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ON COMMUTATIVITY OF NILPOTENT ELEMENTS AT ZERO

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Abstract. The reversible property of rings was initially introduced by Habeb and plays a role in noncommutative ring theory. In this note we study the reversible ring property on nilpotent elements, introducing the concept of *commutativity of nilpotent elements at zero* (simply, a *CNZ* ring) as a generalization of reversible rings. We first find the CNZ property of 2 by 2 full matrix rings over fields, which provides a basis for studying the structure of CNZ rings. We next observe various kinds of CNZ rings including ordinary ring extensions.

1. Introduction

Throughout this note every ring is an associative ring with identity unless otherwise stated. Given a ring R, $N^*(R)$ and $N(R)$ denote the upper nilradical (i.e., the sum of nil ideals) and the set of all nilpotent elements in R, respectively. Note $N^*(R) \subseteq N(R)$. The polynomial (resp., power series) ring with an indeterminate x over R is denoted by $R[x]$ (resp., $R[[x]]$). $\mathbb{Z}_n$ denotes the ring of integers modulo n. Denote the n by n ($n \geq 2$) full (resp., upper triangular) matrix ring over R by $Mat_n(R)$ (resp., $U_n(R)$). Use E_{ij} for the matrix with (i,j)-entry 1 and elsewhere 0.

Following the literature, a ring is called *reduced* if it has no nonzero nilpotent elements. It is easily checked that if R is a reduced ring, then the following condition holds:

$$ab = 0 \text{ implies } ba = 0 \text{ for } a, b \in R.$$

Cohn [5] called a ring R *reversible* if this condition holds. Anderson and Camillo [3], observed the rings whose zero products commute, and used the term ZC_2 for what is called reversible. Prior to Cohn's work, reversible rings were studied under the names of *completely reflexive* and *zero commutative* by Mason [22] and Habeb [9], respectively. While, Tuganbaev [29] investigated

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reversible rings in his monograph on distributive lattices arising in ring theory, using the name of *commutative at zero* in place of reversible. It is obvious that commutative rings and reduced rings are reversible. A ring is called *abelian* if every idempotent is central. It is simply checked that reversible rings are abelian. Recently, various generalized conditions of reversible rings have studied by many authors, and the results obtained were applied to many sorts of problems arising in noncommutative ring theory.

In this paper, of particular interest will be the commutativity of nilpotent elements at zero. Various results obtained in this work also can provide a sort of bridge between commutative and noncommutative ring theory.

2. Basic properties and examples

We study in this section the basic properties of CNZ rings and investigate their related examples in the process. Our central tool is the following notion.

Definition 2.1. A ring R is said to satisfy the *commutativity of nilpotent elements at zero* if $ab = 0$ for $a, b \in N(R)$ implies $ba = 0$. For simplicity, we will call a ring *CNZ* if it satisfies the commutativity of nilpotent elements at zero.

Reversible rings are clearly CNZ, but there exist many CNZ rings which are not abelian (and so not reversible) as we see in the following procedure. We first consider an important example of such CNZ rings.

Example 2.2. Let K be a field and $R = \mathrm{Mat}_2(K)$. Note that R is obviously not abelian and hence not reversible. Let

$$0 \neq A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in N(R).$$

Then $A^2 = 0$ by help of Cayley-Hamilton Theorem. Since $A^2 = (a + d)A$, $A^2 = 0$ and $a + d = 0$ and so

$$\begin{pmatrix} a^2 + bc & 0 \\ 0 & d^2 + bc \end{pmatrix} = 0.$$

We conclude all the possible forms of A, using this result.

If $b = 0$ then $A = \left(\begin{smallmatrix} 0 & 0 \\ c & 0 \end{smallmatrix}\right)$ with $c \neq 0$.

If $c = 0$ then $A = \left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right)$ with $b \neq 0$.

If $b \neq 0$ and $c \neq 0$, then $A = \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)$ with $a^2 = -bc \neq 0$, since $\det(A) = 0$. Therefore $N(R)\backslash\{0\}$ is the union of the following three subsets:

$$M_1 = \left\{ \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix} \mid c \neq 0 \right\},\ M_2 = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \neq 0 \right\} \text{ and}$$

$$M_3 = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a \neq 0, b \neq 0, c \neq 0 \text{ and } a^2 = -bc \neq 0 \right\}.$$

We next show that R is a CNZ ring. Let $AB = 0$ for any nonzero $A, B \in N(R)$ to see that. It is easily shown that $AB \neq 0$ if $A \in M_i$ and $B \in M_j$ for

i, j with $i \neq j$. So A and B must be contained in an M_i together. If $A, B \in M_1$ or $A, B \in M_2$ then clearly $BA = 0$. So let $A, B \in M_3$, say

$$A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \text{ and } B = \begin{pmatrix} a' & b' \\ c' & -a' \end{pmatrix}.$$

From

$$AB = \begin{pmatrix} aa' + bc' & ab' - ba' \\ ca' - ac' & cb' + aa' \end{pmatrix} = 0,$$

we have

$$aa' + bc' = 0,\ ab' = ba',\ ca' = ac' \ \text{ and } \ aa' + cb' = 0.$$

These entail that $BA = 0$, and therefore $R = \mathrm{Mat}_2(K)$ is a CNZ ring.

A ring R is called *directly finite* if $ab = 1$ implies $ba = 1$ for $a, b \in R$. The fact that abelian rings (e.g., reversible rings) are directly finite is well-known. So one may conjecture that CNZ rings are directly finite.

