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GENERALIZED BI-QUASI-VARIATIONAL-LIKE INEQUALITIES ON NON-COMPACT SETS

Yeol Je Cho, Mohammad S. R. Chowdhury, and Je Ai Ha

Abstract. In this paper, we prove some existence results of solutions for a new class of generalized bi-quasi-variational-like inequalities (GBQVLI) for (η-h)-quasi-pseudo-monotone type I and strongly (η-h)-quasi-pseudo-monotone type I operators defined on non-compact sets in locally convex Hausdorff topological vector spaces. To obtain our results on GBQVLI for (η-h)-quasi-pseudo-monotone type I and strongly (η-h)-quasi-pseudo-monotone type I operators, we use Chowdhury and Tan's generalized version of Ky Fan's minimax inequality as the main tool.

1. Introduction

Let E, F be topological spaces and let $g : E \to 2^F$ be a multi-valued mapping.

The mapping g is said to be *upper semi-continuous* on E if, for all $x_0 \in E$ and for each open set G in F with $g(x_0) \subset G$, there exists an open neighborhood $N(x_0)$ of x_0 such that $g(x) \subset G$ for all $x \in N(x_0)$. The mapping g is said to be *lower semi-continuous* on E if, for all $x_0 \in E$ and for each open set G in F with $g(x_0) \cap G \neq \emptyset$, there exists an open neighborhood $N(x_0)$ of x_0 such that $g(x) \cap G \neq \emptyset$ for all $x \in N(x_0)$. The mapping g is said to be *continuous* on E if g is both upper semi-continuous and lower semi-continuous on E.

Note that a multi-valued mapping g is upper semi-continuous (resp., lower semi-continuous) if the inverse image of a closed set (resp., an open set) is closed (resp., open), where, if $A \subset E$, then the set

$$g(A) = \cup_{x \in A} g(x) = \{y \in F : g^{-1}(y) \cap A \neq \emptyset\}$$

is called the *image* of A under g. If $B \subset F$, the set

$$g^{-1}(B) = \cup_{y \in B} g^{-1}(y) = \{x \in E : g(x) \cap B \neq \emptyset\}$$

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is called the *inverse image* of B under g.

Let E be a topological vector space over the field Φ, F be a vector space over Φ and $\langle \cdot, \cdot \rangle : F \times E \to \Phi$ be a bilinear functional. For each $x_0 \in E$, for each nonempty subset A of E and $\varepsilon > 0$, let

$$W(x_0; \varepsilon) = \{y \in F : |\langle y, x_0 \rangle| < \varepsilon\}$$

and

$$U(A; \varepsilon) = \{y \in F : \sup_{x \in A} |\langle y, x \rangle| < \varepsilon\}.$$

Let $\sigma\langle F, E\rangle$ be the topology on F generated by the family

$$\{W(x_0; \varepsilon) : x \in E, \, \varepsilon > 0\}$$

as a subbase for the neighborhood system at 0 and let $\delta\langle F, E\rangle$ be the topology on F generated by the family

$$\{U(A; \varepsilon) : A \text{ is a nonempty compact subset of } E, \, \varepsilon > 0\}$$

as a base for the neighborhood system at

We note then that F, when equipped with the topology $\sigma\langle F, E\rangle$ or the topology $\delta\langle F, E\rangle$, becomes a locally convex topological vector space, but not necessarily a Hausdorff topological vector space. Furthermore, for a net $\{y_\alpha\}$ in F and $y \in F$, we have the following:

(1) $y_\alpha \to y$ in $\sigma\langle F, E\rangle$ if and only if $\langle y_\alpha, x\rangle \to \langle y, x\rangle$ for each $x \in E$;

(2) $y_\alpha \to y$ in $\delta\langle F, E\rangle$ if and only if $\langle y_\alpha, x\rangle \to \langle y, x\rangle$ uniformly for each $x \in A$, where A is a nonempty compact subset of E.

Definition 1.1. Let X be a nonempty subset of E. A mapping $T : X \to 2^F$ is said to be *monotone* with respect to the bilinear functional $\langle \cdot, \cdot \rangle$ if, for any $x, y \in X$, $\forall u \in T(x)$ and $\forall w \in T(y)$,

$$Re\langle w - u, y - x\rangle \geq 0.$$

Remark 1.1. (1) When $F = E^*$, the vector space of all continuous linear functionals on E, and $\langle \cdot, \cdot \rangle$ is the usual pairing between E^* and E, then the monotonicity notion coincides with the usual definition, i.e.,

$$Re\langle Ty - Tx, y - x\rangle \geq 0$$

for any $x, y \in X$, when $T : X \to E^*$ is single-valued, and

$$Re\langle w - u, y - x\rangle \geq 0$$

for any $x, y \in X$, $\forall u \in T(x)$ and $\forall w \in T(y)$, when $T : X \to 2^{E^*}$ is set-valued.

(2) A mapping $T : X \to 2^F$ is monotone if and only if its graph $G(T) = \{(x, y) : y \in T(x)\}$ is a monotone subset of $X \times F$, i.e., for all $(x_1, y_1), (x_2, y_2) \in G(T)$,

$$Re\langle y_2 - y_1, x_2 - x_1\rangle \geq 0.$$

In 1989, Shih and Tan [30] introduced the following problem:
Let E and F be vector spaces over Φ, $\langle\cdot,\cdot\rangle : F\times E \to \Phi$ be a bilinear functional and X be a nonempty subset of E.

If $S : X \to 2^X$ and $M, T : X \to 2^F$, then the *generalized bi-quasi-variational inequality problem* (GBQVI) for the triple (S, M, T) is as follows:

Find $\hat{y} \in X$ such that

(1) $\hat{y} \in S(\hat{y})$;

(2) $\inf_{w\in T(\hat{y})} Re\langle f - w, \hat{y} - x\rangle \leq 0$ for all $x \in S(\hat{y})$ and $f \in M(\hat{y})$.

If T is a single-valued mapping, then a generalized bi-quasi-variational inequality problem will be called a *bi-quasi-variational inequality problem*.

We have the following special cases of the problem (GBQVI):

Suppose the E is a topological vector space, $F = E^*$, the vector space of all continuous linear functionals on E and $\langle\cdot,\cdot\rangle$ is the usual duality pairing between E^* and E.

(I) If $T = 0$, then a generalized bi-quasi-variational inequality problem for $(S, M, 0)$ becomes a generalized quasi-variational inequality problem:

Find $\hat{y} \in X$ such that

(1) $\hat{y} \in S(\hat{y})$;

(2) $Re\langle f, \hat{y} - x\rangle \leq 0$ for all $x \in S(\hat{y})$ and $f \in M(\hat{y})$.

This problem was studied by Chan and Pang [7] in the finite-dimensional case and, by Shih and Tan [31], in the infinite-dimensional case.

(II) If $T = 0$ and M is single-valued, then a generalized bi-quasi-variational inequality problem for $(S, M, 0)$ becomes a *quasi-variational inequality problem*:

Find $\hat{y} \in S(\hat{y})$ such that

$$Re\langle M(\hat{y}), \hat{y} - x\rangle \leq 0$$

for all $x \in S(\hat{y})$.

This problem was introduced by Bensoussan and Lions in 1973 in connection with impulse control (see Aubin [1], Baiocchi and Capelo [3], Bensoussan and Lions [4]).

(III) If $S(x) = X$, $M = 0$ and T is single-valued, then a generalized bi-quasi-variational inequality problem becomes a *variational inequality problem*:

Find $\hat{y} \in X$ such that

$$Re\langle T(\hat{y}), \hat{y} - x\rangle \geq 0$$

for all $x \in X$.

This problem was introduced by Stampacchia [32].

(IV) If $S(x) = X$ and $M = 0$, then a generalized bi-quasi-variational inequality problem becomes a *generalized variational inequality problem*:

Find $\hat{y} \in S(\hat{y})$ and $w \in T(\hat{y})$ such that

$$Re\langle w, \hat{y} - x\rangle \leq 0$$

for all $x \in S(x)$.

This problem was studied by Browder [6] and Yen [34].

Also, Shih and Tan proved the following theorems:

Theorem 1.1. *Let E be a locally convex Hausdorff topological vector space over Φ, X be a nonempty compact convex subset of E and F be a topological vector space over Φ. Let $\langle \cdot, \cdot \rangle : F \times E \to \Phi$ be a bilinear functional which is continuous on compact subsets of $F \times X$. Suppose that*

(a) $S : X \to 2^X$ *is an upper semi-continuous mapping such that each* $S(x)$ *is closed convex*;

(b) $M : X \to 2^F$ *is a monotone mapping with respect to* $\langle \cdot, \cdot \rangle$;

(c) $T : X \to 2^F$ *is an upper semi-continuous mapping such that each* $T(x)$ *is compact*;

(d) *the set*

$$\Sigma = \{y \in X : \sup_{x \in S(y)} \sup_{f \in M(x)} \inf_{w \in T(y)} Re\langle f - w, y - x \rangle > 0\}$$

is open in X.

Then there exists a point $\hat{y} \in X$ *such that*

(1) $\hat{y} \in S(\hat{y})$;

(2) $\inf_{w \in T(\hat{y})} Re\langle f - w, \hat{y} - x \rangle \leq 0$ *for all* $x \in S(\hat{y})$ *and* $f \in M(x)$.

In addition, if M *is lower semi-continuous along the line segments in* X *to the topology* $\sigma\langle F, E \rangle$ *on* F, *then*

(3) $\inf_{w \in T(\hat{y})} Re\langle f - w, \hat{y} - x \rangle \leq 0$ *for all* $x \in S(\hat{y})$ *and* $f \in M(\hat{y})$.

