otimes m})$ is the domain, the range and the kernel of $\bar{\partial}$, respectively.
- $H^{r,s}(X, E^{\otimes m}) = \frac{C_{r,s}^\infty(X, E^{\otimes m}) \cap \ker(\bar{\partial}, E^{\otimes m})}{\bar{\partial}(C_{r,s-1}^\infty(X, E^{\otimes m}))}$.
- $C_{r,s}^\infty(b\Omega, E^{\otimes m}) = C_{r,s}^\infty(\bar{\Omega}, E^{\otimes m}) / \mathcal{D}_{r,s}(\Omega, E^{\otimes m})$.
- We put

$$\pi_{r,s} : C_{r,s}^\infty(\bar{\Omega}, E^{\otimes m}) \longrightarrow C_{r,s}^\infty(b\Omega, E^{\otimes m}),$$

$$\sigma_{r,s} : \oplus_{p,q} C_{(p,q)}^\infty(\bar{\Omega}, E^{\otimes m}) \longrightarrow C_{r,s}^\infty(b\Omega, E^{\otimes m})$$

the natural projections. For simplicity we put

$$\pi_{r,s}(u) = u|_{b\Omega}.$$

- The $\bar{\partial}_b$ -operator

$$\bar{\partial}_b : C_{r,s}^\infty(b\Omega, E^{\otimes m}) \longrightarrow C_{r,s+1}^\infty(b\Omega, E^{\otimes m})$$

is defined by

$$\bar{\partial}_b = \sigma_{r,s+1} \circ d \circ (\pi_{r,s})^{-1}.$$

Differentiable functions f on $b\Omega$ satisfying $\bar{\partial}_b f = 0$ are called \mathcal{CR} functions on $b\Omega$. It is clear that f is \mathcal{CR} if there exists a differentiable function F on $\bar{\Omega}$ satisfying $F|_{b\Omega} = f$ and $\bar{\partial}F = 0$. Then the space $C_{r,s}^\infty(b\Omega, E)$ and the operator

$$\bar{\partial}_b : C_{r,s}^\infty(b\Omega, E^{\otimes m}) \longrightarrow C_{r,s+1}^\infty(b\Omega, E^{\otimes m})$$

are defined similarly as above.

For $\alpha, u \in C_{r,s}^\infty(X, E^{\otimes m})$, the local inner product $(\alpha, u)_m$ is defined by

$$(\alpha, u)_m dV = \alpha_j \wedge \star h^m \bar{u}_j = \alpha_j \wedge \star \#_{E^{\otimes m}} u_j.$$

For $\alpha, u \in C_{r,s}^\infty(X, E^{\otimes m})$, the global inner product $\langle \alpha, u \rangle_{m,\Omega}$ and the norm $\|\alpha\|_{m,\Omega}$ are defined by

$$\begin{aligned} \langle \alpha, u \rangle_{m,\Omega} &= \int_{\Omega} \alpha \wedge \star \#_{E^{\otimes m}} u, \\ \|\alpha\|_{m,\Omega}^2 &= \langle \alpha, \alpha \rangle_{m,\Omega}. \end{aligned}$$

For $\alpha \in C_{r,s}^\infty(\Omega, E^{\otimes m})$ and $\eta \in \mathcal{D}_{r,s-1}(\Omega, E^{\otimes m})$, the formal adjoint operator ϑ_m of the operator $\bar{\partial} : C_{r,s-1}^\infty(\Omega, E^{\otimes m}) \rightarrow C_{r,s}^\infty(\Omega, E^{\otimes m})$ is defined by:

$$(2.1) \quad \begin{aligned} \langle \vartheta_m \alpha, \eta \rangle_{m,\Omega} &= \langle \alpha, \bar{\partial} \eta \rangle_{m,\Omega}, \\ \vartheta_m &= -\#_{E^{\otimes m}} \star \bar{\partial} \star \#_{E^{\otimes m}}. \end{aligned}$$

Other notations are the following:

- $L_{r,s}^2(\Omega, E^{\otimes m})$ is the Hilbert space obtained by completing $C_{r,s}^\infty(\bar{\Omega}, E^{\otimes m})$ under the norm $\|\alpha\|_{m,\Omega}^2$.
- $\bar{\partial} : L_{r,s-1}^2(\Omega, E^{\otimes m}) \rightarrow L_{r,s}^2(\Omega, E^{\otimes m})$ is the maximal closed extension of the the Cauchy-Riemann operator $\bar{\partial}$ and $\bar{\partial}_m^*$ its Hilbert space adjoint.
- $\square^m = \square_{r,s}^m = \bar{\partial} \bar{\partial}_m^* + \bar{\partial}_m^* \bar{\partial} : \text{dom}(\square_{r,s}^m, E^{\otimes m}) \rightarrow L_{r,s}^2(\Omega, E^{\otimes m})$ is the Laplace-Beltrami operator \square^m for $E^{\otimes m}$ -valued forms, where

$$\begin{aligned} \text{dom}(\square_{r,s}^m, E^{\otimes m}) &= \{u \in L_{r,s}^2(\Omega, E^{\otimes m}) : u \in \text{dom}(\bar{\partial}, E^{\otimes m}) \cap \text{dom}(\bar{\partial}_m^*, E^{\otimes m}); \\ &\quad \bar{\partial} u \in \text{dom}(\bar{\partial}_m^*, E^{\otimes m}) \text{ and } \bar{\partial}_m^* u \in \text{dom}(\bar{\partial}, E^{\otimes m})\}. \end{aligned}$$