Proposition 2.3. *Every CNZ ring is directly finite.*

Proof. Let R be a CNZ ring and assume on the contrary that R is not directly finite. Say that $ab = 1$ and $ba \neq 1$ for some $a, b \in R$. Note first that if $b^k = b^{k+1}a$ for $k \geq 1$ then $1 = a^k b^k = a^k b^{k+1} a = ba$, a contradiction. Thus $b^k \neq b^{k+1}a$ for all $k \geq 1$.

Next, consider two elements

$$x = b(1 - ba) \text{ and } y = b^2(1 - ba)a.$$

Then $x^2 = 0$ and $y^2 = 0$. Suppose that $x = 0$. Then $b = b^2 a$ and $1 = ab = ab^2 a = ba$, a contradiction. Suppose that $y = 0$. Then $b^2 a = b^3 a^2$ and $1 = a^2(b^2 a)b = a^2(b^3 a^2)b = ba$, a contradiction. So $0 \neq x, y \in N(R)$. Then we have

$$xy = [b(1 - ba)][b^2(1 - ba)a] = 0 \text{ since } \ (1 - ba)b = 0,$$

but

$$yx = [b^2(1 - ba)a][b(1 - ba)] = b^2 - b^3 a \neq 0.$$

This implies that R is not CNZ, a contradiction. Therefore R is directly finite. □

The following proposition provides some examples of CNZ rings.

Lemma 2.4. (1) *Let R be a ring. If $N(R)^2 = 0$, then R is CNZ.*

(2) *The class of CNZ rings is closed under subrings.*

(3) *Let R be a ring such that $(ab)^2 = 0$ implies $ab = 0$ for all $a, b \in N(R)$. Then R is CNZ.*

(4) *The ring $\mathrm{Mat}_2(D)$ over a commutative domain D is CNZ.*

Proof. The proofs of (1) and (2) follow from the definition of a CNZ ring directly.

(3) Assume that $ab = 0$ for $a, b \in N(R)$. Since $(ba)^2 = 0$, we have $ba = 0$, showing that R is CNZ.

(4) Let Q be the field of quotients of D. Then $\mathrm{Mat}_2(Q)$ is CNZ by Example 2.2, and so $\mathrm{Mat}_2(D)$ is CNZ by (2). □

The converses of Lemma 2.4(1),(3) need not hold by the following example.

Example 2.5. (1) Let K be a field and $A = K\langle a, b\rangle$ be the free algebra with noncommuting indeterminates a, b over K. Let I be the ideal of A generated by

$$a^m, ab, ba, \text{ and } b^n \text{ for } m, n \geq 3.$$

Set $R = A/I$ and let a, b coincide with their images in R for simplicity. Then R is commutative and hence CNZ. Note that

$$N(R) = \left\{ \sum_{i=0}^{m-1} h_i a^i + \sum_{j=0}^{n-1} k_j b^j \mid h_i, k_j \in K \text{ for all } i, j \right\}.$$

But $N(R)^{m+n} = 0$ and $0 \neq (a+b)^2 \in N(R)^2$ for $a + b \in N(R)$.

(2) Consider a ring

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z}_4 \right\}.$$

Then R is reversible by [17, Proposition 1.6] and hence CNZ. For

$$a = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, b = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in N(R),$$

we obtain $(ab)^2 = 0$ but

$$0 \neq ab = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}.$$

Remark 2.6. (1) The condition "the commutativity" for the domain D in Lemma 2.4(4) cannot be dropped. Indeed, there exists a non-commutative domain (and so a CNZ ring) D such that $\mathrm{Mat}_2(D)$ is not directly finite by [28, Theorem 1.0], and hence $\mathrm{Mat}_2(D)$ is not a CNZ ring by Proposition 2.3.

(2) Both $U_n(A)$ and $\mathrm{Mat}_n(A)$ over any ring A are not CNZ for $n \geq 3$. Indeed, for $E_{12}, E_{23} \in N(U_n(A))$ and $n \geq 3$ we get $E_{23}E_{12} = 0$ but $E_{12}E_{23} \neq 0$, showing that $U_n(A)$ is not CNZ for $n \geq 3$. Thus $\mathrm{Mat}_n(A)$ is not CNZ for $n \geq 3$ by Lemma 2.4(2).

It is evident that $U_2(A)$ over any ring A is not reversible, but we have the following useful result for CNZ rings.

Theorem 2.7. *A ring R is reduced if and only if $U_2(R)$ is a CNZ ring.*

Proof. Suppose that R is a reduced ring. Then $U_2(R)$ is CNZ by Lemma 2.4(1), noting

$$N(U_2(R)) = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in R \right\}.$$

Conversely, let $U_2(R)$ be CNZ and assume on the contrary that R is not reduced. Then there exists $0 \neq a \in R$ with $a^2 = 0$. Consider two matrices

$$M_1 = \begin{pmatrix} a & 1 \\ 0 & 0 \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

in $U_2(R)$. Then $M_1, M_2 \in N(U_2(R))$. But $M_1M_2 = 0$ and $M_2M_1 = aE_{12} \neq 0$, entailing that $U_2(R)$ is not CNZ. This induces a contradiction, and so such a cannot exist. Thus R is reduced. □

Regarding to Theorem 2.7, the next example illuminates that the ring $U_2(R)$ is not CNZ any more when we take the weaker condition "R is a reversible ring" in place of the condition "R is a reduced ring".

