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EXPLICIT EVALUATION OF HARMONIC SUMS

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Abstract. In this paper, we obtain some formulae for harmonic sums, alternating harmonic sums and Stirling number sums by using the method of integral representations of series. As applications of these formulae, we give explicit formula of several quadratic and cubic Euler sums through zeta values and linear sums. Furthermore, some relationships between harmonic numbers and Stirling numbers of the first kind are established.

1. Introduction

In this paper, the generalized harmonic numbers and alternating harmonic numbers of order k are defined respectively as

$$\zeta_n(k) := \sum_{j=1}^{n} \frac{1}{j^k} \quad \text{and} \quad L_n(k) := \sum_{j=1}^{n} \frac{(-1)^{j-1}}{j^k} \quad (n, k \in \mathbb{N}), \tag{1.1}$$

where $H_n \equiv \zeta_n(1)$ denotes the classical harmonic number.

From [15, 33], we know that the classical Euler sums are the infinite sums whose general term is product of 1 or $(-1)^{n-1}$, harmonic numbers and alternating harmonic numbers of index n and a power of n^{-1}. Therefore, the Euler sums of index

$$\pi_1 := \left(\underbrace{k_1, \ldots, k_1}_{q_1}, \ldots, \underbrace{k_{m_1}, \ldots, k_{m_1}}_{q_{m_1}} \right),$$

$$\pi_2 := \left(\underbrace{h_1, \ldots, h_1}_{l_1}, \ldots, \underbrace{h_{m_2}, \ldots, h_{m_2}}_{l_{m_2}} \right) \quad \text{and}$$

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p are defined by

$$S_{\pi_1,\pi_2,p} \equiv S_{\prod\limits_{i=1}^{m_1} k_i^{q_i}, \prod\limits_{j=1}^{m_2} h_j^{l_j},p} := \sum_{n=1}^{\infty} \frac{\prod\limits_{i=1}^{m_1} \zeta_n^{q_i}(k_i) \prod\limits_{j=1}^{m_2} L_n^{l_j}(h_j)}{n^p},$$

$$\bar{S}_{\pi_1,\pi_2,p} \equiv \bar{S}_{\prod\limits_{i=1}^{m_1} k_i^{q_i}, \prod\limits_{j=1}^{m_2} h_j^{l_j},p} := \sum_{n=1}^{\infty} \frac{\prod\limits_{i=1}^{m_1} \zeta_n^{q_i}(k_i) \prod\limits_{j=1}^{m_2} L_n^{l_j}(h_j)}{n^p}(-1)^{n-1},$$

where the quantity $\omega := \sum_{i=1}^{m_1}(k_i q_i) + \sum_{j=1}^{m_2}(h_j l_j) + p$ being called the weight and the quantity $d := \sum_{i=1}^{m_1} q_i + \sum_{j=1}^{m_2} l_j$ being the degree. Here p $(p > 1), m_1, m_2, q_i, k_i, h_j, l_j$ are non-negative integers, $1 \le k_1 < k_2 < \cdots < k_{m_1} \in \mathbb{N}$, $1 \le h_1 < h_2 < \cdots < h_{m_2} \in \mathbb{N}$. For example,

$$S_{1^3 2^5 3^2, 1^2 2^3, 4} = \sum_{n=1}^{\infty} \frac{H_n^3 \zeta_n^5(2)\zeta_n^2(3) L_n^2(1) L_n^3(2)}{n^4},$$

$$\bar{S}_{0,p_1 p_2,p} = \sum_{n=1}^{\infty} \frac{L_n(p_1) L_n(p_2)}{n^p}(-1)^{n-1}$$

$$\bar{S}_{1^2 2^4 3, 1^2 2^3, 5} = \sum_{n=1}^{\infty} \frac{H_n^2 \zeta_n^4(2)\zeta_n(3) L_n^2(1) L_n^3(2)}{n^5}(-1)^{n-1},$$

$$S_{p_1 p_2, 0, p} = \sum_{n=1}^{\infty} \frac{\zeta_n(p_1)\zeta_n(p_2)}{n^p}.$$