Moreover, if $S(x) = X$ *for all* $x \in X$, *then* E *is not required to be locally convex and, if* $T \equiv 0$, *then the continuity assumption on* $\langle \cdot, \cdot \rangle$ *can be weakened to the assumption that, for each* $f \in F$, *the mapping* $x \mapsto \langle f, x \rangle$ *is continuous on* X.

Theorem 1.2. *Let E be a locally convex Hausdorff topological vector space over Φ, X be a nonempty compact convex subset of E and F be a topological vector space over Φ. Let $\langle \cdot, \cdot \rangle : F \times E \to \Phi$ be a bilinear functional which is continuous on X and let F equip with the topology $\delta\langle F, E \rangle$. Suppose that*

(a) $S : X \to 2^X$ *is an upper semi-continuous mapping such that each* $S(x)$ *is closed convex*;

(b) $M : X \to 2^F$ *is a monotone mapping with respect to* $\langle \cdot, \cdot \rangle$ *and lower semi-continuous*;

(c) $T : X \to 2^F$ *is an upper semi-continuous mapping such that each* $T(x)$ *is compact.*

Then there exists a point $\hat{y} \in X$ *such that*

(1) $\hat{y} \in S(\hat{y})$;

(2) $\inf_{w \in T(\hat{y})} Re\langle f - w, \hat{y} - x \rangle \leq 0$ *for all* $x \in S(\hat{y})$ *and* $f \in M(\hat{y})$.

Remark 1.2. Since the results of Shih and Tan, some authors have obtained many results on generalized (quasi-)variational inequalities, generalized (quasi-)variational-like inequalities and generalized bi-quasi-variational inequalities in topological vector spaces (see [9–24]).

In this paper, we obtain some existence results for solutions for a new class of generalized bi-quasi-variational-like inequalities (GBQVLI) for (η, h)-quasi-pseudo-monotone type I and strongly (η, h)-quasi-pseudomonotone type I operators defined on non-compact sets in locally convex Hausdorff topological vector spaces. In fact, the generalized bi-quasi-variational-like inequalities (GBQVLI) are the extensions of the generalized bi-quasi-variational inequalities (GBQVI) which was first introduced by Shih and Tan [31] in 1989.

2. Preliminaries

In 2010, Chowdhury and Tan [17] obtained the generalized bi-quasi-variational inequalities for quasi-pseudomonotone type I and strongly quasi-pseudomonotone type I operators on non-compact sets. As we have mentioned above, we are going to obtain some results for solutions for a new class of generalized bi-quasi-variational-like inequalities (GBQVLI) for (η, h)-quasi-pseudomonotone type I and strongly (η, h)-quasi-pseudomonotone type I operators on non-compact sets. For this, we now introduce the following definition of generalized bi-quasi-variational-like inequality (GBQVLI):

Let $S : X \to 2^X$ be a set-valued mapping, $M, T : X \to 2^F$ be two set-valued mappings and $\eta : X \times X \to E$ be a single-valued mapping. The *generalized bi-quasi-variational-like inequality problem* (GBQVLI) is as follows:

Find a point $\hat{y} \in X$ and a point $\hat{w} \in T(\hat{y})$ such that
(1) $\hat{y} \in S(\hat{y})$;
(2) $Re\langle f - \hat{w}, \eta(\hat{y}, x)\rangle \leq 0$ for all $x \in S(\hat{y})$ and $f \in M(\hat{y})$;
or
Find a point $\hat{y} \in X$, a point $\hat{w} \in T(\hat{y})$ and a point $\hat{f} \in M(\hat{y})$ such that
(1) $\hat{y} \in S(\hat{y})$;
(3) $Re\langle \hat{f} - \hat{w}, \eta(\hat{y}, x)\rangle \leq 0$ for all $x \in S(\hat{y})$.

If $\eta(\hat{y}, x) = \hat{y} - x$, then the generalized bi-quasi-variational-like inequality (GBQVLI) is equivalent to the generalized bi-quasi-variational inequality (GBQVI) introduced by Chowdhury and Tan in [14] and Shih and Tan in [31].

Now, we first introduce the following definition of (η, h)-quasi-pseudomonotone (resp., strongly (η, h)-quasi-pseudomonotone) type I operators which is a slight modification of the quasi-pseudomonotone (resp., strongly quasi-pseudomonotone) type I operators (see Definition 1.1 in [15] given by Chowdhury and Tan in 2010):

Definition 2.1. Let E be a topological vector space over Φ, X be a non-empty subset of E and F be a topological vector space over Φ which is equipped with the $\sigma\langle F, E\rangle$ topology. Let $\langle \cdot, \cdot \rangle : F \times E \to \Phi$ be a bilinear functional. Consider the following four mappings:
(1) $M : X \to 2^F$ is a multi-valued mapping;
(2) $T : X \to 2^F$ is a multi-valued mapping;
(3) $h : E \times E \to \mathbb{R}$ is a single-valued mapping;
(4) $\eta : X \times X \to E$ is a single-valued mapping.

Then the mapping T is said to be an (η, h)-*quasi-pseudomonotone type I* (resp., *strongly* (η, h)-*quasi-pseudomonotone type I*) *operator* if, for each $y \in X$ and net $\{y_\alpha\}_{\alpha\in\Gamma}$ in X converging to y (resp., weakly to y) with

$$\limsup_{\alpha}[\inf_{f\in M(y)}\inf_{u\in T(y_\alpha)} Re\langle f-u, \eta(y_\alpha, y)\rangle + h(y_\alpha, y)] \le 0,$$

we have

$$\begin{aligned}&\limsup_{\alpha}[\inf_{f\in M(x)}\inf_{u\in T(y_\alpha)} Re\langle f-u, \eta(y_\alpha, x)\rangle + h(y_\alpha, x)]\\ \ge&\inf_{f\in M(x)}\inf_{w\in T(y)} Re\langle f-w, \eta(y, x)\rangle + h(y, x)\end{aligned}$$

for all $x \in X$.

Remark 2.1. The above operator T reduces to an h-quasi-pseudomonotone type I (resp., strongly h-quasi-pseudomonotone type I) operator due to Chowdhury and Tan in [17] if T is an (η, h)-quasi-pseudomonotone type I (resp., strongly (η, h)-quasi-pseudomonotone type I) operator with $\eta(x, y) = x - y$ for all $x, y \in X$ and, for some $h' : E \to \mathbb{R}$, $h(x, y) = h'(x) - h'(y)$ for all $x, y \in E$.

Also, T reduces to a quasi-pseudomonotone type I (resp., strongly quasi-pseudomonotone type I) operator due to Chowdhury and Tan in 15] if T is an h-quasi-pseudomonotone type I (resp., strongly h-quasi-pseudomonotone type I) operator with $h \equiv 0$.

Remark 2.2. (1) When $M \equiv 0$ and T is replaced by $-T$, an h-quasi-pseudomonotone type I operator is reduced to an h-pseudomonotone (or an h-demimonotone) operator defined in [10].

(2) The h-pseudomonotone (or h-demi-monotone) operators defined in [10] are slightly more general than the definition of h-pseudomonotone operators given in [13].

(3) Later, in the year 2000, Chowdhury renamed the above h-pseudomonotone (or h-demi-monotone) operators as *pseudomonotone type I operators* [8]. The pseudomonotone type I operators are set-valued generalization of the classical (single-valued) pseudomonotone operators with slight variations. The classical definition of a single-valued pseudomonotone operator was introduced by Brézis et al. in [5].

(4) The authors first introduced quasi-pseudomonotone type I operators in [15, Definition 1.1] as a generalization of pseudomonotone type I operators.

We state the following result given in [17]:

Proposition 2.1. *Let X be a non-empty subset of a topological vector space E. Let $T : X \to E^*$ and $M : X \to E^*$ be two single-valued maps. Suppose that the operator T is monotone, and both M and T are continuous maps from the relative weak topology on X to the weak* topology on E^*. Then T is both quasi-pseudomonotone type I and strongly quasi-pseudomonotone type I operator.*

For the proof, see in [17, pp. 424–425].

The following result justifies the validity of an (η-h)-quasi-pseudo-monotone type I and strongly (η-h)-quasi-pseudo-monotone type I operators:

Proposition 2.2. *Let X be a non-empty subset of a topological vector space E. Let $T : X \to E^*$ and $M : X \to E^*$ be two single-valued maps. Suppose that $h : X \times X \to \mathbb{R}$ is a real valued function such that for each $y \in X$, $h(\cdot, y)$ is continuous and $h(X \times X)$ is bounded. Let $\eta : X \times X \to E$ be a continuous mapping.*

Further suppose that the operators T and M are η-monotone (i.e., for each $x, y \in X$, we have $Re\langle T(y) - T(x), \eta(y, x)\rangle \geq 0$ (respectively, $Re\langle M(y) - M(x), \eta(y, x)\rangle \geq 0$)), and also both M and T are continuous mappings from the relative weak topology on X to the weak* topology on E^*. Then T is both (η-h)-quasi-pseudo-monotone type I and strongly (η-h)-quasi-pseudo-monotone type I operator.