Example 2.8. Let $R = \mathbb{Z}_4$ be the ring of integers modulo 4. Then R is a reversible (hence CNZ) ring but not reduced. For

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } b = \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix} \in U_2(R),$$

we have $a^2 = 0$ and $b^3 = 0$, and so $a, b \in N(U_2(R))$. Then $ab = 0$ but

$$0 \neq ba = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}.$$

This shows that $U_2(R)$ is not CNZ.

Let R be a ring and $n \geq 2$. Following the literature, consider the ring extension

$$D_n(R) = \{(a_{ij}) \in U_n(R) \mid a_{11} = \cdots = a_{nn}\}$$

of R. For any ring A and $n \geq 3$, $D_n(A)$ is not CNZ by the same argument as in the proof of Remark 2.6(2). This fact leads to [17, Example 1.5]. We will use this fact without reference.

Given a ring R and an (R, R)-bimodule M, the *trivial extension* of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication:$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$. This is isomorphic to the ring of all matrices $\left(\begin{smallmatrix} r & m \\ 0 & r \end{smallmatrix}\right)$, where $r \in R$ and $m \in M$ and the usual matrix operations are used. Note $D_2(R) = T(R, R)$.

Note that $T(R, R)$ of a reduced ring R is CNZ by Lemma 2.4(2) and Theorem 2.7, but not conversely: In fact, letting R be a non-reduced commutative ring (e.g., $\mathbb{Z}_{n^k}$ for $n, k \geq 2$), we have that $T(R, R)$ is commutative (hence CNZ).

The following example also shows that the trivial extension of a CNZ ring need not be CNZ.

Example 2.9. We use the ring and apply the argument in [17, Example 2.1]. Let

$$A = \mathbb{Z}_2\langle a_0, a_1, a_2, b_0, b_1, b_2, c\rangle$$

be the free algebra generated by noncommuting indeterminates a_0, a_1, a_2, b_0, b_1, b_2, c over $\mathbb{Z}_2$. Next let I be the ideal of A generated by

$$a_0b_0, a_0b_1 + a_1b_0, a_0b_2 + a_1b_1 + a_2b_0, a_1b_2 + a_2b_1, a_2b_2, a_0rb_0, a_2rb_2,$$
$$b_0a_0, b_0a_1 + b_1a_0, b_0a_2 + b_1a_1 + b_2a_0, b_1a_2 + b_2a_1, b_2a_2, b_0ra_0, b_2ra_2,$$
$$(a_0+a_1 + a_2)r(b_0 + b_1 + b_2), (b_0 + b_1 + b_2)r(a_0 + a_1 + a_2), \text{ and } r_1r_2r_3r_4,$$

where the constant terms of $r, r_1, r_2, r_3, r_4 \in A$ are zero. Now set $R = A/I$. Then R is a reversible ring by the argument in [17, Example 2.1], and so R is CNZ.

We identity $a_0, a_1, a_2, b_0, b_1, b_2, c$ with their images in R for simplicity. Consider the trivial extension $T(R, R)$ of R, and take

$$M_1 = \begin{pmatrix} a_0 & a_1 \\ 0 & a_0 \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} b_0c & b_1c \\ 0 & b_0c \end{pmatrix}$$

in $T(R, R)$. Then $M_1, M_2 \in N(T(R, R))$ and $M_1M_2 = \begin{pmatrix} a_0b_0c & (a_0b_1+a_1b_0)c \\ 0 & a_0b_0c \end{pmatrix} = 0$, but

$$M_2M_1 = \begin{pmatrix} b_0ca_0 & b_0ca_1 + b_1ca_0 \\ 0 & b_0ca_0 \end{pmatrix} = \begin{pmatrix} 0 & b_0ca_1 + b_1ca_0 \\ 0 & 0 \end{pmatrix} \neq 0$$

since $b_0ca_1 + b_1ca_0 \notin I$. Thus $T(R, R)$ is not CNZ.

For a ring R and $n \geq 2$, let $V_n(R)$ be the ring of all matrices (a_{ij}) in $D_n(R)$ such that $a_{st} = a_{(s+1)(t+1)}$ for $s = 1, \ldots, n-2$ and $t = 2, \ldots, n-1$. Note that $V_n(R) \cong \frac{R[x]}{x^nR[x]}$.

For a reduced ring R and $n \geq 2$, $V_n(R)$ is reversible by [17, Theorem 2.5] and so it is CNZ. However, Example 2.9 shows that there exists a reversible ring R such that $V_2(R)$ need not be CNZ, and consequently $V_n(R)$ need not be CNZ for $n \geq 2$ by Lemma 2.4(2).

Example 2.10. (1) The class of CNZ rings is not closed under homomorphic images.

Let K be a field and $R = K\langle a, b\rangle$ be the free algebra with noncommuting indeterminates a, b over K. Then R is a domain and hence it is CNZ. Now, let I be the ideal of R generated by

$$ab, a^2 \text{ and } b^2.$$

Let $\bar{r} = r + I$ for $r \in R$. Then $\bar{a}, \bar{b} \in N(R/I)$ and $\bar{a}\bar{b} = 0$ but $\bar{b}\bar{a} \neq 0$ by the construction of I. Thus R/I is not CNZ.

(2) There exists a non-CNZ ring R such that R/I is CNZ for any nil ideal I of R.