From the definition of Euler sums, there are altogether four types of linear sums:

$$S_{p,0,q} = \sum_{n=1}^{\infty} \frac{\zeta_n(p)}{n^q}, \ \bar{S}_{p,0,q} = \sum_{n=1}^{\infty} \frac{\zeta_n(p)}{n^q}(-1)^{n-1},$$

$$S_{0,p,q} = \sum_{n=1}^{\infty} \frac{L_n(p)}{n^q}, \ \bar{S}_{0,p,q} = \sum_{n=1}^{\infty} \frac{L_n(p)}{n^q}(-1)^{n-1}.$$

The evaluation of linear sums in terms of values of Riemann zeta function and polylogarithm function at positive integers is known when $(p,q) = (1,3), (2,2)$, or $p+q$ is odd [6, 15]. For instance, we have

$$\begin{aligned} \bar{S}_{1,0,3} &= \sum_{n=1}^{\infty} \frac{H_n}{n^3}(-1)^{n-1} \\ &= -2\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{11}{4}\zeta(4) + \frac{1}{2}\zeta(2)\ln^2 2 - \frac{1}{12}\ln^4 2 - \frac{7}{4}\zeta(3)\ln 2, \\ \bar{S}_{0,1,3} &= \sum_{n=1}^{\infty} \frac{L_n(1)}{n^3}(-1)^{n-1} = \frac{3}{2}\zeta(4) + \frac{1}{2}\zeta(2)\ln^2 2 - \frac{1}{12}\ln^4 2 - 2\mathrm{Li}_4\left(\frac{1}{2}\right). \end{aligned}$$

In [15], Flajolet and Salvy gave explicit reductions to zeta values for all linear sums

$$S_{p,0,q},\ \bar{S}_{p,0,q},\ S_{0,p,q} \quad \text{and} \quad \bar{S}_{0,p,q}$$

when $p+q$ is an odd weight. The relationship between the values of the Riemann zeta function and Euler sums has been studied by many authors, for example see [2, 3, 6–11, 13–21, 23, 27, 28, 30, 31, 33]. The Riemann zeta function and alternating Riemann zeta function are defined respectively by [1, 4, 5]

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \Re(s) > 1,$$

and

$$\bar{\zeta}(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s},\ \Re(s) \geq 1.$$

The general multiple zeta functions is defined as

$$\zeta(s_1, s_2, \ldots, s_m) := \sum_{n_1 > n_2 > \cdots > n_m > 0} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_m^{s_m}},$$

where $s_1 > 1, s_j > 0\ (j = 2, 3, \ldots, m)$ and the quantities $w := s_1 + \cdots + s_m$ and m are called the weight the multiplicity, respectively. In this paper, we show that the Euler-type sums with harmonic numbers:

$$\sum_{n=1}^{\infty} \frac{\zeta_n(m)}{n(n+k)},\ \sum_{n=1}^{\infty} \frac{L_n(m)}{n(n+k)} \quad (m, k \in \mathbb{N})$$

can be expressed in terms of series of Riemann zeta values and harmonic numbers. We also provide an explicit evaluation of

$$(p-1)! \sum_{n=p-1}^{\infty} \frac{s(n+1, p)}{n!\, n(n+k)} \quad (2 \leq p \in \mathbb{N},\ k \in \mathbb{N})$$

in a closed form in terms of zeta values and the complete exponential Bell polynomial $Y_k(n)$. Here $s(n,k)$ stands for the Stirling number of the first kind. Specifically, we investigate closed-form representations for sums of the following form:

$$S_{1^2,0,p} = \sum_{n=1}^{\infty} \frac{H_n^2}{n^p},\ S_{1,1,p} = \sum_{n=1}^{\infty} \frac{H_n L_n(1)}{n^p},\ \bar{S}_{1,1,p} = \sum_{n=1}^{\infty} \frac{H_n L_n(1)}{n^p} (-1)^{n-1},$$

$$\bar{S}_{1^2,0,p} = \sum_{n=1}^{\infty} \frac{H_n^2}{n^p} (-1)^{n-1},\ S_{0,1^2,p} = \sum_{n=1}^{\infty} \frac{L_n^2(1)}{n^p},\ \bar{S}_{0,1^2,p} = \sum_{n=1}^{\infty} \frac{L_n^2(1)}{n^p} (-1)^{n-1}.$$