Proof. Suppose that $\{y_\alpha\}_{\alpha\in\Gamma}$ is a net in X and $y \in X$ with $y_\alpha \to y$ (respectively, $y_\alpha \to y$ weakly) and that

$$\limsup_{\alpha} Re\langle M(y) - T(y_\alpha), \eta(y_\alpha, y)\rangle + h(y_\alpha, y) \leq 0.$$

Let $x \in X$ be arbitrarily fixed. Then

$$\begin{aligned} &\limsup_{\alpha}[Re\langle M(x) - T(y_\alpha), \eta(y_\alpha, x)\rangle + h(y_\alpha, x)] \\ \geq\ &\limsup_{\alpha}[Re\langle M(x) - T(y_\alpha), \eta(y_\alpha, x)\rangle] + \liminf_{\alpha} h(y_\alpha, x). \end{aligned} \tag{2.1}$$

Since M and T are η-monotone, we have

$$Re\langle (M(x) - T(y_\alpha)) - (M(x) - T(y)), \eta(y_\alpha, x)\rangle \geq 0.$$

Thus we have

$$Re\langle M(x) - T(y_\alpha), \eta(y_\alpha, x)\rangle \geq Re\langle M(x) - T(y), \eta(y_\alpha, x)\rangle.$$

Hence, we have,

$$\begin{aligned} &\limsup_{\alpha}[Re\langle M(x) - T(y_\alpha), \eta(y_\alpha, x)] \\ \geq\ &\limsup_{\alpha}[Re\langle M(x) - T(y), \eta(y_\alpha, x)\rangle]. \end{aligned} \tag{2.2}$$

Therefore, from equations (2.1) and (2.2) we have,

$$\begin{aligned} &\limsup_{\alpha}[Re\langle M(x) - T(y_\alpha), \eta(y_\alpha, x)\rangle + h(y_\alpha, x)] \\ \geq\ &\limsup_{\alpha}[Re\langle M(x) - T(y), \eta(y_\alpha, x)\rangle] + \liminf_{\alpha} h(y_\alpha, x) \\ =\ &Re\langle M(x) - T(y), \eta(y, x)\rangle + h(y, x) \end{aligned}$$

for all $x \in X$.

Consequently, T is both (η-h)-quasi-pseudo-monotone type I and strongly (η-h)-quasi-pseudo-monotone type I operator. □

In this paper, we obtain some general theorems on solutions for a new class of generalized bi-quasi-variational-like inequalities for (η, h)-quasi-pseudomonotone type I and strongly (η, h)-quasi-pseudomonotone type I operators defined on non-compact spaces in topological vector spaces. To obtain these results, we mainly use the following generalized version of Ky Fan's minimax inequality [27] due to Chowdhury and Tan [10] which was stated and proved as Theorem 2.1 in [16] and is a slight modification of Theorem 1 in [10]:

Theorem 2.3. *Let E be a topological vector space, X be a nonempty convex subset of E, $\mathcal{F}(X)$ denote the family of all non-empty finite subsets of X and $f : X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ be such that*

(a) *for each $A \in \mathcal{F}(X)$ and fixed $x \in co(A)$, $y \mapsto f(x, y)$ is lower semi-continuous on $co(A)$;*

(b) *for each $A \in \mathcal{F}(X)$ and $y \in co(A)$, $\min_{x\in A} f(x, y) \le 0$;*

(c) *for each $A \in \mathcal{F}(X)$ and $x, y \in co(A)$, every net $\{y_\alpha\}_{\alpha\in\Gamma}$ in X converging to y with $f(tx+(1-t)y, y_\alpha) \le 0$ for all $\alpha \in \Gamma$ and $t \in [0, 1]$, we have $f(x, y) \le 0$;*

(d) *there exist a nonempty closed and compact subset K of X and $x_0 \in K$ such that $f(x_0, y) > 0$ for all $y \in X \setminus K$.*

Then there exists $\hat{y} \in K$ such that $f(x, \hat{y}) \le 0$ for all $x \in X$.

Definition 2.2. A function $\phi : X \times X \to \mathbb{R} \cup \{\pm\infty\}$ is said to be 0-*diagonally concave* (in short, 0-DCV) in the second argument [26] if, for any finite set $\{x_1, \dots, x_n\} \subset X$ and $\lambda_i \ge 0$ with $\sum_{i=1}^n \lambda_i = 1$, we have $\sum_{i=1}^n \lambda_i \phi(y, x_i) \le 0$, where $y = \sum_{i=1}^n \lambda_i x_i$.

Let E be a topological vector space over Φ, F be a vector space over Φ and X be a non-empty subset of E. Let $\langle \cdot, \cdot \rangle : F \times E \to \Phi$ be a bilinear functional. Throughout this paper, Φ denotes either the real field $\mathbb{R}$ or the complex field $\mathcal{C}$.

Now, we state the following definition given in [25]:

Definition 2.3. Let X, E, F be the sets defined above and $T : X \to 2^F$, $\eta : X \times X \to E$, $g : X \to E$ be mappings.

(1) The mappings T and η are said to have 0-*diagonally concave relation* (in short, 0-DCVR) if the function $\phi : X \times X \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$\phi(x, y) = \inf_{w\in T(x)} Re\langle w, \eta(x, y)\rangle$$

is 0-DCV in y;

(2) The mappings T and g are said to have *0-diagonally concave relation* if T and $\eta(x, y) = g(x) - g(y)$ have the 0-DCVR.

We first state the following result which is Lemma 1 of Shih and Tan in [25, pp. 334–335]:

Lemma 2.4. *Let X be a nonempty subset of a Hausdorff topological vector space E and $S : X \to 2^E$ be an upper semi-continuous mapping such that $S(x)$*

is a bounded subset of E for each $x \in X$. Then, for each continuous linear functional p on E, the functional $f_p : X \to \mathbb{R}$ defined by

$$f_p(y) = \sup_{x\in S(y)} Re\langle p, x\rangle$$

is upper semi-continuous, i.e., for each $\lambda \in \mathbb{R}$, the set

$$\{y \in X : f_p(y) = \sup_{x\in S(y)} Re\langle p, x\rangle < \lambda\}$$

is open in X.

The following result is Lemma 3 of Takahashi in [33, pp. 177] (see also Lemma 3 in [31, pp. 71–72]):

Lemma 2.5. *Let X and Y be topological spaces, $f : X \to \mathbb{R}$ be non-negative and continuous and $g : Y \to \mathbb{R}$ be lower semi-continuous. Then the functional $F : X \times Y \to \mathbb{R}$ defined by*

$$F(x, y) = f(x)g(y)$$

for all $(x, y) \in X \times Y$ is lower semi-continuous.

The following result, which was stated and proved as Lemma 2.2 in [16], follows from slight modification of Lemma 3 of Chowdhury and Tan given in [10]:

Lemma 2.6. *Let E be a Hausdorff topological vector space over Φ, $A \in \mathcal{F}(E)$ and $X = co(A)$ where $co(A)$ denotes the convex hull of A. Let F be a vector space over Φ and $\langle , \rangle : F \times E \to \phi$ be a bilinear functional such that $\langle , \rangle$ separates points in F. We equip F with the $\sigma\langle F, E\rangle$-topology. Suppose that, for each $w \in F$, $x \mapsto Re\langle w, x\rangle$ is continuous. Let $\eta : X \times X \to E$ be continuous. Let $T : X \to 2^F$ be upper semi-continuous from X into 2^F such that each $T(x)$ is $\sigma\langle F, E\rangle$-compact. Let $f : X \times X \to \mathbb{R}$ be defined by*

$$f(x, y) = \inf_{w\in T(y)} Re\langle w, \eta(y, x)\rangle$$

for all $x, y \in X$.

Suppose that $\langle , \rangle$ is continuous on the (compact) subset $[\cup_{y\in X}T(y)] \times \eta(X \times X)$ of $F \times E$. Then, for each fixed $x \in X$, $y \mapsto f(x, y)$ is lower semi-continuous on X.

For completeness we include the proof here given in [16]:

Proof. Let $\lambda \in \mathbb{R}$ be given and let $x \in X = co(A)$ be arbitrarily fixed. Let $A_\lambda = \{y \in X : f(x, y) \le \lambda\}$. Suppose that $\{y_\alpha\}_{\alpha\in\Gamma}$ is a net in A_λ and $y_0 \in co(A) = X$ such that $y_\alpha \to y_0$. Then for each $\alpha \in \Gamma$,

$$\lambda \ge f(x, y_\alpha) = \inf_{w\in T(y_\alpha)} Re\langle w, \eta(y_\alpha, x)\rangle.$$

Since F is equipped with the $\sigma\langle F,E\rangle$-topology, for each $x\in E$, the function $w\mapsto Re\langle w,x\rangle$ is continuous. Also, $\eta(y_\alpha,x)\to\eta(y_0,x)$ because $\eta(\cdot,x)$ is continuous. By the $\sigma\langle F,E\rangle$-compactness of $T(y_\alpha)$, there exists $w_\alpha\in T(y_\alpha)$ such that

$$\lambda\geq\inf_{w\in T(y_\alpha)}Re\langle w,\eta(y_\alpha,x)\rangle=Re\langle w_\alpha,\eta(y_\alpha,x)\rangle.$$

Since T is upper semi-continuous from $X=co(A)$ to the $\sigma\langle F,E\rangle$-topology on F, X is compact, and each $T(z)$ is $\sigma\langle F,E\rangle$-compact, $\cup_{z\in X}T(z)$ is also $\sigma\langle F,E\rangle$-compact by Proposition 3.1.11 of Aubin and Ekeland [2]. Thus there is a subnet $\{w_{\alpha'}\}_{\alpha'\in\Gamma'}$ of $\{w_\alpha\}_{\alpha\in\Gamma}$ and $w_0\in\cup_{z\in X}T(z)$ such that $w_{\alpha'}\to w_0$ in the $\sigma\langle F,E\rangle$-topology. Again, as T is upper semi-continuous with the $\sigma\langle F,E\rangle$-closed values, $w_0\in T(y_0)$.