- $\mathcal{H}_{r,s}^m(E^{\otimes m}) = \{u \in \text{dom}(\square_{r,s}^m, E^{\otimes m}) : \bar{\partial} u = \bar{\partial}_m^* u = 0\}$ is a closed subspace of $\text{dom}(\square_{r,s}^m, E^{\otimes m})$ since $\square_{r,s}^m$ is a closed operator.
- The $\bar{\partial}$ -Neumann operator $N^m = N_{r,s}^m : L_{r,s}^2(\Omega, E^{\otimes m}) \rightarrow L_{r,s}^2(\Omega, E^{\otimes m})$ is defined as the inverse of the restriction of $\square_{r,s}^m$ to $(\mathcal{H}_{r,s}^m(E^{\otimes m}))^\perp$, i.e.,

$$N_{r,s}^m u = \begin{cases} 0 & \text{if } u \in \mathcal{H}_{r,s}^m(E^{\otimes m}), \\ v & \text{if } u = \square_{r,s}^m v, \text{ and } v \perp \mathcal{H}_{r,s}^m(E^{\otimes m}). \end{cases}$$

In other words, $N_{r,s}^m u$ is the unique solution v to the equations $\square_{r,s}^m v = u - \Pi_{r,s}^m u$ and $\Pi_{r,s}^m v = 0$, where $\Pi_{r,s}^m : L_{r,s}^2(\Omega, E^{\otimes m}) \rightarrow \mathcal{H}_{r,s}^m(E^{\otimes m})$ is the orthogonal projection from the space $L_{r,s}^2(\Omega, E^{\otimes m})$ onto the space $\mathcal{H}_{r,s}^m(E^{\otimes m})$.

The following proposition is due to Hörmander [7] Propositions 1.2.3 and 1.2.4.

Proposition 1. $\mathcal{B}_{r,s}(\bar{\Omega}, E^{\otimes m})$ is dense in $\text{dom}(\bar{\partial}_m^*, E^{\otimes m})$ (resp. $\text{dom}(\bar{\partial}, E^{\otimes m}) \cap \text{dom}(\bar{\partial}_m^*, E^{\otimes m})$) with respect to the graph norm $(\|\alpha\|_m^2 + \|\bar{\partial}_m^* \alpha\|_m^2)^{1/2}$ (resp. $(\|\alpha\|_m^2 + \|\bar{\partial} \alpha\|_m^2 + \|\bar{\partial}_m^* \alpha\|_m^2)^{1/2}$).

The curvature form associated to the metric h is defined by $\Theta = \{\Theta_j\}$,

$$\Theta_j = \sqrt{-1} \bar{\partial} \partial \log h_j = \sqrt{-1} \sum_{\alpha, \beta=1}^n \Theta_{j\alpha\bar{\beta}} dz_j^\alpha \wedge d\bar{z}_j^\beta,$$

where $\Theta_{j\alpha\bar{\beta}} = -\frac{\partial^2 \log h_j}{\partial z_j^\alpha \partial \bar{z}_j^\beta}$ is the coefficients of the curvature form Θ associated to the metric h .

Definition 1. A holomorphic line bundle $\pi : E \rightarrow X$ is said to be positive on a subset Ω of X if there exist a coordinate cover $\{U_j\}_{j \in J}$ of X such that $\pi^{-1}(U_j)$ are trivial and a hermitian metric $h = \{h_j\}$ along the fibres of E such that $-\log h_j$ is strictly plurisubharmonic on $U_j \cap \Omega$ for any $j \in J$.

By a complex tensor calculus for Kähler manifolds with boundary, one obtain the following theorem (see [15]).

Proposition 2. Let X be a Kähler manifold of dimension n and let $\Omega \Subset X$ be an open subset of X . Assume that E is a holomorphic line bundle over X and $E^{\otimes m}$ is the m -times tensor product of E for positive integer m . Let U^* be a neighborhood of $b\Omega$ and let $\bar{\nabla}$ be the covariant differentiation associated to ds^2 . If $m \geq 1$, we have

$$(2.2) \quad \begin{aligned} & \|\bar{\partial}\alpha\|_m^2 + \|\bar{\partial}_m^* \alpha\|_m^2 \\ &= \|\bar{\nabla}\alpha\|_m^2 + \int_{b\Omega} h_j^m |\text{grad } \rho|^{-1} \sum_{\beta, \gamma=1}^n \frac{\partial^2 \rho}{\partial z^\beta \partial \bar{z}^\gamma} \alpha_{jC_r \bar{B}_{s-1}}^\beta \overline{\alpha_j^{C_r \gamma B_{s-1}}} dS \\ & \quad + \int_m h_j^m \sum_{\beta, \gamma=1}^n s \left(\delta_\tau^\sigma [m\Theta_\alpha^{\bar{\beta}} + R_\alpha^{\bar{\beta}}] - R_{\tau\bar{\alpha}}^{\sigma\bar{\beta}} \right) \times \alpha_{jC_r \bar{B}_{s-1}}^\beta \overline{\alpha_j^{C_r \gamma B_{s-1}}} dV \end{aligned}$$

for $\alpha \in \mathcal{B}_{r,s}(\bar{\Omega}, E^{\otimes m})$, such that $\text{supp } \alpha \Subset U^*$, $r \geq 0$, and $s \geq 1$, where

$$\begin{aligned} \|\bar{\nabla}\alpha\|_m^2 &= \int_\Omega \sum_{\alpha, \beta=1}^n g_j^{\bar{\beta}\alpha} \bar{\nabla}_\beta \alpha_{jC_r \bar{D}_s} \overline{\bar{\nabla}_\alpha \alpha_j^{C_r D_s}} dV, \\ R_{\beta\bar{\nu}\gamma}^\alpha &= -\frac{\partial}{\partial \bar{z}_j^\nu} \left(\sum g_j^{\bar{\sigma}\alpha} \frac{\partial}{\partial z_j^\gamma} g_{j\beta\bar{\sigma}} \right) \text{ is the Riemann curvature tensor,} \\ R_{\alpha\bar{\nu}} &= -\frac{\partial^2}{\partial z_j^\alpha \partial \bar{z}_j^\nu} (\log \det g_{j\alpha\bar{\beta}}) \text{ is the Ricci curvature tensor,} \\ \Theta_{\alpha\bar{\nu}} &= -\frac{\partial^2}{\partial z_j^\alpha \partial \bar{z}_j^\nu} (\log h) \text{ is the curvature tensor of } E \text{ and,} \\ \delta_\tau^\sigma &\text{ denotes the Kronecker's delta.} \end{aligned}$$