Furthermore, we evaluate several other series involving harmonic numbers. For example

$$\sum_{n=1}^{\infty} \frac{H_n^2}{n(n+k)} = \frac{1}{k} \left\{ \begin{array}{l} 3\zeta(3) + \frac{H_k^3 + 3H_k\zeta_k(2) + 2\zeta_k(3)}{3} \\ -\frac{H_k^2 + \zeta_k(2)}{k} - \sum_{i=1}^{k-1} \frac{H_i}{i^2} + \zeta(2) H_{k-1} \end{array} \right\}. \tag{1.2}$$

Note that the result (1.2) are given in Sofo's paper [26] and Xu's paper [34] with M. Zhang and W. Zhu.

2. Main theorems and proof

In this section, we use certain integral of polylogarithm function representations to evaluate several series with alternating (or non-alternating) harmonic numbers. The polylogarithm function defined as follows

$$\mathrm{Li}_p(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^p}, \Re(p) > 1,\ |x| \le 1,$$

when x takes 1 and -1, then the function $\mathrm{Li}_p(x)$ are reducible to Riemann zeta function and alternating Riemann zeta function, respectively.

Theorem 2.1. *For positive integers m and k, we have*

(2.1)
$$\sum_{n=1}^{\infty} \frac{\zeta_n(m)}{n(n+k)} = \frac{1}{k}\left\{\zeta(m+1) + \sum_{j=1}^{m-1} (-1)^{j-1} \zeta(m+1-j)\,\zeta_{k-1}(j) + (-1)^{m-1} \sum_{i=1}^{k-1} \frac{H_i}{i^m}\right\},$$

(2.2)
$$\sum_{n=1}^{\infty} \frac{L_n(m)}{n(n+k)} = \frac{1}{k}\left\{\begin{array}{l} \bar{\zeta}(m+1) + \sum_{j=1}^{m-1} (-1)^{j-1} \bar{\zeta}(m+1-j)\,\zeta_{k-1}(j) \\ +(-1)^{m-1} \ln 2\,(\zeta_{k-1}(m) + L_{k-1}(m)) + (-1)^m \sum_{i=1}^{k-1} \frac{(-1)^{i-1}}{i^m} L_i(1) \end{array}\right\}.$$

Proof. By the definition of polylogarithm function and Cauchy product formula of power series, we can verify that

$$\frac{\mathrm{Li}_m(x)}{1-x} = \sum_{n=1}^{\infty} \zeta_n(m)\, x^n,\ x \in (-1,1), \tag{2.3}$$

$$-\frac{\mathrm{Li}_m(-x)}{1-x} = \sum_{n=1}^{\infty} L_n(m)\, x^n,\ x \in (-1,1). \tag{2.4}$$

Multiplying (2.3) and (2.4) by $x^{r-1} - x^{k-1}$ and integrating over (0,1), we obtain

(2.5)
$$\int_0^1 \left(x^{r-1} - x^{k-1}\right) \frac{\mathrm{Li}_m(x)}{1-x} dx = (k-r) \sum_{n=1}^{\infty} \frac{\zeta_n(m)}{(n+r)(n+k)} \quad (0 \le r < k,\ r, k \in \mathbb{Z}),$$

(2.6)
$$\int_0^1 \left(x^{k-1} - x^{r-1}\right) \frac{\mathrm{Li}_m(-x)}{1-x} dx = (k-r) \sum_{n=1}^{\infty} \frac{L_n(m)}{(n+r)(n+k)} \quad (0 \le r < k,\ r, k \in \mathbb{Z}).$$