Consider $R = D_3(F)$ over a division ring F. Then R is not CNZ as noted before. All the nonzero proper nil ideals of R are

$$I_1 = \begin{pmatrix} 0 & F & F \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix}, I_2 = \begin{pmatrix} 0 & F & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$I_3 = \begin{pmatrix} 0 & 0 & F \\ 0 & 0 & F \\ 0 & 0 & 0 \end{pmatrix}, I_4 = \begin{pmatrix} 0 & 0 & F \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{and}$$

$$I_5 = \left\{ \begin{pmatrix} 0 & \alpha b & c \\ 0 & 0 & \alpha d \\ 0 & 0 & 0 \end{pmatrix} \mid b, c, d, \alpha \in F \ \text{ and } \ b \neq 0, d \neq 0 \ \text{ are fixed} \right\}.$$

However, every R/I_i for $i = 1, 2, 3, 4, 5$ is reversible by [19, Example 2.9] and so all of these rings are CNZ.

But we have an affirmative answer if we add some condition as in the following.

Proposition 2.11. *Let R be a ring and I be a proper ideal of R.*

(1) *If R is a CNZ ring and I consists of all nilpotent element of bounded index $\leq n$ in R, then R/I is CNZ.*

(2) *If R/I is CNZ and I is reduced as a ring without identity, then R is CNZ.*

Proof. We denote $\bar{r} = r + I$ for any $r \in R$. (1) Suppose that R is a CNZ ring and I consists of all nilpotent element of bounded index $\leq n$. Let $\bar{a}\bar{b} = \bar{0}$ for $\bar{a}, \bar{b} \in N(R/I)$. Note that $a, b \in N(R)$ and $ab \in I$. Then $(ab)^n = 0$ and $0 = (ab)^n = a(ba)^{n-1}b$ implies $0 = ba(ba)^{n-1} = (ba)^n$ since $a(ba)^{n-1} \in I \subseteq N(R)$ and $b \in N(R)$. Hence $ba \in I$. This implies that $\bar{b}\bar{a} = \bar{0}$, showing that R/I is CNZ.

(2) Assume that R/I is CNZ and I is reduced and let $ab = 0$ for $a, b \in N(R)$. Then $\bar{a}, \bar{b} \in N(R/I)$ and $\bar{a}\bar{b} = \bar{0}$. Since R/I is CNZ, $ba \in I$ by assumption. Then $(ba)^2 = b(ab)a = 0$ and so $ba = 0$ since I is reduced. Therefore R is CNZ. □

Proposition 2.12. (1) *The class of CNZ rings is closed under direct products and direct sums*

(2) *Let $e \in R$ be a central idempotent. Then R is CNZ if and only if eR and $(1-e)R$ are CNZ rings.*

(3) *If R is a ring whose units form an Abelian group, then R is CNZ.*

Proof. (1) The proof of direct product case is follows from $N(R) \subseteq \prod_{\gamma \in \Gamma} N(R_\gamma)$ where $R = \prod_{\gamma \in \Gamma} R_\gamma$ for any family $\{R_\gamma \mid \gamma \in \Gamma\}$ of rings. The proof of direct sum case comes from Lemma 2.4(2) and the above.

(2) It follows directly from (1) and Lemma 2.4(2), since $R \cong eR \oplus (1-e)R$.

(3) Let $ab = 0$ for $a, b \in N(R)$. Then $1 - a$ and $1 - b$ are invertible, and so $(1-a)(1-b) = (1-b)(1-a)$. This implies $1-a-b = 1-a-b+ab = 1-b-a+ba$, entailing $ba = 0$. Thus R is CNZ. □

A ring R is called *semiperfect* if R is semilocal and idempotents can be lifted modulo the Jacobson radical $J(R)$ of R. Local rings are abelian and semilocal. We can see that the classes of abelian rings and CNZ rings do not imply each other by Example 3.1 to follow. But it can be obtained that for an abelian ring R, R is CNZ and semiperfect if and only if R is a finite direct sum of local CNZ rings by the same argument as in [20] with Proposition 2.12(1,2). Moreover, if R is a minimal noncommutative CNZ ring, then R is of order 8 and is isomorphic to $U_2(\mathbb{Z}_2)$ (here by minimal we mean having smallest cardinality) by similar computation to [7], noting that $U_2(\mathbb{Z}_2)$ is a CNZ ring by Theorem 2.7.

3. Relations to CNZ rings

We study in this section the related rings and the extension rings of CNZ rings. Following Bell [4], a ring R is said to satisfy the *Insertion-of-Factors-Property* (simply, an *IFP* ring) if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. Reversible rings are IFP, and IFP rings are also abelian through a simple computation, but not conversely in each case. One may have a question whether IFP rings are CNZ, but this is impossible by Example 3.1(2) below. But we get that a ring R is reduced if and only if $U_2(R)$ is a CNZ ring if and only if $D_3(R)$ is an IFP ring by Theorem 2.7 and [13, Proposition 2.8].

In [24, Definition 2.1], a ring R is called *nil-IFP* if $ab = 0$ for any $a, b \in N(R)$ implies $aRb = 0$. Clearly IFP rings are nil-IFP (We change over from "nil-semicommutative in [24] to "nil-IFP", so as to cohere with the above). Nil-IFP rings need not be abelian by [24, Example 2.2]. The classes of CNZ rings and nil-IFP rings do not imply each other by the following example.