For a given boundary point $z_0 \in b\Omega$, we consider a boundary complex frame which means an orthonormal basis $dz^1, \dots, dz^n = \partial\rho$ of $(1,0)$ -forms with C^∞ coefficients on a small neighborhood U of z_0 . We denote by $\left(\frac{\partial^2 \rho(z)}{\partial z_j \partial \bar{z}_j} \right)$, $1 \leq i, j \leq n-1$, the matrix of the Levi form $\partial\bar{\partial}\rho(z)$ in the complex tangential direction at z with respect to the basis dz^1, \dots, dz^n . Let $\lambda_1(z) \leq \dots \leq \lambda_{n-1}(z)$ be the eigenvalues of $\left(\frac{\partial^2 \rho(z)}{\partial z_j \partial \bar{z}_j} \right)$.

Definition 2 (cf. [6]). Let Ω be a smooth domain in \mathbb{C}^n and ρ be its defining function, Ω is weakly q -convex ($q \geq 1$) if at every point $z \in b\Omega$ we have

$$\sum_{|K|} \sum_{j,k} \rho_{jk} u_{jK} \overline{u_{kK}} \geq 0 \text{ for every } (0, q)\text{-form } u = \sum_{|J|=q} u_J \overline{w}^J \text{ such that}$$

$$\sum_{j=1}^n L_j(\rho) u_{jK} = 0 \text{ for all } |K| = q - 1.$$

Lemma 1 (cf. [6]). Let Ω be a smooth domain in \mathbb{C}^n and ρ be its defining function. The following two conditions are equivalent:

- (1) Ω is weakly q -convex.
- (2) For any $z \in b\Omega$ the sum of any q eigenvalues $\rho_{i_1}, \dots, \rho_{i_q}$, with distinct subscripts, of the Levi-form at z satisfies $\sum_{j=1}^q \rho_{i_j} \geq 0$.

Definition 3. $\alpha \in L_{r,s}^2(\Omega, E^{\otimes m})$ is supported in $\overline{\Omega}$ ($\text{supp } \alpha \subset \overline{\Omega}$) or α vanishes to infinite order at the boundary of Ω if α vanishes on $b\Omega$.

To prove the basic estimate (3.6), the following lemma which is Theorem 1.1.3 of [7] is needed.

Lemma 2. Let H_j ($j = 1, 2, 3$) be three Hilbert spaces and $T : H_1 \rightarrow H_2$ and $S : H_2 \rightarrow H_3$ be closed linear operators with dense domains such that $ST = 0$. Assume that for any sequence $\{f_\nu\}$ such that $f_\nu \in H_2 \cap \text{dom } S \cap \text{dom } T$, $\|\alpha_\nu\|_{H_2}^2 \leq 1$ and $\lim_{\nu \rightarrow \infty} \|S\alpha_\nu\|_{H_3}^2 = 0$, $\lim_{\nu \rightarrow \infty} \|T\alpha_\nu\|_{H_1}^2 = 0$, one can choose a strongly convergent subsequence of $\{f_\nu\}$. Then $\text{range}(T)$ is closed and $\mathcal{H}(S)/\text{range}(T)$ is a finite dimensional vector space.

3. Proof of Theorem 1

Let X be an n -dimensional complex manifold and let $\Omega \Subset X$ be a weakly q -convex domain with smooth boundary $b\Omega$. Let $E \rightarrow X$ be a holomorphic line bundle which is positive on a neighborhood V of $b\Omega$. Let $h = \{h_j\}$ be the metric of E on X which gives the positivity of E on V with respect to a suitable covering $\{U_j\}_{j \in J}$ of X . Then the curvature form $\sum_{\alpha, \beta=1}^n \left(-\frac{\partial^2 \log h_j}{\partial z_j^\alpha \partial \overline{z}_j^\beta} \right) dz^\alpha \wedge d\overline{z}^\beta$ of E provides a Kähler metric $d\sigma^2 = \sum_{\alpha, \beta=1}^n \left(-\frac{\partial^2 \log h_j}{\partial z_j^\alpha \partial \overline{z}_j^\beta} \right) dz^\alpha d\overline{z}^\beta$ on V . We may assume that the defining function ρ of $b\Omega$ is constructed from the geodesic distance with respect to the metric $d\sigma^2$ and we obtain the following lemma.

Lemma 3. There exist neighborhoods V and V' of $b\Omega$, a coordinate covering $\{U_j\}_{j \in J}$ of X , a fibre metric $h = \{h_j\}$ of E on X and a hermitian metric $ds^2 = \sum_{\alpha, \beta=1}^n g_{j\alpha\overline{\beta}}(z) dz_j^\alpha d\overline{z}_j^\beta$ on X such that

- 1) $V \Subset V'$ and \overline{V}' is contained in a smooth product neighborhood of $b\Omega$,
- 2) $\pi^{-1}(\overline{U}_j)$ is trivial for any $j \in J$ and $U_j \Subset V$ if $U_j \cap b\Omega \neq \emptyset$,
- 3) E is positive on V' with respect to h ,

4) the restriction of ds^2 onto V' coincides with the Kähler metric $d\sigma^2$.