We now evaluate the integral on the left side of (2.5) and (2.6). Noting that

$$\int_0^1 \left(x^{r-1} - x^{k-1}\right) \frac{\mathrm{Li}_m(x)}{1-x} dx = \sum_{i=1}^{k-r} \int_0^1 x^{r+i-2} \mathrm{Li}_m(x)\, dx, \tag{2.7}$$

$$\int_0^1 \left(x^{k-1}-x^{r-1}\right)\frac{\mathrm{Li}_m\left(-x\right)}{1-x}dx=\sum_{i=1}^{k-r}\left(-1\right)^{r+i}\int_0^{-1}x^{r+i-2}\mathrm{Li}_m\left(x\right)dx. \tag{2.8}$$

Using integration by parts we have

(2.9)

$$\begin{aligned}\int_0^x t^{n-1}\mathrm{Li}_q\left(t\right)dt=&\sum_{i=1}^{q-1}\left(-1\right)^{i-1}\frac{x^n}{n^i}\mathrm{Li}_{q+1-i}\left(x\right)+\frac{\left(-1\right)^q}{n^q}\ln\left(1-x\right)\left(x^n-1\right)\\&-\frac{\left(-1\right)^q}{n^q}\left(\sum_{k=1}^n\frac{x^k}{k}\right).\end{aligned}$$

Taking $r=0$ in (2.5)-(2.8), substituting (2.9) into (2.5)-(2.8), we can obtain the results (2.1) and (2.2). □

Note that the result (2.1) is given in Sofo's paper [22] by another method. Taking $r\geq 1$ in (2.5)-(2.8) and using (2.9), we have

(2.10)

$$\sum_{n=1}^\infty\frac{\zeta_n\left(m\right)}{\left(n+r\right)\left(n+k\right)}=\frac{1}{k-r}\left\{\begin{array}{l}\sum_{j=1}^{m-1}\left(-1\right)^{j-1}\zeta\left(m+1-j\right)\left(\zeta_{k-1}\left(j\right)-\zeta_{r-1}\left(j\right)\right)\\+\left(-1\right)^{m-1}\left(\sum_{i=1}^{k-1}\frac{H_i}{i^m}-\sum_{i=1}^{r-1}\frac{H_i}{i^m}\right)\end{array}\right\},$$

(2.11)

$$\sum_{n=1}^\infty\frac{L_n\left(m\right)}{\left(n+r\right)\left(n+k\right)}=\frac{1}{k-r}\left\{\begin{array}{l}\sum_{j=1}^{m-1}\left(-1\right)^{j-1}\bar{\zeta}\left(m+1-j\right)\left(\zeta_{k-1}\left(j\right)-\zeta_{r-1}\left(j\right)\right)\\+\left(-1\right)^{m-1}\ln 2\left(\zeta_{k-1}\left(m\right)-\zeta_{r-1}\left(m\right)+L_{k-1}\left(m\right)-L_{r-1}\left(m\right)\right)\\+\left(-1\right)^m\sum_{i=r}^{k-1}\frac{\left(-1\right)^{i-1}}{i^m}L_i\left(1\right)\end{array}\right\}.$$

Letting $m=1$ in (2.1) and (2.2), we conclude that

$$\sum_{n=1}^\infty\frac{H_n}{n\left(n+k\right)}=\frac{1}{k}\left(\frac{1}{2}H_k^2+\frac{1}{2}\zeta_k\left(2\right)+\zeta\left(2\right)-\frac{H_k}{k}\right), \tag{2.12}$$

$$\sum_{n=1}^\infty\frac{L_n\left(1\right)}{n\left(n+k\right)}=\frac{1}{k}\left\{\begin{array}{l}\bar{\zeta}\left(2\right)+\ln 2\left(H_k+L_k\left(1\right)\right)-\ln 2\frac{1+\left(-1\right)^{k-1}}{k}\\-\frac{1}{2}\left(L_k^2\left(1\right)+\zeta_k\left(2\right)\right)+\frac{L_k\left(1\right)}{k}\left(-1\right)^{k-1}\end{array}\right\}. \tag{2.13}$$

Note that formula (2.12) were also proved in [22, 34].