Suppose that $A=\{a_1,a_2,\ldots,a_n\}$ and let $t_1,t_2,\ldots,t_n\geq 0$ with $\sum_{i=1}^n t_i=1$ such that $y_0=\sum_{i=1}^n t_ia_i$. For each $\alpha'\in\Gamma$, let $t_1^{\alpha'},t_2^{\alpha'},\ldots,t_n^{\alpha'}\geq 0$ with $\sum_{i=1}^n t_i^{\alpha'}=1$ such that $y_{\alpha'}=\sum_{i=1}^n t_i^{\alpha'}a_i$. Since E is Hausdorff and $y_{\alpha'}\to y_0$, we must have $t_i^{\alpha'}\to t_i$ for each $i=1,2,\ldots,n$. Thus

$$\begin{aligned}\lambda&\geq Re\langle w_{\alpha'},\eta(y_{\alpha'},x)\rangle=Re\langle w_{\alpha'},\eta(\sum_{i=1}^n t_i^{\alpha'}a_i,x)\rangle\\ &\to Re\langle w_0,\eta(\sum_{i=1}^n t_ia_i,x)\rangle \qquad (2.1)\\ &=Re\langle w_0,\eta(y_0,x)\rangle\geq\inf_{w\in T(y_0)}Re\langle w,\eta(y_0,x)\rangle=f(x,y_0),\end{aligned}$$

where (2.1) is true since $\eta(\cdot,x)$ is continuous on X and $\langle\,,\,\rangle$ is continuous on the compact subset $[\cup_{y\in X}T(y)]\times\eta(X\times X)$ of $F\times E$.

Hence $y_0\in A_\lambda$. Thus A_λ is closed in $X=co(A)$ for each $\lambda\in\mathbb{R}$. Therefore $y\mapsto f(x,y)$ is lower semi-continuous on X. □

By a slight modification of Lemma 4.2 in [12], we obtain below a further modification of the result given in [24, Lemma 2.3]:

Lemma 2.7. *Let E be a topological vector over ϕ, X a nonempty convex subset of E and F a vector space over ϕ. Let $\langle\cdot,\cdot\rangle:F\times E\to\Phi$ be a bilinear functional such that $\langle\cdot,\cdot\rangle$ separates points in F. We equip F with the $\sigma\langle F,E\rangle$-topology such that for each $w\in F$, the function $x\mapsto Re\langle w,x\rangle$ is continuous. Let $\eta:X\times X\to E$ be such that for each fixed $y\in X$, $\eta(\cdot,y)$ is continuous and for each fixed $x\in X$, $\eta(x,\cdot)$ is affine. Let $h:X\times X\to\mathbb{R}$ be a mapping such that for each fixed $y\in X$, $h(\cdot,y)$ is lower semi-continuous and convex on $co(A)$ for each $A\in\mathcal{F}(X)$, and for each fixed $x\in X$, $h(x,\cdot)$ is concave, and $h(x,x)=0$, $\eta(x,x)=0$, and T and η have the 0-DCVR.*

Suppose that $S:X\to 2^X$ is a mapping, $M:X\to 2^F$ is a lower semi-continuous mapping along line segments in X to the $\sigma\langle F,E\rangle$-topology on F and $T:X\to 2^F$ is an upper hemi-continuous mapping along line segments in X.

Suppose further that there exists $\hat{y} \in X$ *such that* $\hat{y} \in S(\hat{y})$, $S(\hat{y})$ *is convex and*

$$\inf_{f \in M(x)} \inf_{w \in T(\hat{y})} Re\langle f - w, \eta(\hat{y}, x)\rangle \leq h(x, \hat{y})$$

for all $x \in S(\hat{y})$. *Then*

$$\inf_{f \in M(\hat{y})} \inf_{w \in T(\hat{y})} Re\langle f - w, \eta(\hat{y}, x)\rangle \leq h(x, \hat{y})$$

for all $x \in S(\hat{y})$.

For completeness we give the detailed proof below:

Proof. Suppose that

$$\inf_{f \in M(x)} \inf_{w \in T(\hat{y})} Re\langle f - w, \eta(\hat{y}, x)\rangle \leq h(x, \hat{y}) \text{ for all } \ x \in S(\hat{y}).$$

Let $x \in S(\hat{y})$ be arbitrarily fixed. Let $z_t = tx + (1-t)\hat{y} = \hat{y} - t(\hat{y} - x)$ for all $t \in [0, 1]$. Then $z_t \in S(\hat{y})$ as $S(\hat{y})$ is convex.

Let $L = \{z_t : t \in [0, 1]\}$. Thus for every $t \in [0, 1]$

$$\inf_{f \in M(z_t)} \inf_{w \in T(\hat{y})} Re\langle f - w, \eta(\hat{y}, z_t)\rangle \leq h(z_t, \hat{y}).$$

Since for each $y \in S(\hat{y})$, $h(\cdot, y)$ is convex and for each $x \in S(\hat{y})$, $h(x, \cdot)$ is affine, we have

$$\begin{aligned} &\inf_{f \in M(z_t)} \inf_{w \in T(\hat{y})} Re\langle f - w, \eta(\hat{y}, tx + (1-t)\hat{y})\rangle \\ &\leq h(tx + (1-t)\hat{y}, \hat{y}) \leq t(h(x, \hat{y})) + (1-t)h(\hat{y}, \hat{y}) \end{aligned}$$

for all $t \in (0, 1]$; thus we have,

$$\inf_{f \in M(z_t)} \inf_{w \in T(\hat{y})} [Re\langle f - w, t\eta(\hat{y}, x) + (1-t)\eta(\hat{y}, \hat{y})\rangle] \leq t(h(x, \hat{y}));$$

therefore we have,

$$\inf_{f \in M(z_t)} \inf_{w \in T(\hat{y})} t[Re\langle f - w, \eta(\hat{y}, x)\rangle] \leq t(h(x, \hat{y})).$$

This implies that $\inf_{f \in M(z_t)} \inf_{w \in T(\hat{y})} Re\langle f - w, \eta(\hat{y}, x)\rangle \leq h(x, \hat{y})$ for all $t \in (0, 1]$. Since T is upper hemi-continuous on L, and M is lower semi-continuous on L, the function $f_{\eta(\hat{y},x)} : L \to \mathbb{R} \cup \{+\infty\}$, defined by

$$f_{\eta(\hat{y},x)}(z_t) = \inf_{f \in M(z_t)} \inf_{w \in T(\hat{y})} Re\langle f - w, \eta(\hat{y}, x)\rangle \ \text{ for each } z_t \in L,$$

is lower semi-continuous on L. Thus the set

$$A = \{z_t \in L : f_{\eta(\hat{y},x)}(z_t) \leq h(x, \hat{y})\}$$

is closed in L. Now $z_t \to \hat{y}$ in L as $t \to 0^+$. Since $z_t \in A$ for all $t \in (0, 1]$ we have $\hat{y} \in A$. Hence $f_{\eta(\hat{y},x)}(\hat{y}) = \inf_{f \in M(\hat{y})} \inf_{w \in T(\hat{y})} Re\langle f - w, \eta(\hat{y}, x)\rangle \leq h(x, \hat{y})$. Since $x \in S(\hat{y})$ is arbitrary, we have

$$\inf_{f \in M(\hat{y})} \inf_{w \in T(\hat{y})} Re\langle w, \eta(\hat{y}, x)\rangle \leq h(x, \hat{y}) \text{ for all } \ x \in S(\hat{y}).$$

□

We need the following Kneser's minimax theorem in [28, pp. 2418–2420] (see also Aubin [1, pp. 40–41]):

Theorem 2.8. *Let X be a nonempty convex subset of a vector space and Y be a nonempty compact convex subset of a Hausdorff topological vector space. Suppose that f is a real-valued function on $X \times Y$ such that, for each fixed $x \in X$, the mapping $y \mapsto f(x,y)$, i.e., $f(x,\cdot)$ is lower semi-continuous and convex on Y and, for each fixed $y \in Y$, the map $x \mapsto f(x,y)$, i.e., $f(\cdot, y)$ is concave on X. Then*

$$\min_{y\in Y}\sup_{x\in X} f(x,\, y) = \sup_{x\in X}\min_{y\in Y} f(x,\, y).$$

3. Generalized bi-quasi-variational-like inequalities

In this section, we obtain and prove some existence theorems for the solutions to the generalized bi-quasi-variational-like inequalities for (η, h)-quasi-pseudomonotone type I and strongly (η, h)-quasi-pseudomonotone type I operators T with non-compact domain in locally convex Hausdorff topological vector spaces. Our results extend and generalize the corresponding results in [31].

We first establish the following result:

Theorem 3.1. *Let E be a locally convex Hausdorff topological vector space over Φ, X be a nonempty para-compact convex and bounded subset of E and F a Hausdorff topological vector space over Φ. Let $\langle\cdot,\cdot\rangle : F \times E \to \Phi$ be a bilinear functional which is continuous on compact subsets of $F \times X$. Suppose that*

(a) *$S : X \to 2^X$ is upper semi-continuous such that each $S(x)$ is compact and convex;*

(b) *$h : E \times E \to \mathbb{R}$ is convex and $h(X \times X)$ is bounded;*

(c) *$T : X \to 2^F$ is an $(\eta$-$h)$-quasi-pseudo-monotone type I (respectively, strongly $(\eta$-$h)$-quasi-pseudo-monotone type I) operator and is upper semi-continuous such that each $T(x)$ is compact (respectively, weakly compact) and convex and $T(X)$ is strongly bounded;*

(d) *$T : X \to 2^F$, and $\eta : X \times X \to E$ have the 0-DCVR and $\eta : X \times X \to E$ is convex and continuous;*

(e) *$M : X \to 2^F$ is a linear mapping in X (and is therefore single-valued for each $x \in X$);*

(f) *for each fixed $y \in X$, $x \mapsto h(x,y)$, i.e., $h(\cdot,y)$ is lower semi-continuous on $co(A)$ for each $A \in \mathcal{F}(X)$ and, for each fixed $x \in X$, $h(x,\cdot)$ and $\eta(x,\cdot)$ are concave, and $\eta(x,\cdot)$ is affine and $h(x,x) = 0$, $\eta(x,x) = 0$;*

(g) *the set*

$$\Sigma = \{y \in X : \sup_{x\in S(y)} (\inf_{w\in T(y)} Re\langle M(x) - w, \eta(y,x)\rangle + h(y,x) > 0\}$$

is open in X.