Under the situation of Lemma 3, one obtain the following estimate (see Appendix II in [16]).

Proposition 3. *There exist a positive constant C not depending on m and a positive integer m_0 such that for any $m \geq m_0$, $r \geq 0$, $s \geq q$, we have*

$$(3.1) \quad \|\bar{\nabla}\alpha\|_{m,\Omega \setminus K}^2 + (m - m_0)\|\alpha\|_{m,\Omega \setminus K}^2 \leq C(\|\bar{\partial}\alpha\|_{m,\Omega}^2 + \|\bar{\partial}_m^*\alpha\|_{m,\Omega}^2 + \|\alpha\|_{m,K}^2),$$

where K is the compact subset of Ω defined by $K = \Omega \setminus (\Omega \cap V)$ and $\bar{\nabla}$ is the covariant differentiation of type $(0, 1)$ associated to the metric ds^2 .

Proof. In the situation of Lemma 3, assume that χ is a C^∞ -function on X such that $\text{supp } \chi \Subset V'$ and $\chi = 1$ on \bar{V} . Then one can apply the formula (2.2) to $\chi\alpha$. Since the third term of the right-hand side of (2.2) is non-negative by the weakly q -convexity of $b\Omega$ for $s \geq q$, one obtain

$$(3.2) \quad \|\bar{\nabla}(\chi\alpha)\|_m^2 + \int_m h^m \sum_{\beta,\gamma=1}^n s \left(\delta_\tau^\sigma [m\Theta_\alpha^{\bar{\beta}} + R_\alpha^{\bar{\beta}}] - rR_{\tau\bar{\alpha}}^{\sigma\bar{\beta}} \right) \times (\chi\alpha)_{j,C_r\bar{B}_{s-1}}^\beta \overline{(\chi\alpha)_j^{\bar{C}_r\gamma B_{s-1}}} dV \leq \|\bar{\partial}(\chi\alpha)\|_m^2 + \|\bar{\partial}_m^*(\chi\alpha)\|_m^2.$$

Since the integrand of the first term of the left-hand side of (3.2) is nonnegative on V' , one obtain

$$(3.3) \quad \|\bar{\nabla}\alpha\|_{m,\Omega \setminus K}^2 \leq \|\bar{\nabla}(\chi\alpha)\|_m^2,$$

where $K = \Omega \setminus (\Omega \cap V)$. From the construction of ds^2 , the matrix $(g_{j\alpha\bar{\beta}})$ coincides with the one $(\Theta_{\alpha\bar{\beta}})$ at each point of V' . Hence

$$\Theta_{\alpha\bar{\beta}}^{\bar{\beta}} = \sum_{\gamma=1}^n g_j^{\bar{\beta}\gamma} \Theta_{\gamma\bar{\alpha}} = \delta_\alpha^\beta.$$

Also, at each point of $\text{supp } \chi$, there exists a positive constant C not depending on m such that the hermitian form

$$\sum_{\beta,\gamma=1}^n s \left(\delta_\tau^\sigma R_\alpha^{\bar{\beta}} - rR_{\tau\bar{\alpha}}^{\sigma\bar{\beta}} \right) (\chi\alpha)_{j,C_{r-1}\bar{D}_{s-1}}^\beta \overline{(\chi\alpha)_j^{\bar{C}_{r-1}\alpha D_{s-1}}}$$

is greater than

$$-C \sum (\chi\alpha)_{j,C_r\bar{D}_s}^\beta \overline{(\chi\alpha)_j^{\bar{C}_r D_s}}.$$

Setting $m_0 = [C] + 1$ for every $m \geq m_0$, one obtain

$$(3.4) \quad \begin{aligned} (m - m_0)\|\alpha\|_{m,\Omega \setminus K}^2 &\leq (m - m_0)\|\chi\alpha\|_m^2 \\ &\leq \int_m h^m \sum_{\beta,\gamma=1}^n s \left(\delta_\tau^\sigma [m\Theta_\alpha^{\bar{\beta}} + R_\alpha^{\bar{\beta}}] - rR_{\tau\bar{\alpha}}^{\sigma\bar{\beta}} \right) \\ &\quad \times (\chi\alpha)_{j,C_r\bar{B}_{s-1}}^\beta \overline{(\chi\alpha)_j^{\bar{C}_r\gamma B_{s-1}}} dV. \end{aligned}$$

Moreover we have

$$\begin{aligned}
(3.5) \quad & \|\bar{\partial}(\chi\alpha)\|_m^2 + \|\bar{\partial}_m^*(\chi\alpha)\|_{m,\Omega}^2 \\
& \leq 2(\|\bar{\partial}\chi \wedge \alpha\|_m^2 + \|\bar{\partial}\chi \wedge \star\alpha\|_m^2 + \|\chi\bar{\partial}\alpha\|_m^2 + \|\chi\bar{\partial}_m^*\alpha\|_m^2) \\
& \leq C(\|\bar{\partial}\alpha\|_m^2 + \|\bar{\partial}_m^*\alpha\|_m^2 + \|\alpha\|_{m,\Omega \setminus K}^2)
\end{aligned}$$

for a positive constant $C \geq 4 \cdot \max\{l, c_0 \cdot \sup |\text{grad } \chi|_{ds^2}(x)\}$ and $m \geq 1$ where c_0 is a positive constant depending only on the dimension of X . From (3.3), (3.4) and (3.5) into (3.2), we obtain the desired estimate. \square

Proposition 4. *There exists a positive constant m_* such that for any $m \geq m_*$, the harmonic space $\mathcal{H}_{r,s}^m(E^{\otimes m})$ has finite dimension and there exists a positive constant C_m depending on m such that*

$$(3.6) \quad \|\alpha\|_m^2 \leq C_m(\|\bar{\partial}\alpha\|_m^2 + \|\bar{\partial}_m^*\alpha\|_m^2)$$

for $\alpha \in \text{dom}(\bar{\partial}, E^{\otimes m}) \cap \text{dom}(\bar{\partial}_m^*, E^{\otimes m})$ with $s \geq q$.