Example 3.1. (1) Let $R = \mathrm{Mat}_2(\mathbb{Z}_2)$. Then R is CNZ by Example 2.2. However, R is not nil-IFP, since $E_{21}^2 = 0$ but $0 \neq E_{21}E_{12}E_{21} \in E_{21}RE_{21}$ for $E_{21} \in N(R)$.

(2) Let A be a reduced ring. The ring $R = D_3(A)$ is IFP by [17, Proposition 1.2] and so nil-IFP, but R is not CNZ as noted earlier.

Following Marks [21], a ring R is called *NI* if $N^*(R) = N(R)$. It is well-known that a ring R is NI if and only if $N(R)$ forms an ideal if and only if $R/N^*(R)$ is reduced. Every nil-IFP ring is NI but not conversely by [24, Theorem 2.5 and Example 2.8]. Moreover, the concepts of CNZ rings and NI rings are independent of each other by Example 3.1. In fact, the ring $R = \mathrm{Mat}_2(\mathbb{Z}_2)$ in Example 3.1(1) is not NI by [21, Example 2.1].

But we obtain the following.

Proposition 3.2. *Let R be a CNZ ring. Then R is NI if and only if R is nil-IFP.*

Proof. Suppose that R is NI and let $ab=0$ for $a,b\in N(R)$. Since R is CNZ, $ba=0$ and so $bar=0$ for any $r\in R$. Since $aR\subseteq N(R)$, $arb=0$ by hypothesis. Thus R is nil-IFP. □

Let A be an algebra over a commutative ring S. Due to Dorroh [6], the *Dorroh extension* of A by S is the Abelian group $A\times S$ with multiplication given by $(r_1,s_1)(r_2,s_2)=(r_1r_2+s_1r_2+s_2r_1,s_1s_2)$ for $r_i\in A$ and $s_i\in S$. We use $A\times S$ to denote the Dorroh extension of A by S.

Theorem 3.3. *Let R be an algebra with identity over a commutative reduced ing S. Then R is CNZ if and only if the Dorroh extension $D=R\times S$ is CNZ.*

Proof. It can be easily checked that $N(D)=(N(R),0)$ since S is a commutative reduced ring. For any $(r_1,0),(r_2,0)\in N(D)$,

$$(r_1,0)(r_2,0)=(0,0) \text{ if and only if } r_1r_2=0.$$

This implies that R is CNZ if and only if the Dorroh extension D is CNZ. □

An element u of a ring R is *right regular* if $ur=0$ implies $r=0$ for $r\in R$. Similarly, *left regular* elements can be defined. An element is *regular* if it is both left and right regular (i.e., not a zero divisor).

A multiplicatively closed subset S of a ring R is said to satisfy the *right Ore condition* if for each $a\in R$ and $b\in S$, there exist $a_1\in R$ and $b_1\in S$ such that $ab_1=ba_1$. It is shown by [23, Theorem 2.1.12] that S satisfies the right Ore condition and S consists of regular elements if and only if the right quotient ring $Q(R)$ of R with respect to S exists.

Theorem 3.4. *Let S be a multiplicatively closed subset of a ring R, and suppose that S satisfies the right Ore condition and S consists of regular elements. Assume that the right quotient ring $Q=Q(R)$ of R with respect to S is NI. Then R is CNZ if and only if Q is CNZ.*

Proof. Let Q be an NI ring. It suffices to show that the right quotient ring Q of R is CNZ by Lemma 2.4(2) when R is CNZ. Assume that R is CNZ. Then R is nil-IFP by Proposition 3.2 and [11, Lemma 2.1]. Let $\alpha\beta=0$ for $\alpha=ab^{-1},\beta=cd^{-1}\in N(Q)$. Set I and J be the ideals of Q generated by α and β, respectively. Then both I and J are nil since Q is NI, with $a=\alpha b\in I$ and $c=\beta d\in J$, and so $a,c\in N(R)$. Since S satisfies the right Ore condition, $b^{-1}c=c_1b_1^{-1}$ for some $c_1\in R$ and $b_1\in S$. Then $0=\alpha\beta=ab^{-1}cd^{-1}=ac_1b_1^{-1}d^{-1}$ and so $ac_1=0$. Since R is nil-IFP, $0=abc_1=acb_1$ and hence $ac=0$. Since S satisfies the right Ore condition, $d^{-1}a=a_1d_1^{-1}$ for some $a_1\in R$ and $d_1\in S$. From $0=ac$, we get $0=ad_1c=da_1c$ and so $a_1c=0$. Here, $a_1\in N(R)$ since $da_1=ad_1\in I$. Thus $ca_1=0$ by hypothesis, and thus $\beta\alpha=cd^{-1}ab^{-1}=ca_1d_1^{-1}b^{-1}=0$. Therefore Q is CNZ. □

The following is a similar result to Theorem 3.4.

Proposition 3.5. *Let M be a multiplicatively closed subset of a ring R consisting of central regular elements. Then R is CNZ if and only if $M^{-1}R$ is CNZ.*

Proof. It comes from the fact that $N(M^{-1}R) = M^{-1}N(R)$. □

Recall the ring of *Laurent polynomials* in x, written by $R[x, x^{-1}]$. Letting $M = \{1, x, x^2, \ldots\}$, M is clearly a multiplicatively closed subset of central regular elements in $R[x]$ such that $R[x, x^{-1}] = M^{-1}R[x]$. So Proposition 3.5 yields the following.