Suppose further that there exist a nonempty closed and compact (respectively, weakly closed and weakly compact) subset K of X and a point $x_0 \in X$ such that $x_0 \in K \cap S(y)$ and $\inf_{w \in T(y)} Re\langle M(x_0) - w, \eta(y, x_0)\rangle + h(y, x_0) > 0$ for all $y \in X \setminus K$.
Then there exists a point $\hat{y} \in X$ such that
(1) $\hat{y} \in S(\hat{y})$;
(2) *there exists a point $\hat{w} \in T(\hat{y})$ with $Re\langle M(\hat{y}) - \hat{w}, \eta(\hat{y}, x)\rangle + h(\hat{y}, x) \leq 0$ for all $x \in S(\hat{y})$.*

Moreover, if $S(x) = X$ for all $x \in X$, then E is not required to be locally convex and, if $T \equiv 0$, then the continuity assumption on $\langle \cdot, \cdot \rangle$ can be weakened to the assumption that, for each $f \in F$, the mapping $x \mapsto \langle f, x\rangle$ is continuous (resp., weakly continuous) on X.

Proof. We divide the proof into three steps:
Step 1. There exists a point $\hat{y} \in X$ such that $\hat{y} \in S(\hat{y})$ and

$$\sup_{x \in S(\hat{y})} [\inf_{w \in T(\hat{y})} Re\langle M(x) - w, \eta(\hat{y}, x)\rangle + h(\hat{y}, x)] \leq 0.$$

Suppose the contrary. Then for each $y \in X$, either $y \notin S(y)$ or there exists $x \in S(y)$ such that

$$\inf_{w \in T(y)} Re\langle M(x) - w, \eta(y, x)\rangle + h(y, x) > 0,$$

that is, for each $y \in X$, either $y \notin S(y)$ or $y \in \Sigma$.

If $y \notin S(y)$, then, by a separation theorem for convex sets in locally convex Hausdorff topological vector spaces, there exist $p \in E^*$ and $\alpha \in \mathbb{R}$ such that $Re\langle p, x\rangle < \alpha < Re\langle p, y\rangle$ for all $x \in S(y)$. Therefore,

$$\sup_{x \in S(y)} Re\langle p, x\rangle \leq \alpha < Re\langle p, y\rangle.$$

Hence we have, $Re\langle p, y\rangle - \sup_{x \in S(y)} Re\langle p, x\rangle > 0$. Let

$$\gamma(y) = \sup_{x \in S(y)} \inf_{w \in T(y)} Re\langle M(x) - w, \eta(y, x)\rangle + h(y, x),$$

$$V_0 := \{y \in X | \gamma(y) > 0\} = \Sigma$$

and, for each $p \in E^*$, set

$$V_p := \{y \in X : Re\langle p, y\rangle - \sup_{x \in S(y)} Re\langle p, x\rangle > 0\}.$$

Then $X = V_0 \cup \bigcup_{p \in E^*} V_p$. Since each V_p is open in X by Lemma 2.1 and V_0 is open in X by hypothesis, $\{V_0, V_p : p \in E^*\}$ is an open covering for X. Since X is para-compact, there is a continuous partition of unity $\{\beta_0, \beta_p : p \in E^*\}$ for X subordinated to the open cover $\{V_0, V_p : p \in E^*\}$ (see Theorem VIII, 4.2 of Dugundji in [23]), that is, for each $p \in E^*$, $\beta_p : X \to [0, 1]$ and $\beta_0 : X \to [0, 1]$ are continuous functions such that, for each $p \in E^*$, $\beta_p(y) = 0$ for all $y \in X \setminus V_p$, $\beta_0(y) = 0$ for all $y \in X \setminus V_0$, $\{\text{support } \beta_0, \text{support } \beta_p : p \in E^*\}$ is locally finite and $\beta_0(y) + \Sigma_{p\in E^*}\beta_p(y) = 1$ for each $y \in X$. Note that, for each $A \in \mathcal{F}(X)$, h is continuous on $co(A)$ (see [29, Corollary 10.1.1, p. 83]).

Define a function $\phi : X \times X \to \mathbb{R}$ by

$$\phi(x,y) = \beta_0(y)[\inf_{w\in T(y)} Re\langle M(x) - w, \eta(y,x)\rangle + h(y,x)] + \sum_{p\in E^*} \beta_p(y) Re\langle p, y - x\rangle$$

for each $x, y \in X$. Then we have the following:

(1) Since E is Hausdorff, for each $A \in \mathcal{F}(X)$ and fixed $x \in co(A)$, the mapping

$$y \mapsto \inf_{w\in T(y)} Re\langle M(x) - w, \eta(y,x)\rangle + h(y,x)$$

is lower semi-continuous (resp., weakly lower semi-continuous) on $co(A)$ by Lemma 2.6 and the fact that h is continuous on $co(A)$ and therefore the map

$$y \mapsto \beta_0(y)[\inf_{w\in T(y)} Re\langle M(x) - w, \eta(y,x)\rangle + h(y,x)]$$

is lower semi-continuous (resp., weakly lower semi-continuous) on $co(A)$ by Lemma 2.5. Also, for each fixed $x \in X$,

$$y \mapsto \sum_{p\in E^*} \beta_p(y) Re\langle p, y - x\rangle$$

is continuous on X. Hence, for each $A \in \mathcal{F}(X)$ and fixed $x \in co(A)$, the mapping $y \mapsto \phi(x,y)$ is lower semi-continuous (resp., weakly lower semi-continuous) on $co(A)$.

(2) For each $A \in \mathcal{F}(X)$ and $y \in co(A)$, $\min_{x\in A} \phi(x,y) \le 0$. Indeed, if this were false, then, for some $A = \{x_1, x_2, \ldots, x_n\} \in \mathcal{F}(X)$ and $y \in co(A)$ (say $y = \sum_{i=1}^n \lambda_i x_i$, where $\lambda_1, \lambda_2, \ldots, \lambda_n \ge 0$ with $\sum_{i=1}^n \lambda_i = 1$), we have $\min_{1\le i\le n} \phi(x_i, y) > 0$. Then, for each $i = 1, 2, \ldots, n$,

$$\beta_0(y)[\inf_{w\in T(y)} Re\langle M(x_i) - w, \eta(y,x_i)\rangle + h(y,x_i)] + \sum_{p\in E^*} \beta_p(y) Re\langle p, y - x_i\rangle > 0$$

and so

$$\begin{aligned} 0 &= \phi(y,y) \\ &= \beta_0(y)[\inf_{w\in T(y)} Re\langle M(\sum_{i=1}^n \lambda_i x_i) - w, \eta(y, \sum_{i=1}^n \lambda_i x_i)\rangle + h(y, \sum_{i=1}^n \lambda_i x_i)] \\ &\quad + \sum_{p\in E^*} \beta_p(y) Re\langle p, y - \sum_{i=1}^n \lambda_i x_i\rangle \\ &= \beta_0(y)[\inf_{w\in T(y)} Re\langle \sum_{i=1}^n \lambda_i M(x_i) - w, \eta(y, \sum_{i=1}^n \lambda_i x_i)\rangle + h(y, \sum_{i=1}^n \lambda_i x_i)] \\ &\quad + \sum_{p\in E^*} \beta_p(y) Re\langle p, y - \sum_{i=1}^n \lambda_i x_i\rangle \end{aligned}$$

$$\geq \sum_{i=1}^{n} \lambda_i(\beta_0(y)[\inf_{w\in T(y)} Re\langle M(x_i) - w, \eta(y, x_i)\rangle + h(y, x_i)] + \sum_{p\in E^*} \beta_p(y) Re\langle p, y - x_i\rangle) > 0,$$

which is a contradiction.

(3) Suppose that $A \in \mathcal{F}(X)$, $x, y \in co(A)$ and $\{y_\alpha\}_{\alpha\in\Gamma}$ is a net in X converging to y (resp., weakly to y) with $\phi(tx + (1-t)y, y_\alpha) \leq 0$ for all $\alpha \in \Gamma$ and $t \in [0, 1]$.

Case (1): $\beta_0(y) = 0$.

Note that $\beta_0(y_\alpha) \geq 0$ for each $\alpha \in \Gamma$ and $\beta_0(y_\alpha) \to 0$. Since $T(X)$ is strongly bounded and $\{y_\alpha\}_{\alpha\in\Gamma}$ is a bounded net, it follows that

$$\limsup_{\alpha}[\beta_0(y_\alpha)(\min_{w\in T(y_\alpha)} Re\langle M(x) - w, \eta(y_\alpha, x)\rangle + h(y_\alpha, x)] = 0. \tag{3.1}$$

Also, we have

$$\beta_0(y)[\min_{w\in T(y)} Re\langle M(x) - w, \eta(y, x)\rangle + h(y, x)] = 0.$$

Thus it follows from (3.1) that

$$\begin{aligned}
&\limsup_{\alpha}[\beta_0(y_\alpha)(\min_{w\in T(y_\alpha)} Re\langle M(x) - w, \eta(y_\alpha, x)\rangle + h(y_\alpha, x)] \\
&\quad + \sum_{p\in E^*}^{n} \beta_p(y) Re\langle p, y - x\rangle \\
&= \sum_{p\in E^*} \beta_p(y) Re\langle p, y - x\rangle \qquad (3.2)\\
&= \beta_0(y)[\min_{w\in T(y)} Re\langle M(x) - w, \eta(y, x)\rangle + h(y, x)] \\
&\quad + \sum_{p\in E^*} \beta_p(y) Re\langle p, y - x\rangle.
\end{aligned}$$