Proof. Let m_0, C and K be the same as in Proposition 3, then we determine a positive integer m_* as $m_* = m_0 + 1$. As in Proposition 3, let χ be a real-valued C^∞ -function on X such that $\text{supp } \chi \Subset X$ and $\chi = 1$ on K . If $m \geq m_*$ and $\alpha \in \mathcal{B}_{r,s}(\bar{\Omega}, E^{\otimes m})$, then from (3.1), we obtain the following estimate:

$$\|\alpha\|_m^2 \leq C_m(\|\bar{\partial}\alpha\|_m^2 + \|\bar{\partial}_m^*\alpha\|_m^2 + \|\chi\alpha\|_m^2),$$

where C_m is a positive constant depending on m .

Take any sequence $\{\alpha_\nu\}$ such that $\alpha_\nu \in \text{dom } \bar{\partial} \cap \text{dom } \bar{\partial}_m^*$, $\|\alpha_\nu\|^2 \leq 1$ and $\lim_{\nu \rightarrow \infty} \|\bar{\partial}\alpha_\nu\|_m^2 = 0$, $\lim_{\nu \rightarrow \infty} \|\bar{\partial}_m^*\alpha_\nu\|_m^2 = 0$. Then, from Lemma 2 there exists a subsequence $\{\alpha_{\nu_k}\}$ of $\{\alpha_\nu\}$ which converges strongly on Ω . In fact ds^2 is complete, $\mathcal{D}_{r,s}(\Omega, E^{\otimes m})$ is dense in $\text{dom } \bar{\partial} \cap \text{dom } \bar{\partial}_m^*$ with respect to the norm $\|\alpha\|_m^2 + \|\bar{\partial}\alpha\|_m^2 + \|\bar{\partial}_m^*\alpha\|_m^2$ ([17], Theorem 1.1). Hence we may assume $\chi\alpha_\nu \in \mathcal{D}_{r,s}(\Omega, E^{\otimes m})$. Thus

$$\|\bar{\partial}(\chi\alpha_\nu)\|_m^2 + \|\bar{\partial}_m^*(\chi\alpha_\nu)\|_m^2 + \|(\chi\alpha_\nu)\|_m^2 = \langle \square^m(\chi\alpha_\nu), \chi\alpha_\nu \rangle_m + \langle \chi\alpha_\nu, \chi\alpha_\nu \rangle_m$$

is bounded by the assumption. From coerciveness of elliptic differential operator \square^m on $\mathcal{D}_{r,s}(\Omega, E^{\otimes m})$ (cf. [4], (2.2.1) Theorem) and Rellich's lemma (cf. [4], Appendix (A.1.6) Proposition), it follows that $\{\alpha_\nu\}$ has a subsequence $\{\alpha_{\nu_k}\}$ which is strongly convergent on compact subset K of Ω . By (3.1), we conclude that $\{\alpha_{\nu_k}\}$ converges strongly on Ω . Thus, by Hörmander [7] Theorem 1.1.2 and Theorem 1.1.3, there exists a positive constant C_m such that

$$(3.7) \quad \|\alpha\|_m^2 \leq C_m(\|\bar{\partial}\alpha\|_m^2 + \|\bar{\partial}_m^*\alpha\|_m^2)$$

for $\alpha \in \text{dom}(\bar{\partial}, E^{\otimes m}) \cap \text{dom}(\bar{\partial}_m^*, E^{\otimes m})$ with $\alpha \perp \mathcal{H}_{r,s}^m(E^{\otimes m})$, while each element α in $\mathcal{H}_{r,s}^m(E^{\otimes m})$ is a solution of the operator \square^m . Namely α is a harmonic form with valued in $E^{\otimes m}$. Now, from (3.1), α vanishes identically on $\Omega \setminus K$. Since any connected component of Ω is not contained in K , by the above unique continuation property, α vanishes on each connected component and so

α vanishes identically on Ω . Hence $\mathcal{H}_{r,s}^m(E^{\otimes m})$ is the null space. Combining this with (3.7), the proof is completed. \square

Remark 1. If there exists a strongly plurisubharmonic function ϕ on a neighborhood V of $b\Omega$, then any line bundle E is positive on a relatively compact neighborhood of $b\Omega$. In fact let h be a metric of E over X and extend ϕ to a C^∞ -function Φ on X without changing the original near $b\Omega$ in a suitable manner. Then there exists a positive integer m^* such that $h_m = he^{-m\Phi}$ gives the positivity of E on a relatively compact neighborhood $V'(\subseteq V)$ of $b\Omega$ for every $m \geq m^*$.

Remark 2. There are pseudoconvex domains with smooth boundary $b\Omega$ not possessing such a strongly plurisubharmonic function on any neighborhood of $b\Omega$ but possessing a line bundle which is positive on a neighborhood of $b\Omega$ (cf. [5]).