Corollary 3.6. *Let R be a ring. Then $R[x]$ is CNZ if and only if $R[x, x^{-1}]$ is CNZ.*

A ring R is called (*von Neumann*) *regular* [8] if for each $a \in R$ there exists $b \in R$ such that $a = aba$. It is well-known that a regular ring R is reversible if and only if it is IFP if and only if it is abelian. But there exists a regular CNZ ring which is not abelian. For example, the CNZ ring $\mathrm{Mat}_2(\mathbb{Z}_2)$, in Example 3.1(1), is regular but not abelian, obviously.

Due to Nielsen [26] and Rege and Chhawchharia [27], a ring R is called *right* (resp. *left*) *McCoy* when $f(x)g(x) = 0$ implies $f(x)r = 0$ (resp. $rg(x) = 0$) for some nonzero $r \in R$, where $f(x), g(x)$ are nonzero polynomials in $R[x]$. If a ring is both left and right McCoy we say that the ring is a McCoy ring. It is shown that if R is a reversible ring then R is a McCoy ring by [26, Theorem 2], and the converse holds for regular rings by [18, Theorem 20]. However, the regular CNZ ring $\mathrm{Mat}_2(\mathbb{Z}_2)$ as above is neither left nor right McCoy by [12, Proposition 1.6].

It is easily shown that the ring properties of reducedness and commutativity can extend to polynomial rings. But the reversible ring property does not extend to polynomial rings by [17, Example 2.1]. So one may ask whether the polynomial rings over CNZ rings are CNZ. However the answer is negative by the following.

Example 3.7. We use the ring and apply the argument in [17, Example 2.1] and Example 2.9. Let R be the CNZ ring in Example 2.9. Note that

$$N(R) = N^*(R) = \mathbb{Z}_2\langle a_0, a_1, a_2, b_0, b_1, b_2, c\rangle$$

and

$$N(R)[x] = N(R[x]) = N^*(R[x]) = \mathbb{Z}_2\langle a_0, a_1, a_2, b_0, b_1, b_2, c\rangle[x] \text{ and}$$
$$\frac{R}{N^*(R)}[x] \cong \frac{R[x]}{N^*(R[x])} \cong \mathbb{Z}_2,$$

where $\mathbb{Z}_2\langle a_0, a_1, a_2, b_0, b_1, b_2, c\rangle$ means the set of all elements in S of zero constants.

Now consider $R[x]$ and take

$$f(x) = a_0 + a_1x + a_2x^2 \text{ and } g(x) = b_0c + b_1cx + b_2cx^2$$

in $R[x]$. Then $f(x), g(x) \in N(R[x])$ such that $f(x)g(x) = (a_0+a_1x+a_2x^2)(b_0+b_1x+b_2x^2)c = 0$. But

$$g(x)f(x) = (b_0c + b_1cx + b_2cx^2)(a_0 + a_1x + a_2x^2) = (b_0ca_1 + b_1ca_0)x + \cdots \neq 0$$

since $b_0ca_1 + b_1ca_0 \notin I$. Thus $R[x]$ is not CNZ.

We next find a condition under which the CNZ property extends to polynomial rings and power series rings.

Theorem 3.8. *Let R be a nil-IFP ring such that $(ab)^2 = 0$ implies $ab = 0$ for all $a, b \in N(R)$. Then we have the following.*

(1) *If $aba = 0$ or $ab^2 = 0$ for any $a, b \in N(R)$, then $ab = 0$.*

(2) *Both $R[[x]]$ and $R[x]$ are CNZ.*

Proof. Note that R is CNZ by Lemma 2.4(3).

(1) If $aba = 0$ for $a, b \in N(R)$, then $(ab)^2 = 0$ and so $ab = 0$ by hypothesis. Now suppose that $ab^2 = 0$ for $a, b \in N(R)$. Since $ab \in N(R)$ by Proposition 3.2, $ab^2 = (ab)b = 0 \Rightarrow 0 = (ab)ab = (ab)^2$ and thus $ab = 0$.

(2) It is enough to show that $R[[x]]$ is CNZ by Lemma 2.4(2). Let $f(x) = \sum_{i=0}^{\infty} a_i x^i$, $g(x) = \sum_{j=0}^{\infty} b_j x^j \in N(R[[x]])$ with $f(x)g(x) = 0$. Then $f(x), g(x) \in N(R)[[x]]$, since $N(R[[x]]) \subseteq N(R)[[x]]$ by help of [10, Lemma 2] and [24, Theorem 2.5]. From $f(x)g(x) = 0$, we get

$$\text{(I)} \qquad \sum_{k=0}^{\infty}\left(\sum_{i+j=k} a_i x^i b_j x^j\right) = \sum_{k=0}^{\infty}\left(\sum_{i+j=k} a_i b_j x^{i+j}\right) = 0.$$

We claim that $a_ib_j = 0$ for all i, j, proceeding by induction on $i + j$. First we obtain $a_0b_0 = 0$. This proves the claim for $i + j = 0$. Now suppose that our claim is true for $i + j \leq n - 1$. From the equality (I), we have

$$\text{(II)} \qquad \sum_{l=0}^{n} a_l b_{n-l} = 0.$$

Multiplying the equality (II) by b_0 on the right side, we obtain $a_nb_0b_0 = 0$ by the inductive hypothesis, and so $a_nb_0 = 0$ by (1). The equality (II) becomes,

$$\text{(III)} \qquad a_0b_n + a_1b_{n-1} + \cdots + a_{n-1}b_1 = 0.$$

Multiplying the equality (III) by $b_1, b_2, \ldots, b_n$ on the right side, we get

$$a_{n-1}b_1 = 0, \ldots, a_1b_{n-1} = 0, a_0b_n = 0$$

in turn by the similar argument to above, and so $a_ib_j = 0$ for $i + j = n$. Inductively, $a_ib_j = 0$ for all i, j. Since R is CNZ and $f(x), g(x) \in N(R)[[x]]$, $b_ja_i = 0$ for all i, j and thus $g(x)f(x) = 0$. This concludes that $R[[x]]$ is CNZ. □