Theorem 2.2 ([33]). *For integers $n\geq 1, k\geq 0$, we have*

$$\int_0^1 t^{n-1}\ln^k\left(1-t\right)dt=\left(-1\right)^k\frac{Y_k\left(n\right)}{n}, \tag{2.14}$$

where

$$Y_k\left(n\right)=Y_k\left(\zeta_n\left(1\right),1!\zeta_n\left(2\right),2!\zeta_n\left(3\right),\ldots,\left(r-1\right)!\zeta_n\left(r\right),\ldots\right),Y_k\left(x_1,x_2,\ldots\right)$$

stands for the complete exponential Bell polynomial is defined by (see [12])

$$\exp\left(\sum_{m\geq 1}x_m\frac{t^m}{m!}\right)=1+\sum_{k\geq 1}Y_k\left(x_1,x_2,\ldots\right)\frac{t^k}{k!}. \tag{2.15}$$

Proof. Using the definition of the complete exponential Bell polynomial, it is easily shown that

$$1+\sum_{k\geq 1} Y_k\left(n\right)\frac{t^k}{k!}=\frac{1}{\left(1-t\right)\left(1-\frac{t}{2}\right)\cdots\left(1-\frac{t}{n}\right)}$$

and

$$\frac{1}{\left(1-t\right)\left(1-\frac{t}{2}\right)\cdots\left(1-\frac{t}{n}\right)}=1+\sum_{m=1}^{n}\frac{t}{m\left(1-t\right)\left(1-\frac{t}{2}\right)\cdots\left(1-\frac{t}{m}\right)}.$$

Therefore we obtain

$$Y_k\left(n\right)=k\sum_{m=1}^{n}\frac{Y_{k-1}\left(m\right)}{m},\ Y_0\left(n\right)=1. \tag{2.16}$$

It is easily shown using integration by parts that

$$\int_0^x t^{n-1}\ln\left(1-t\right)dt=\frac{1}{n}\left\{x^n\ln\left(1-x\right)-\sum_{j=1}^{n}\frac{x^j}{j}-\ln\left(1-x\right)\right\},-1\leq x<1.$$

Letting $x\to 1^-$, we have

$$\lim_{x\to 1^-}\int_0^x t^{n-1}\ln\left(1-t\right)dt=-\frac{H_n}{n}=-\frac{Y_1(n)}{n}. \tag{2.17}$$

Using integration by parts and (2.17), we can find that

$$\begin{aligned}\int_0^x t^{n-1}\ln^2\left(1-t\right)dt&=\frac{1}{n}(x^n-1)\ln^2\left(1-x\right)\\&\quad-\frac{2}{n}\sum_{k=1}^{n}\frac{1}{k}\left\{x^k\ln\left(1-x\right)-\sum_{j=1}^{k}\frac{x^j}{j}-\ln\left(1-x\right)\right\}.\end{aligned} \tag{2.18}$$

Letting x tend to 1^- in (2.18), we deduce that

$$\lim_{x\to 1^-}\int_0^x t^{n-1}\ln^2\left(1-t\right)dt=\frac{2}{n}\sum_{k=1}^{n}\frac{1}{k}\sum_{j=1}^{k}\frac{1}{j}=\frac{Y_2\left(n\right)}{n}. \tag{2.19}$$

Further, it is easily verify that

$$\begin{aligned}\lim_{x\to 1^-}\int_0^x t^{n-1}\ln^m\left(1-t\right)dt&=m!\frac{(-1)^m}{n}\sum_{k_1=1}^{n}\frac{1}{k_1}\sum_{k_2=1}^{k_1}\frac{1}{k_2}\cdots\sum_{k_m=1}^{k_{m-1}}\frac{1}{k_m}\\&=\left(-1\right)^m\frac{Y_m\left(n\right)}{n},\ 1\leq m\in\mathbb{Z}.\end{aligned} \tag{2.20}$$