When $t = 1$, we have $\phi(x, y_\alpha) \leq 0$ for all $\alpha \in \Gamma$, i.e.,

$$\begin{aligned}
&\beta_0(y_\alpha)[\min_{w\in T(y_\alpha)} Re\langle M(x) - w, \eta(y_\alpha, x)\rangle + h(y_\alpha, x)] \\
&\quad + \sum_{p\in E^*} \beta_p(y_\alpha) Re\langle p, y_\alpha - x\rangle \leq 0
\end{aligned} \tag{3.3}$$

for all $\alpha \in \Gamma$. Therefore, by (3.3), we have

$$\begin{aligned}
&\limsup_{\alpha}[\beta_0(y_\alpha)\min_{w\in T(y_\alpha)} Re\langle M(x) - w, \eta(y_\alpha, x)\rangle + h(y_\alpha, x)] \qquad (3.4)\\
&\quad + \liminf_{\alpha}[\sum_{p\in E^*} \beta_p(y_\alpha) Re\langle p, y_\alpha - x\rangle] \\
&\leq \limsup_{\alpha}[\beta_0(y_\alpha)\min_{w\in T(y_\alpha)} Re\langle M(x) - w, \eta(y_\alpha, x)\rangle + h(y_\alpha, x)
\end{aligned}$$

$$+\sum_{p\in E^*}\beta_p(y_\alpha)Re\langle p, y_\alpha - x\rangle] \leq 0$$

and thus, by (3.4),

$$\limsup_\alpha[\beta_0(y_\alpha)\min_{w\in T(y_\alpha)} Re\langle M(x)-w, \eta(y_\alpha, x)\rangle + h(y_\alpha, x)]$$
$$+\sum_{p\in E^*}\beta_p(y)Re\langle p, y-x\rangle \leq 0.$$

Hence, by (3.2) and (3.4), we have $\phi(x, y) \leq 0$.

Case (2): $\beta_0(y) > 0$.

Since $\beta_0(y_\alpha) \to \beta_0(y)$, there exists $\lambda \in \Gamma$ such that $\beta_0(y_\alpha) > 0$ for all $\alpha \geq \lambda$. When $t = 0$, we have $\phi(y, y_\alpha) \leq 0$ for all $\alpha \in \Gamma$, i.e.,

$$\beta_0(y_\alpha)[\inf_{w\in T(y_\alpha)} Re\langle M(y)-w, \eta(y_\alpha, y)\rangle + h(y_\alpha, y)]$$
$$+\sum_{p\in E^*}\beta_p(y_\alpha)Re\langle p, y_\alpha - y\rangle \leq 0$$

for all $\alpha \in \Gamma$ and thus

$$\begin{aligned}&\limsup_\alpha[\beta_0(y_\alpha)(\inf_{w\in T(y_\alpha)} Re\langle M(y)-w, \eta(y_\alpha, y)\rangle + h(y_\alpha, y)\\ &+\sum_{p\in E^*}\beta_p(y_\alpha)Re\langle p, y_\alpha - y\rangle] \leq 0.\end{aligned} \tag{3.5}$$

Hence it follows from (3.5) that

$$\begin{aligned}&\limsup_\alpha[\beta_0(y_\alpha)(\inf_{w\in T(y_\alpha)} Re\langle M(y)-w, \eta(y_\alpha, y\rangle + h(y_\alpha, y)]\\ &+\liminf_\alpha[\sum_{p\in E^*}\beta_p(y_\alpha)Re\langle p, y_\alpha - y\rangle]\\ \leq\ &\limsup_\alpha[\beta_0(y_\alpha)(\inf_{w\in T(y_\alpha)} Re\langle M(y)-w, \eta(y_\alpha, y\rangle + h(y_\alpha, y)\\ &+\sum_{p\in E^*}\beta_p(y_\alpha)Re\langle p, y_\alpha - y\rangle] \leq 0.\end{aligned}$$

Since $\liminf_\alpha[\sum_{p\in E^*}\beta_p(y_\alpha)Re\langle p, y_\alpha - y\rangle] = 0$, we have

$$\limsup_\alpha[\beta_0(y_\alpha)(\min_{w\in T(y_\alpha)} Re\langle M(y)-w, \eta(y_\alpha, y)\rangle + h(y_\alpha, y)] \leq 0. \tag{3.6}$$

Since $\beta_0(y_\alpha) > 0$ for all $\alpha \geq \lambda$, it follows that

$$\begin{aligned}&\beta_0(y)\limsup_\alpha[\min_{w\in T(y_\alpha)} Re\langle M(y)-w, \eta(y_\alpha, y)\rangle + h(y_\alpha, y)]\\ =\ &\limsup_\alpha[\beta_0(y_\alpha)(\min_{w\in T(y_\alpha)} Re\langle M(y)-w, \eta(y_\alpha, y)\rangle + h(y_\alpha, y)].\end{aligned} \tag{3.7}$$

Since $\beta_0(y) > 0$, by (3.6) and (3.7), we have

$$\limsup_\alpha[\min_{w\in T(y_\alpha)} Re\langle M(y)-w, \eta(y_\alpha, y)\rangle + h(y_\alpha, y)] \leq 0.$$

Since T is an $(\eta$-$h)$-quasi-pseudo-monotone type I (respectively, strongly $(\eta$-$h)$-quasi-pseudo-monotone type I) operator, we have

$$\begin{aligned}&\limsup_{\alpha}[\min_{w\in T(y_\alpha)} Re\langle M(x)-w,\eta(y_\alpha,x)\rangle + h(y_\alpha,x)]\\ \geq &\min_{w\in T(y)} Re\langle M(x)-w,\eta(y,x)\rangle + h(y,x)\end{aligned}$$

for all $x \in X$. Since $\beta_0(y) > 0$, we have

$$\begin{aligned}&\beta_0(y)[\limsup_{\alpha}(\min_{w\in T(y_\alpha} Re\langle M(x)-w,\eta(y_\alpha,x)\rangle + h(y_\alpha,x)]\\ \geq &\beta_0(y)[\min_{w\in T(y)} Re\langle M(x)-w,\eta(y,x)\rangle + h(y,x)]\end{aligned}$$

and thus

$$\begin{aligned}&\beta_0(y)[\limsup_{\alpha}(\min_{w\in T(y_\alpha)} Re\langle M(x)-w,\eta(y_\alpha,x)\rangle + h(y_\alpha,x)]\\ &+\sum_{p\in E^*}\beta_p(y)Re\langle p,y-x\rangle\\ \geq &\beta_0(y)[\min_{w\in T(y)} Re\langle M(x)-w,\eta(y,x)\rangle + h(y,x)]\\ &+\sum_{p\in E^*}\beta_p(y)Re\langle p,y-x\rangle.\end{aligned}\tag{3.8}$$

When $t = 1$, we have $\phi(x, y_\alpha) \leq 0$ for all $\alpha \in \Gamma$, i.e.,

$$\begin{aligned}&\beta_0(y_\alpha)[\min_{w\in T(y_\alpha)} Re\langle M(x)-w,\eta(y_\alpha,x)\rangle + h(y_\alpha,x)]\\ &+\sum_{p\in E^*}\beta_p(y_\alpha)Re\langle p,y_\alpha-x\rangle \leq 0\end{aligned}$$

for all $\alpha \in \Gamma$ and so, by (3.8),

$$\begin{aligned}0 \geq &\limsup_{\alpha}[\beta_0(y_\alpha)\min_{w\in T(y_\alpha} Re\langle M(x)-w,\eta(y_\alpha,x)\rangle + h(y_\alpha,x)\\ &+\sum_{p\in E^*}\beta_p(y_\alpha)Re\langle p,y_\alpha-x\rangle]\\ \geq &\limsup_{\alpha}[\beta_0(y_\alpha)\min_{w\in T(y_\alpha)} Re\langle M(x)-w,\eta(y_\alpha,x)\rangle + h(y_\alpha,x)]\\ &+\liminf_{\alpha}[\sum_{p\in E^*}\beta_p(y_\alpha)Re\langle p,y_\alpha-x\rangle]\\ = &\beta_0(y)[\limsup_{\alpha}\{\min_{w\in T(y_\alpha)} Re\langle M(x)-w,\eta(y_\alpha,x)\rangle + h(y_\alpha,x)\}]\\ &+\sum_{p\in E^*}\beta_p(y)Re\langle p,y-x\rangle\\ \geq &\beta_0(y)[\min_{w\in T(y)} Re\langle M(x)-w,\eta(y.x)\rangle + h(y,x)]\\ &+\sum_{p\in E^*}\beta_p(y)Re\langle p,y-x\rangle.\end{aligned}\tag{3.9}$$

Hence we have $\phi(x,y) \le 0$.

(4) By hypothesis, there exist a nonempty compact and therefore closed (respectively, weakly closed and weakly compact) subset K of X and a point $x_0 \in X$ such that

$$x_0 \in K \cap S(y), \qquad \inf_{w\in T(y)}[Re\langle M(x_0) - w, \eta(y, x_0)\rangle + h(y, x_0)] > 0$$

for all $y \in X \setminus K$. Thus it follows that, for all $y \in X \setminus K$,

$$\beta_0(y)[\inf_{w\in T(y)} Re\langle M(x_0) - w, \eta(y, x_0)\rangle + h(y, x_0)] > 0$$

whenever $\beta_0(y) > 0$, and $Re\langle p, y - x_0\rangle > 0$ whenever $\beta_p(y) > 0$ for $p \in E^*$. Consequently, we have

$$\phi(x_0, y) = \beta_0(y)[\inf_{w\in T(y)} Re\langle M(x_0) - w, \eta(y, x_0)\rangle + h(y, x_0)] + \sum_{p\in E^*} \beta_p(y) Re\langle p, y - x_0\rangle > 0$$

for all $y \in X \setminus K$. (If T is a strongly (η-h)-quasi-pseudo-monotone type I operator, we equip E with the weak topology.) Thus ϕ satisfies all the hypotheses of Theorem 1.1. Hence, by Theorem 1.1, there exists a point $\hat{y} \in K$ such that $\phi(x, \hat{y}) \le 0$ for all $x \in X$, i.e.,

$$\begin{aligned} &\beta_0(\hat{y})[\inf_{w\in T(\hat{y})} Re\langle M(x) - w, \eta(\hat{y}.x)\rangle + h(\hat{y}, x)] \\ &\quad + \textstyle\sum_{p\in E^*} \beta_p(\hat{y}) Re\langle p, \hat{y} - x\rangle \\ &\le 0 \end{aligned} \tag{3.10}$$

for all $x \in X$.