Theorem 2. *Let X be a complex manifold of dimension $n \geq 2$ and let $\Omega \Subset X$ be a weakly q -convex domain with smooth boundary in X . Assume that E is a holomorphic line bundle over X and $E^{\otimes m}$ is the m -times tensor product of E for positive integer m . Suppose that there exists a strongly plurisubharmonic function on a neighborhood of $b\Omega$. Then there exists a positive integer m^* such that, for $m \geq m^*$, $r \geq 0$, $s \geq q$, there exists a bounded linear operator $N^m : L_{r,s}^2(\Omega, E^{\otimes m}) \rightarrow L_{r,s}^2(\Omega, E^{\otimes m})$ such that*

- (i) $\text{range}(N^m, E^{\otimes m}) \subset \text{dom}(\square^m, E^{\otimes m})$,
 $N^m \square^m = I - \Pi^m$ on $\text{dom}(\square^m, E^{\otimes m})$,
- (ii) for $\alpha \in L_{r,s}^2(\Omega, E^{\otimes m})$, we have

$$\alpha = \bar{\partial} \bar{\partial}_m^* N^m \alpha \oplus \bar{\partial}_m^* \bar{\partial} N^m \alpha \oplus \Pi^m \alpha,$$

- (iii) $N^m \bar{\partial} = \bar{\partial} N^m$ on $\text{dom}(\bar{\partial}, E^{\otimes m})$ and
- (iv) $N^m \bar{\partial}_m^* = \bar{\partial}_m^* N^m$ on $\text{dom}(\bar{\partial}_m^*, E^{\otimes m})$,
- (v) $N^m, \bar{\partial} N^m, \bar{\partial}_m^* N^m$ are bounded operators on $L_{r,s}^2(\Omega, E^{\otimes m})$.

Proof. From (3.6), we obtain

$$(3.8) \quad \|\alpha\|_m \leq C_m \|\square^m \alpha\|_m$$

for $\alpha \in \text{dom}(\bar{\partial}, E^{\otimes m}) \cap \text{dom}(\bar{\partial}_m^*, E^{\otimes m}) \cap \text{dom} \bar{\partial}_m^*$ with $s \geq q$. Since \square^m is a linear closed densely defined operator, then, from [7]; Theorem 1.1.1, $\text{range}(\square^m, E^{\otimes m})$ is closed. Thus, from (1.1.1) in [7] and the fact that \square^m is self adjoint, we have the Hodge decomposition

$$L_{r,s}^2(\Omega, E^{\otimes m}) = \bar{\partial} \bar{\partial}_m^* \text{dom} \square^m \oplus \bar{\partial}_m^* \bar{\partial} \text{dom} \square^m.$$

Since $\square^m : \text{dom}(\square^m, E^{\otimes m}) \rightarrow \text{range}(\square^m, E^{\otimes m}) = L_{r,s}^2(\Omega, E^{\otimes m})$ is one to one on $\text{dom}(\square^m, E^{\otimes m})$ from (3.98), there exists a unique bounded inverse operator

$$N^m : L_{r,s}^2(\Omega, E^{\otimes m}) \rightarrow \text{dom}(\square^m, E^{\otimes m})$$

such that $N^m \square^m \alpha = \alpha$ on $\text{dom}(\square^m, E^{\otimes m})$. Also, from the definition of N^m , we obtain $\square^m N^m = I$ on $L^2_{r,s}(\Omega, E^{\otimes m})$. Thus (i) and (ii) are satisfied. To show that $\bar{\partial}_m^* N^m = N^m \bar{\partial}_m^*$ on $\text{dom}(\bar{\partial}_m^*, E^{\otimes m})$, by using (ii), we have $\bar{\partial}_m^* \alpha = \bar{\partial}_m^* \bar{\partial}_m^* N^m \alpha$ for $\alpha \in \text{dom}(\bar{\partial}_m^*, E^{\otimes m})$. Thus

$$N^m \bar{\partial}_m^* \alpha = N^m \bar{\partial}_m^* \bar{\partial}_m^* N^m \alpha = N^m (\bar{\partial}_m^* \bar{\partial} + \bar{\partial} \bar{\partial}_m^*) \bar{\partial}_m^* N^m \alpha = \bar{\partial}_m^* N^m \alpha.$$

A similar argument shows that $\bar{\partial} N^m = N^m \bar{\partial}$ on $\text{dom} \bar{\partial}$. By using (iii) and the condition on α , $\bar{\partial} \alpha = 0$, we have $\bar{\partial} N^m \alpha = N^m \bar{\partial} \alpha = 0$. Then, by using (ii), we obtain $\alpha = \bar{\partial} \bar{\partial}_m^* N^m \alpha$. Thus the form $u = \bar{\partial}_m^* N^m \alpha$ satisfies the equation $\bar{\partial} u = \alpha$. Since $\text{Rang}(N^m, E^{\otimes m}) \subset \text{dom}(\square^m, E^{\otimes m})$, then by applying (3.6) to $N^m \alpha$ instead of α , we obtain

$$\begin{aligned} \|N^m \alpha\|_m &\leq C_m \|\alpha\|_m, \\ \|\bar{\partial} N^m \alpha\|_m + \|\bar{\partial}_m^* N^m \alpha\|_m &\leq 2\sqrt{C_m} \|\alpha\|_m. \end{aligned}$$

Thus the proof follows. \square

Theorem 3. *Under the same assumption of Theorem 2, for $\alpha \in L^2_{r,s}(X, E^{\otimes m})$, $\text{supp} \alpha \subset \bar{\Omega}$, with $s \geq q$, satisfying $\bar{\partial} \alpha = 0$ in the distribution sense in X , there exists $u \in L^2_{r,s-1}(X, E^{\otimes m})$, $\text{supp} u \subset \bar{\Omega}$ such that $\bar{\partial} u = \alpha$ in the distribution sense in X .*