The condition "a ring R such that$(ab)^2 = 0$ implies $ab = 0$ for all $a, b \in N(R)$" in Theorem 3.8 is not superfluous. Consider the reversible ring R in

Example 2.9. Then R is nil-IFP but $R[x]$ is not CNZ. Note that $(a_0b_1)^2 = 0$ for $a_0, b_1 \in N(R)$ but $a_0b_1 \neq 0$ by the construction of I.

Rege and Chhawchharia [27] called a ring R *Armendariz* if whenever any polynomials $f(x) = \sum_{i=0}^{m} a_ix^i$, $g(x) = \sum_{j=0}^{n} b_jx^j \in R[x]$ satisfy $f(x)g(x) = 0$, $a_ib_j = 0$ for all i, j. This nomenclature was used by them since it was Armendariz [2, Lemma 1] who initially showed that a reduced ring always satisfies this condition.

The Armendariz property of polynomial rings is extended to power series rings by Kim et al. [15]. A ring R is called *power-serieswise Armendariz* if $ab = 0$ for all $a \in C_{f(x)}$ and $b \in C_{g(x)}$ whenever $f(x), g(x) \in R[[x]]$ satisfy $f(x)g(x) = 0$. Every power-serieswise Armendariz ring is obviously Armendariz by definition, but not conversely by [15, Example 2.1].

Proposition 3.9. (1) *If R is an Armendariz ring, then R is CNZ if and only if $R[x]$ is CNZ.*

(2) *If R is a power-serieswise Armendariz ring, then the following are conditions equivalent*:

(i) *R is CNZ*; (ii) *$R[x]$ is CNZ*; (iii) *$R[[x]]$ is CNZ.*

Proof. (1) It is enough to show that $R[x]$ is CNZ when so is R, by Lemma 2.4(2). Assume that R is Armendariz and CNZ. Let $f(x)g(x) = 0$ for $f(x) = \sum_{i=0}^{m} a_ix^i, g(x) = \sum_{j=0}^{n} a_jx^j \in N(R[x])$. Then $f(x), g(x) \in N(R)[x]$ because $N(R[x]) = N(R)[x]$ by [1, Corollary 5.2]. Since R is Armendariz, $a_ib_j = 0$ for all i and j. This implies that $b_ja_i = 0$ for all i and j by hypothesis and so $g(x)f(x) = 0$. Thus $R[x]$ is CNZ.

(2) Let R be a power-serieswise Armendariz ring. Then it suffices to show that $R[[x]]$ is CNZ when so is R by (1) and Lemma 2.4(2). Note that $N(R[[x]]) \subseteq N(R)[[x]]$ for a power-serieswise Armendariz ring R by [15, Lemma 2.3(2)] and [10, Lemma 2]. Hence, it can be proved that $R[[x]]$ is CNZ if R is CNZ by the similar computation to the proof of (1). □

Note that $D_3(R)$ over a reduced ring R is a power-serieswise Armendariz ring by [15, Corollary 3.6(2)], but $D_3(R)$ is not CNZ as noted above, and the CNZ ring $R = \mathrm{Mat}_2(K)$ over a field K, in Example 2.2, is not Armendariz by [16, Example 1]. However, a ring R is reduced (i.e., $U_2(R)$ is a CNZ ring) if and only if $D_3(R)$ is an Armendariz ring if and only if $D_3(R)$ is a power-serieswise Armendariz ring by help of [13, Proposition 2.8].

As parallel extensions to Proposition 3.9, we finally consider the concept of CNZ ring property for skew (Laurent) polynomial rings and skew (Laurent) power series rings.

For a ring R with an endomorphism α, we denote $R[x;\alpha]$ a *skew polynomial* ring (also called an *Ore extension of endomorphism type*) whose elements are the polynomials $\sum_{i=0}^{n} a_ix^i, a_i \in R$, where the addition is defined as usual and the multiplication subject to the relation $xa = \alpha(a)x$ for any $a \in R$. The set $\{x^j\}_{j\geq 0}$ is easily seen to be a left Ore subset of $R[x;\alpha]$, so that one can localize $R[x;\alpha]$ and form the *skew Laurent polynomial* ring $R[x,x^{-1};\alpha]$. Elements of $R[x,x^{-1};\alpha]$ are finite sums of elements of the form $x^{-j}ax^i$ where $a \in R$ and i and j are nonnegative integers. The *skew power series* ring is denoted by $R[[x;\alpha]]$, whose elements are the series $\sum_{i=0}^{\infty} a_i x^i$ for some $a_i \in R$ and nonnegative integers i. The *skew Laurent power series* ring $R[[x,x^{-1};\alpha]]$ which contains $R[[x;\alpha]]$ as a subring, arises as the localization of $R[[x;\alpha]]$ with respect to the Ore set $\{x^j\}_{j\geq 0}$, and when α is an automorphism of R, it consists elements of the form $x^s a_s + x^{s+1}a_{s+1} + \cdots + a_0 + a_1 x + \cdots$, for some $a_i \in R$ and integers $s \leq 0$ and $i \geq s$, where the addition is defined as usual and the multiplication is defined by the rule $xa = \alpha(a)x$ for any $a \in R$.