On the other hand suppose for the above $\hat{y} \in X$, there exists $\hat{x} \in S(\hat{y})$ such that

$$\inf_{w\in T(\hat{y})} Re\langle M(\hat{x}) - w, \eta(\hat{y}, \hat{x})\rangle + h(\hat{y}, \hat{x}) > 0,$$

Then

$$\beta_0(\hat{y})[\inf_{w\in T(\hat{y})} Re\langle M(\hat{x}) - w, \eta(\hat{y}, \hat{x})\rangle + h(\hat{y}, \hat{x}) > 0$$

whenever $\beta_0(\hat{y}) > 0$.

Also if $\beta_p(\hat{y}) > 0$ for all $p \in E^*$, then $\hat{y} \in V_p$ and hence

$$Re\langle p, \hat{y}\rangle - \sup_{x\in S(\hat{y})} Re\langle p, x\rangle > 0.$$

Therefore, $Re\langle p, \hat{y}\rangle > \sup_{x\in S(\hat{y})} Re\langle p, x\rangle \ge Re\langle p, \hat{x}\rangle$. Hence, $Re\langle p, \hat{y} - \hat{x}\rangle > 0$. Then

$$\beta_p(\hat{y}) Re\langle p, \hat{y} - \hat{x}\rangle > 0$$

whenever $\beta_p(\hat{y}) > 0$ for all $p \in E^*$.

Since $\beta_p(\hat{y}) > 0$ for all $p \in E^*$, we have

$$\beta_0(\hat{y})[\inf_{w\in T(\hat{y})} Re\langle M(\hat{x}) - w, \eta(\hat{y}, \hat{x})\rangle + h(\hat{y}, \hat{x})]$$

$$+\sum_{p\in E^*}\beta_p(\hat{y})Re\langle p,\hat{y}-\hat{x}\rangle>0$$

which contradicts (3.10). Therefore Step 1 is proved. Hence we have shown that there exist a point $\hat{y}\in X$ such that $\hat{y}\in S(\hat{y})$ and

$$\sup_{x\in S(\hat{y})}\big[\inf_{w\in T(\hat{y})}Re\langle M(x)-w,\eta(\hat{y},x)\rangle+h(\hat{y},x)\big]\leq 0.$$

Step 2. $\inf_{w\in T(\hat{y})}Re\langle M(\hat{y})-w,\eta(\hat{y},x)\rangle+h(\hat{y},x)\leq 0$ for all $x\in S(\hat{y})$.

From Step 1, we have

$$\hat{y}\in S(\hat{y}),\qquad \inf_{w\in T(\hat{y})}Re\langle M(x)-w,\eta(\hat{y},x)\rangle+h(\hat{y},x)\leq 0$$

for all $x\in S(\hat{y})$. Since $S(\hat{y})$ is a convex subset of X and M is linear and so continuous along line segments in X, by Lemma 2.7, we have

$$\inf_{w\in T(\hat{y})}Re\langle M(\hat{y})-w,\eta(\hat{y},x)\rangle+h(\hat{y},x)\leq 0$$

for all $x\in S(\hat{y})$.

Step 3. There exists a point $\hat{w}\in T(\hat{y})$ such that

$$Re\langle M(\hat{y})-\hat{w},\eta(\hat{y},x)\rangle+h(\hat{y},x)\leq 0$$

for all $x\in S(\hat{y})$.

From Step 2 we have,

$$\sup_{x\in S(\hat{y})}\big[\inf_{w\in T(\hat{y})}Re\langle M(\hat{y})-w,\eta(\hat{y},x)\rangle+h(\hat{y},x)\big]\leq 0 \tag{3.11}$$

i.e.,

$$\sup_{x\in S(\hat{y})}\big[\inf_{(M(\hat{y}),w)\in M(\hat{y})\times T(\hat{y})}Re\langle M(\hat{y})-w,\eta(\hat{y},x)\rangle+h(\hat{y},x)\big]\leq 0$$

where $M(\hat{y})\times T(\hat{y})$ is a $\sigma\langle F,E\rangle$-compact convex subset of the Hausdorff topological vector space $F\times F$ and $S(\hat{y})$ is a convex subset of X.

Let us set $Q=M(\hat{y})\times T(\hat{y})$ and define the mapping $g:S(\hat{y})\times Q\to\mathbb{R}$ by $g(x,q)=g(x,(M(\hat{y}),w))=Re\langle M(\hat{y})-w,\eta(\hat{y},x)\rangle+h(\hat{y},x)$ for each $x\in S(\hat{y})$ and each $q=(M(\hat{y}),w)\in Q=M(\hat{y})\times T(\hat{y})$. Then, for each fixed $x\in S(\hat{y})$, the mapping $(M(\hat{y}),w)\mapsto g(x,(M(\hat{y}),w))$ is lower semi-continuous from the relative product topology on Q to $\mathbb{R}$ and also convex on Q. Clearly, for each fixed $q=(M(\hat{y}),w)\in Q$, the mapping $x\mapsto g(x,q)=g(x,(M(\hat{y}),w))$ is concave on $S(\hat{y})$.

So, we can apply Keneser's Minimax Theorem (Theorem 2.8) and obtain the following:

$$\min_{(M(\hat{y}),w)\in Q}\sup_{x\in S(\hat{y})}g(x,(M(\hat{y}),w))=\sup_{x\in S(\hat{y})}\min_{(M(\hat{y}),w)\in Q}g(x,(M(\hat{y}),w)).$$

Hence, by (3.11), we obtain

$$\min_{(M(\hat{y}),w)\in Q}\sup_{x\in S(\hat{y})}Re\langle M(\hat{y})-w,\eta(\hat{y},x)\rangle+h(\hat{y},x)\leq 0.$$

Since $Q = M(\hat{y}) \times T(\hat{y})$ is compact, there exists $(M(\hat{y}), \hat{w}) \in M(\hat{y}) \times T(\hat{y})$ such that

$$\sup_{x \in S(\hat{y})} [Re\langle M(\hat{y}) - \hat{w}, \eta(\hat{y}, x)\rangle + h(\hat{y}, x)] \leq 0$$

Therefore we have shown that

$$Re\langle M(\hat{y}) - \hat{w}, \eta(\hat{y}, x)\rangle + h(\hat{y}, x) \leq 0$$

for all $x \in S(\hat{y})$. In other words, there exists a point $\hat{w} \in T(\hat{y})$ with

$$Re\langle M(\hat{y}) - \hat{w}, \eta(\hat{y}, x)\rangle + h(\hat{y}, x) \leq 0$$

for all $x \in S(\hat{y})$.

We observe from the above proof that the requirement that E need to be locally convex is needed when and only when the separation theorem is applied to the case $y \notin S(y)$. Thus, if $S : X \to 2^X$ is the constant mapping $S(x) = X$ for all $x \in X$, then E is not required to be locally convex.

Finally, if $T \equiv 0$, in order to show that for each $x \in X$, $y \mapsto \phi(x, y)$ is lower semi-continuous (resp., weakly lower semi-continuous), Lemma 2.6 is no longer needed and the weaker continuity assumption on $\langle \cdot, \cdot \rangle$ that, for each $f \in F$, the mapping $x \mapsto \langle f, x\rangle$ is continuous (resp., weakly continuous) on X is sufficient. This completes the proof. □

Now, we establish our last result of this section:

Theorem 3.2. *Let E be a locally convex Hausdorff topological vector space over Φ, X be a nonempty para-compact convex and bounded subset of E and F a Hausdorff topological vector space over Φ. Let $\langle \cdot, \cdot \rangle : F \times E \to \Phi$ be a bilinear functional which is continuous on compact subsets of $F \times X$. Suppose that*

(a) *$S : X \to 2^X$ is a continuous mapping such that each $S(x)$ is compact and convex;*

(b) *$h : E \times E \to \mathbb{R}$ is convex and $h(X \times X)$ is bounded;*

(c) *$T : X \to 2^F$ is an $(\eta$-$h)$-quasi-pseudo-monotone type I (respectively, strongly $(\eta$-$h)$-quasi-pseudo-monotone type I) operator and is upper semi-continuous such that each $T(x)$ is compact and convex (respectively, weakly compact and convex, i.e., $\sigma\langle F, E\rangle$-compact and convex) and $T(X)$ is strongly bounded;*

(d) *$T : X \to 2^F$ and $\eta : X \times X \to E$ have the 0-DCVR and $\eta : X \times X \to E$ is convex and continuous;*

(e) *$M : X \to 2^F$ is a continuous linear mapping in X and for each $y \in \Sigma = \{y \in X : \sup_{x \in S(y)}[\inf_{w \in T(y)} Re\langle M(x) - w, \eta(y, x)\rangle + h(y, x)] > 0\}$,*

$$\inf_{w \in T(y)} Re\langle M(x) - w, \eta(y, x)\rangle + h(y, x) > 0$$

for some point $x \in S(y)$.