Proof. Let $\alpha \in L^2_{r,s}(X, E^{\otimes m})$, $\text{supp} \alpha \subset \bar{\Omega}$, then $\alpha \in L^2_{r,s}(\Omega, E^{\otimes m})$. Following Theorem 2, $N^m_{n-r, n-s}$ exists for $n - s \geq q$. Thus, one can define $u \in L^2_{r,s-1}(\Omega, E^{\otimes m})$ by

$$(3.9) \quad u = -\star \#_{E^{\otimes m}} \bar{\partial} N^m_{n-r, n-s} \#_{E^{\otimes m}} \star \alpha.$$

Extend u to X by defining $u = 0$ in $X \setminus \bar{\Omega}$. To prove that u satisfies $\bar{\partial} u = \alpha$ in the distribution sense in X , we first prove that $\bar{\partial} u = \alpha$ in the distribution sense in Ω .

For $\eta \in \text{dom}(\bar{\partial}, E^{*\otimes m})$, we have

$$\langle \bar{\partial} \eta, \#_{E^{\otimes m}} \star \alpha \rangle_{m, \Omega} = (-1)^{r+s} \langle \alpha, \#_{E^{*\otimes m}} \star \bar{\partial} \eta \rangle_{m, \Omega}.$$

From the density of the space $\mathcal{B}_{r,s}(\bar{\Omega}, E^{\otimes m})$ in $\text{dom}(\bar{\partial}, E^{\otimes m}) \cap \text{dom}(\bar{\partial}^*, E^{\otimes m})$ in the graph norm (cf. Proposition 1) and since $\vartheta^m = \bar{\partial}_m^*$ on $\mathcal{B}_{r,s}(\bar{\Omega}, E^{\otimes m})$, when ϑ^m acts in the distribution sense, we have from (2.1) that

$$\langle \bar{\partial} \eta, \#_{E^{\otimes m}} \star \alpha \rangle_{m, \Omega} = \langle \alpha, \bar{\partial}_m^* \#_{E^{*\otimes m}} \star \eta \rangle_{m, \Omega}.$$

Since $\text{supp} \alpha \subset \bar{\Omega}$, then we obtain

$$\langle \bar{\partial} \eta, \#_{E^{\otimes m}} \star \alpha \rangle_{m, \Omega} = \langle \alpha, \bar{\partial}_m^* \#_{E^{*\otimes m}} \star \eta \rangle_{m, \Omega} = \langle \bar{\partial} \alpha, \#_{E^{*\otimes m}} \star \eta \rangle_{m, X} = 0.$$

It follows that

$$\bar{\partial}_m^* (\#_{E^{\otimes m}} \star \alpha) = 0 \quad \text{on } \Omega.$$

Using Theorem 2(iv), we have

$$(3.10) \quad \bar{\partial}_m^* N_{n-r, n-s}^m (\#_{E^{\otimes m}} \star \alpha) = N_{n-r, n-s-1}^m \bar{\partial}_m^* (\#_{E^{\otimes m}} \star \alpha) = 0.$$

Thus, in the distribution sense in Ω and from (2.1), (3.9) and (3.10), we obtain

$$(3.11) \quad \begin{aligned} \bar{\partial} u &= -\bar{\partial} \star \#_{E^{\otimes m}} \bar{\partial} N_{n-r, n-s}^m \#_{E^{\otimes m}} \star \alpha \\ &= (-1)^{r+s} \star \#_{E^{\otimes m}} \bar{\partial}_m^* \bar{\partial} N_{n-r, n-s}^m \#_{E^{\otimes m}} \star \alpha \\ &= (-1)^{r+s} \star \#_{E^{\otimes m}} (\bar{\partial}_m^* \bar{\partial} + \bar{\partial} \bar{\partial}_m^*) N_{n-r, n-s}^m \#_{E^{\otimes m}} \star \alpha \\ &= (-1)^{r+s} \star \#_{E^{\otimes m}} \#_{E^{\otimes m}} \star \alpha \\ &= \alpha. \end{aligned}$$

Because $u = 0$ in $X \setminus \Omega$, then for $\eta \in \text{dom}(\bar{\partial}_m^*, E^{\otimes m}) \subset L_{r,s}^2(X, E^{\otimes m})$, one obtain

$$\langle u, \bar{\partial}_m^* \eta \rangle_{m,X} = \langle u, \bar{\partial}_m^* \eta \rangle_{m,\Omega} = \langle \#_{E^{\otimes m}} \star \bar{\partial}_m^* \eta, \#_{E^{\otimes m}} \star u \rangle_{m,\Omega}.$$

Since

$$\#_{E^{\otimes m}} \star u = (-1)^{r+s+1} \bar{\partial} N_{n-r, n-s}^m \#_{E^{\otimes m}} \star \alpha \in \text{dom}(\bar{\partial}_m^*, E^{*\otimes m}).$$

Thus, from (2.1), we obtain

$$\begin{aligned} \langle u, \bar{\partial}_m^* \eta \rangle_{m,X} &= (-1)^{r+s} \langle \bar{\partial} \#_{E^{\otimes m}} \star \eta, \#_{E^{\otimes m}} \star u \rangle_{m,\Omega} \\ &= \langle \#_{E^{\otimes m}} \star \eta, \#_{E^{\otimes m}} \star \bar{\partial} u \rangle_{m,\Omega} = \langle \bar{\partial} u, \eta \rangle_{m,\Omega}. \end{aligned}$$

Thus, from (3.11),

$$\langle u, \bar{\partial}_m^* \eta \rangle_{m,X} = \langle \alpha, \eta \rangle_{m,\Omega} = \langle \alpha, \eta \rangle_{m,X}.$$

Thus $\bar{\partial} u = \alpha$ in the distribution sense in X . \square

4. Solvability of the $\bar{\partial}_b$ -problem

In this section, applications to the solvability of the $\bar{\partial}_b$ -problem are given.