Recall that a ring R with an endomorphism α is called *skew power-serieswise Armendariz* (or SPA for short) [25, Definition 2.1] if for every skew power series $p(x) = \sum_{i=0}^{\infty} a_i x^i, q(x) = \sum_{j=0}^{\infty} b_j x^j \in R[[x;\alpha]]$, $p(x)q(x) = 0 \Leftrightarrow a_i b_j = 0$ for all i, j.

Lemma 3.10. *Let R be an SPA ring and α an endomorphism of R. Then we have the following.*

(1) *For $a, b \in R$, $ab = 0$ if and only if $a\alpha(b) = 0$.*

(2) *If α is an automorphism and S is one of symbols $R[x;\alpha]$, $R[x,x^{-1};\alpha]$, $R[[x;\alpha]]$ or $R[[x,x^{-1};\alpha]]$, then $N(RS) = N(R)S$.*

Proof. By [25, Lemma 2.2(1) and Theorem 2.13]. □

Theorem 3.11. *Let R be an SPA ring and α an automorphism of R. Then the following are equivalent*:

(1) *R is CNZ.*
(2) *$R[x;\alpha]$ is CNZ.*
(3) *$R[x,x^{-1};\alpha]$ is CNZ.*
(4) *$R[[x;\alpha]]$ is CNZ.*
(5) *$R[[x,x^{-1};\alpha]]$ is CNZ.*

Proof. It is enough to show that (1)⇒(5) by Lemma 2.4(2). Assume that (1) holds. Let $p(x)q(x) = 0$ for $p(x) = x^s a_s + x^{s+1}a_{s+1} + \cdots + a_0 + a_1 x + \cdots, q(x) = x^t b_t + x^{t+1}b_{t+1} + \cdots + b_0 + b_1 x + \cdots \in N(R[[x,x^{-1};\alpha]])$ where s and t are integers with $s, t \leq 0$. Then $a_i b_j = 0$ by [25, Proposition 2.8], and moreover $a_i, b_j \in N(R)$ for all $s \leq i$ and $t \leq j$ by Lemma 3.10(2). Then we have $b_j a_i = 0$ and so $b_j \alpha^n(a_i) = 0$ for any nonnegative integer n, by Lemma 3.10(1). Thus $q(x)p(x) = 0$, and therefore $R[[x,x^{-1};\alpha]]$ is CNZ. □

Corollary 3.12. *Let R be a power-serieswise Armendariz ring. The following are equivalent*:

(1) *R is CNZ.*
(2) *$R[x]$ is CNZ.*
(3) *$R[x,x^{-1}]$ is CNZ.*
(4) *$R[[x]]$ is CNZ.*
(5) *$R[[x,x^{-1}]]$ is CNZ.*

For a ring R with a monomorphism α, let $A(R,\alpha)$ be the subset

$$\{x^{-i}rx^i \mid r \in R \text{ and } i \geq 0\}$$

of the skew Laurent polynomial ring $R[x,x^{-1};\alpha]$. Note that for $j \geq 0$, $x^j r = \alpha^j(r)x^j$ implies $rx^{-j} = x^{-j}\alpha^j(r)$ for $r \in R$. This yields that for each $j \geq 0$ we have $x^{-i}rx^i = x^{-(i+j)}\alpha^j(r)x^{i+j}$. It follows that $A(R,\alpha)$ forms a subring of $R[x,x^{-1};\alpha]$ with the following natural operations: $x^{-i}rx^i + x^{-j}sx^j = x^{-(i+j)}(\alpha^j(r) + \alpha^i(s))x^{i+j}$ and $(x^{-i}rx^i)(x^{-j}sx^j) = x^{-(i+j)}\alpha^j(r)\alpha^i(s)x^{i+j}$ for $r,s \in R$ and $i,j \geq 0$. Note that $A(R,\alpha)$ is an over-ring of R, and the map $\bar{\alpha} : A(R,\alpha) \to A(R,\alpha)$ defined by $\bar{\alpha}(x^{-i}rx^i) = x^{-i}\alpha(r)x^i$ is an automorphism of $A(R,\alpha)$. Jordan showed, with the use of left localization of the skew polynomial $R[x;\alpha]$ with respect to the set of powers of x, that for any pair (R,α), such an extension $A(R,\alpha)$ always exists in [14]. This ring $A(R,\alpha)$ is usually said to be the *Jordan extension* of R by α.

Proposition 3.13. *Let R be a ring R with a monomorphism α. Then R is CNZ if and only if the Jordan extension $A = A(R,\alpha)$ of R by α is CNZ.*

Proof. It is enough to show the necessity by Lemma 2.4(2). Suppose that R is CNZ and $cd = 0$ for $c = x^{-i}rx^i, d = x^{-j}sx^j \in N(A)$ for $i,j \geq 0$. Then $r,s \in N(R)$ obviously and so $\alpha^m(r), \alpha^n(s) \in N(R)$ for any nonnegative integers m and n, since $\alpha(N(R)) \subseteq N(R)$. From $cd = 0$, we have $\alpha^j(r)\alpha^i(s) = 0$ and hence $0 = \alpha^i(s)\alpha^j(r)$ by assumption. This implies that $dc = 0$, showing that the Jordan extension A of R by α is CNZ. □

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