(f) *for each fixed $y \in X$, $x \mapsto h(x, y)$, i.e., $h(\cdot, y)$ is lower semi-continuous on $co(A)$ for each $A \in \mathcal{F}(X)$ and, for each fixed $x \in X$, $h(x, \cdot)$ and $\eta(x, \cdot)$ are concave, and $\eta(x, \cdot)$ is affine and $h(x, x) = 0$, $\eta(x, x) = 0$;*

(g) *for each open subset* U *of* X *and* $x, y \in U$, $\eta(x, y) = x - y$ *and there exists* $h' : X \to \mathbb{R}$ *such that* $h(x, y) = h'(x) - h'(y)$;

Suppose further that there exist a nonempty closed and compact (respectively, weakly closed and weakly compact) subset K *of* X *and a point* $x_0 \in X$ *such that*

$$x_0 \in K \cap S(y), \inf_{w \in T(y)} Re\langle M(x_0) - w, \eta(y, x_0)\rangle + h(y, x_0) > 0$$

for all $y \in X \setminus K$.

Then there exists a point $\hat{y} \in X$ *such that*

(1) $\hat{y} \in S(\hat{y})$;

(2) *there exists a point* $\hat{w} \in T(\hat{y})$ *with* $Re\langle M(\hat{y}) - \hat{w}, \eta(\hat{y}, x)\rangle + h(\hat{y}, x) \leq 0$ *for all* $x \in S(\hat{y})$.

Moreover, if $S(x) = X$ *for all* $x \in X$, *then* E *is not required to be locally convex.*

The proof is similar to the proof of Theorem 2 in [14]. For the completeness, we include the proof here.

Proof. The proof will follow from Theorem 3.1 if we can show that the set

$$\Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{w \in T(y)} Re\langle M(x) - w, \eta(y, x)\rangle + h(y, x)] > 0\}$$

is open in X. To show that Σ is open in X, we start as follows:

Let $y_0 \in \Sigma$ be an arbitrary point. We show that there exists an open neighbourhood N_0 of y_0 in X such that $N_0 \subset \Sigma$. Now, by the hypothesis (e), M is a continuous linear mapping on X and at some point x_0 in $S(y_0)$ we have

$$\inf_{w \in T(y_0)} Re\langle M(x_0) - w, \eta(y_0, x_0)\rangle + h(y_0, x_0) > 0.$$

Let

$$\alpha := \inf_{w \in T(y_0)} Re\langle M(x_0) - w, \eta(y_0, x_0)\rangle + h(y_0, x_0).$$

Thus $\alpha > 0$. Again, let

$$W := \{w \in F : \sup_{z_1, z_2 \in X} |\langle w, z_1 - z_2\rangle| < \alpha/6\}.$$

Then W is an open neighbourhood of 0 in F and so $U_1 := T(y_0) + W$ is an open neighbourhood of $T(y_0)$ in F. Since T is upper semi-continuous at y_0, there exists an open neighbourhood N_1 of y_0 in X such that $T(y) \subset U_1$ for all $y \in N_1$.

Let $U_2 := M(x_0) + W$, then U_2 is an open neighbourhood of $M(x_0)$ in F. Since M is continuous at x_0, and therefore upper semi-continuous at x_0, there exists an open neighbourhood V_1 of x_0 in X such that $M(x) \in U_2$ for all $x \in V_1$.

Since the mapping $x \longmapsto \inf_{w \in T(y_0)} Re\langle M(x_0) - w, \eta(x_0, x)\rangle + h(x_0, x)$ is continuous at x_0, there exists an open neighbourhood V_2 of x_0 in X such that

$$|\inf_{w \in T(y_0)} Re\langle M(x_0) - w, \eta(x_0, x)\rangle + h(x_0, x)| < \frac{\alpha}{6} \text{ for all } x \in V_2.$$

Let $V_0 := V_1 \cap V_2$. Then V_0 is an open neighborhood of x_0 in X. Since $x_0 \in V_0 \cap S(y_0) \neq \emptyset$ and S is lower semi-continuous at y_0, there exists an open neighborhood N_2 of y_0 in X such that $S(y) \cap V_0 \neq \emptyset$ for all $y \in N_2$.

Since the mapping $y \longmapsto \inf_{w \in T(y_0)} Re\langle M(x_0) - w, \eta(y, y_0)\rangle + h(y, y_0)$ is continuous at y_0, there exists an open neighborhood N_3 of y_0 in X such that

$$|\inf_{w \in T(y_0)} Re\langle M(x_0) - w, \eta(y, y_0)\rangle + h(y, y_0)| < \frac{\alpha}{6} \quad \text{for all} \quad y \in N_3.$$

Let $N_0 := N_1 \cap N_2 \cap N_3$. Then N_0 is an open neighborhood of y_0 in X such that for each $y_1 \in N_0$, we have the following:

(1) $T(y_1) \subset U_1 = T(y_0) + W$ as $y_1 \in N_1$;

(2) $S(y_1) \cap V_0 \neq \emptyset$ as $y_1 \in N_2$; so we can choose any $x_1 \in S(y_1) \cap V_0$;

(3) $|\inf_{w \in T(y_0)} Re\langle M(x_0) - w, \eta(y_1, y_0)\rangle + h(y_1, y_0)| < \frac{\alpha}{6}$ as $y_1 \in N_3$;

(4) $M(x_1) \in U_2 = M(x_0) + W$ as $x_1 \in V_1$;

(5) $|\inf_{w \in T(y_0)} Re\langle M(x_0) - w, \eta(x_0, x_1)\rangle + h(x_0, x_1)| < \frac{\alpha}{6}$ as $x_1 \in V_2$.

Hence, using the assumption (g) of the theorem and by (1)-(5) above, we can obtain the following by omitting the details:

$$\begin{aligned}
&\inf_{w \in T(y_1)} Re\langle M(x_0) - w, \eta(y_1, x_1)\rangle + h(y_1, x_1) \\
\geq\ &\inf_{[w \in T(y_0) + W]} Re\langle M(x_0) - w, \eta(y_1, x_1)\rangle + h(y_1, x_1) \\
\geq\ &\inf_{w \in T(y_0)} Re\langle M(x_0) - w, \eta(y_1, x_1)\rangle + h(y_1, x_1) \\
&+ \inf_{w \in W} Re\langle M(x_0) - w, \eta(y_1, x_1)\rangle \\
\geq\ &\inf_{w \in T(y_0)} Re\langle M(x_0) - w, y_1 - y_0\rangle + h'(y_1) - h'(y_0) \\
&+ \inf_{w \in T(y_0)} Re\langle M(x_0) - w, y_0 - x_0\rangle + h'(y_0) - h'(x_0) \\
&+ \inf_{w \in T(y_0)} Re\langle M(x_0) - w, x_0 - x_1\rangle + h'(x_0) - h'(x_1) \\
&+ Re\langle M(x_0), y_1 - x_1\rangle + \inf_{w \in W} Re\langle -w, y_1 - x_1\rangle \\
\geq\ &-\frac{\alpha}{6} + \alpha - \frac{\alpha}{6} - \frac{\alpha}{6} - \frac{\alpha}{6} \\
=\ &\frac{\alpha}{3} > 0.
\end{aligned}$$

Consequently, we have

$$\sup_{x \in S(y_1)} [\inf_{w \in T(y_1)} Re\langle M(x_0) - w, \eta(y_1, x)\rangle + h(y_1, x)] > 0$$

since $x_1 \in S(y_1)$. Hence $y_1 \in \Sigma$ for all $y_1 \in N_0$. Therefore, $y_0 \in N_0 \subset \Sigma$. But y_0 was arbitrary. Consequently, Σ is open in X.

Thus all the hypotheses of Theorem 3.1 are satisfied. Hence, the conclusion follows from Theorem 3.1. This completes the proof. □

Remark 3.1. (1) Theorems 3.1 and 3.2 in this paper are the extensions of Theorems 3.2 and 3.3 in [17], respectively, for generalized bi-quasi-variational-like inequalities (GBQVLI).

(2) The first paper on generalized bi-quasi-variational inequalities was written by Shih and Tan in 1989 in [31] and the results were obtained on compact sets where the set-valued mappings were either lower semi-continuous or upper semi-continuous. Our present paper is another extension of the original work in [31] using (η, h)-quasi-pseudomonotone type I and strongly (η, h)-quasi-pseudomonotone type I operators on non-compact spaces. The (η, h)-quasi-pseudomonotone type I and strongly (η, h)-quasi-pseudomonotone type I operators are generalizations of pseudomonotone type I operators introduced first in [10].

(3) In all our results on generalized bi-quasi-variational inequalities, if the operators $M \equiv 0$ and the operator T is replaced by $-T$, then we obtain results on generalized quasi-variational inequalities which generalize the corresponding results in the literature (see [30]).

(4) The results on generalized bi-quasi-variational inequalities given in [21] were obtained for set-valued quasi-semi-monotone and bi-quasi-semi-monotone operators and the corresponding results in [19] were obtained for set-valued upper-hemi-continuous operators introduced in [24]. Our results in this paper are also further extensions of the corresponding results in [21] and [9] using set-valued (η, h)-quasi-pseudomonotone type I and strongly (η, h)-quasi-pseudomonotone type I operators on non-compact spaces.

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Yeol Je Cho
Department of Mathematics Education and RINS
Gyeongsang National University
Jinju 660-701, Korea
E-mail address: yjcho@gnu.ac.kr; yjchomath@gmail.com

Mohammad S. R. Chowdhury
Department of Mathematics
University of Management and Technology (UMT)
C-II, Johar Town, Lahore-54770, Pakistan
E-mail address: msrchowdhury@hotmail.com, ssc.adr@umt.edu.pk

Je Ai Ha
Department of Mathematics Education and RINS
Gyeongsang National University
Jinju 660-701, Korea
E-mail address: olive7179@hanmail.net