Theorem 4. *Let X be a Kähler manifold of dimension $n \geq 2$ and let $\Omega \Subset X$ be a weakly q -convex domain with smooth boundary in X . Let E be a holomorphic line bundle over X and $E^{\otimes m}$ be the m -times tensor product of E for positive integer m . Suppose that there exists a strongly plurisubharmonic function on a neighborhood of $b\Omega$. Then, for $f \in C_{r,s}^\infty(b\Omega, E^{\otimes m})$, $q \leq s \leq n-2$, satisfying $\bar{\partial}_b f = 0$, there exists $F \in C_{r,s}^\infty(\bar{D}, E^{\otimes m})$ such that $F|_{b\Omega} = f$ and $\bar{\partial} F = 0$.*

Proof. The proof follows as in Theorem 4.1 in Saber [12]. \square

Theorem 5. *Under the same assumption of Theorem 4, if $f \in C_{r,s}^\infty(b\Omega, E^{\otimes m})$, $1 \leq s \leq n-2$, with $\bar{\partial}_b f = 0$, there exists $u \in C_{r,s-1}^\infty(b\Omega, E^{\otimes m})$ such that $\bar{\partial}_b u = f$.*

Proof. Let $f \in C_{r,s}^\infty(b\Omega, E^{\otimes m})$, $1 \leq s \leq n-2$, with $\bar{\partial}_b f = 0$. Then from Theorem 4, there exists $F \in C_{r,s}^\infty(\bar{\Omega}, E^{\otimes m})$ such that $F|_{b\Omega} = f$ and $\bar{\partial}F = 0$. Following Theorem 3, there exists $U \in C_{r,s-1}^\infty(\bar{\Omega}, E^{\otimes m})$ satisfying $\bar{\partial}U = f$ in Ω . Then $u = U|_{b\Omega}$ satisfies $\bar{\partial}_b u = f$. \square

Corollary 6. *Let X be a Kähler manifold of dimension $n \geq 2$ and let $D \Subset X$ be a weakly q -concave domain with smooth boundary in X . Let E be a holomorphic line bundle over X and $E^{\otimes m}$ be the m -times tensor product of E for positive integer m . Suppose that there exists a strongly plurisubharmonic function on a neighborhood of $b\Omega$. If $H^{r,s}(X, E^{\otimes m}) = 0$, then, for $f \in C_{r,s}^\infty(\bar{D}, E^{\otimes m})$, $\bar{\partial}f = 0$, $q \leq s \leq n-2$, there exists $u \in C_{r,s-1}^\infty(\bar{D}, E^{\otimes m})$ such that $\bar{\partial}u = f$.*

Proof. The proof follows as in Corollary 4.3 in Saber [12]. \square

The necessary and sufficient condition on $f \in W_{r,s}^{\frac{1}{2}}(b\Omega, E^{\otimes m})$ to have a $\bar{\partial}$ -closed extension F on Ω is summarized as follows.

Theorem 7. *Let Ω , E and X be the same as in Theorem 4.1. For $f \in W_{r,s}^{\frac{1}{2}}(b\Omega, E^{\otimes m})$, $0 \leq r \leq n$, $q \leq s \leq n-2$. We assume that $\bar{\partial}_b f = 0$. Then there exists $F \in L_{r,s-1}^2(\Omega, E^{\otimes m})$ such that $F = f$ on $b\Omega$ and $\bar{\partial}F = 0$ in Ω .*

Proof. The proof follows as in Theorem 4.4 in Saber [12]. \square

5. Extension from the boundary

Let X be a connected complex manifold of dimension $n \geq 2$, and let $\Omega \subset X$ be any domain with C^∞ -smooth boundary. Let E be a holomorphic vector bundle over X . In this section we prove the following results:

Lemma 4. *For any $\alpha \in C_{r,s}^\infty(b\Omega, E)$ satisfying $\bar{\partial}_b \alpha = 0$, there exists $\tilde{\alpha} \in C_{r,s}^\infty(\bar{\Omega}, E)$ such that $\tilde{\alpha}|_{b\Omega} = \alpha$ and that $\bar{\partial}\tilde{\alpha}$ vanishes to the infinite order on $b\Omega$.*

Proof. The proof follows as in Lemma 4 in Oshawa [9]. \square

By virtue of a theory of Kodaira-Andreotti-Vesentini (cf. Kodaira [8], Andreotti and Vesentini [1]), we can show that a sufficient condition for the C^k -extendability can be stated as follows.

Lemma 5. *Let X be a connected Kähler manifold of dimension n and let $\Omega \Subset X$ be a weakly q -convex domain with C^∞ -smooth boundary. Let E be a holomorphic vector bundle over X . Suppose that Ω admits a C^∞ defining function ρ such that*

$$\partial\bar{\partial}(-\log(-\rho)) \geq c(\partial(-\log(-\rho))\bar{\partial}(-\log(-\rho)) + \omega).$$

holds on Ω for some positive constant c . Then, for any $\psi \in C_{r,s}^\infty(b\Omega, E) \cap \ker(\bar{\partial}_b, E)$ with $s < n-1$, and for any nonnegative integer k , there exists a $\bar{\partial}$ -closed E -valued (r, s) -form Ψ_k of class C^k on $\bar{\Omega}$ satisfying $\Psi_k|_{b\Omega} = \psi$.

Proof. The proof follows as in Theorem 5 in Oshawa [9]. \square

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