We complete the proof of (2.14). □

From the definition of the complete exponential Bell polynomial, we have

$$Y_1(n) = H_n,\ Y_2(n) = H_n^2 + \zeta_n(2),\ Y_3(n) = H_n^3 + 3H_n\zeta_n(2) + 2\zeta_n(3),$$
$$Y_4(n) = H_n^4 + 8H_n\zeta_n(3) + 6H_n^2\zeta_n(2) + 3\zeta_n^2(2) + 6\zeta_n(4).$$

The integral of Theorem 2.2 itself is readily available in Sofo's [24].

Theorem 2.3 ([34])**.** *For integer $p \geq 2, k \geq 1$, we have*

$$(p-1)!\sum_{n=p-1}^{\infty}\frac{s(n+1,p)}{n!n(n+k)} = \frac{1}{k}\left\{(p-1)!\zeta(p) + \frac{Y_p(k)}{p} - \frac{Y_{p-1}(k)}{k}\right\}, \tag{2.21}$$

where $s(n,k)$ are the Stirling number of the first kind defined by

$$n!(1+x)\left(1+\frac{x}{2}\right)\cdots\left(1+\frac{x}{n}\right) = \sum_{k=0}^{n} s(n+1,k+1)x^k.$$

From the definition $s(n,k)$, we can rewrite it as

$$\begin{aligned}\sum_{k=0}^{n} s(n+1,k+1)x^k &= n!\exp\left\{\sum_{j=1}^{n}\ln\left(1+\frac{x}{j}\right)\right\}\\ &= n!\exp\left\{\sum_{j=1}^{n}\sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^k}{kj^k}\right\}\\ &= n!\exp\left\{\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\zeta_n(k)x^k}{k}\right\}.\end{aligned}$$

Therefore, we know that $s(n,k)$ is a rational linear combination of products of harmonic numbers. The following identities is easily derived

$$\begin{aligned}s(n,1) &= (n-1)!,\\ s(n,2) &= (n-1)!H_{n-1},\\ s(n,3) &= \frac{(n-1)!}{2}\left[H_{n-1}^2 - \zeta_{n-1}(2)\right],\\ s(n,4) &= \frac{(n-1)!}{6}\left[H_{n-1}^3 - 3H_{n-1}\zeta_{n-1}(2) + 2\zeta_{n-1}(3)\right],\\ s(n,5) &= \frac{(n-1)!}{24}[H_{n-1}^4 - 6\zeta_{n-1}(4) - 6H_{n-1}^2\zeta_{n-1}(2)\\ &\quad + 3\zeta_{n-1}^2(2) + 8H_{n-1}\zeta_{n-1}(3)].\end{aligned}$$

Proof. From [12], we have

$$\ln^p(1-x) = (-1)^p p!\sum_{n=p}^{\infty} s(n,p)\frac{x^n}{n!},\ 1 \leq p \in \mathbb{Z},\ -1 \leq x < 1. \tag{2.22}$$

Differentiating this equality, we obtain

$$\frac{\ln^{p-1}(1-x)}{1-x} = (-1)^{p-1}(p-1)!\sum_{n=p-1}^{\infty} s(n+1,p)\frac{x^n}{n!},\ p \geq 2. \tag{2.23}$$

Let k, r be integers with $k > r \geq 0$, then multiplying (2.23) by $x^{r-1} - x^{k-1}$ and integrating over (0,1), we get

$$\begin{aligned}&\int_0^1 \frac{\ln^{p-1}(1-x)}{1-x}\left(x^{r-1}-x^{k-1}\right)dx\\ &= (-1)^{p-1}(p-1)!\sum_{n=p-1}^{\infty}\frac{(k-r)\,s(n+1,p)}{n!\,(n+r)(n+k)},\ p \geq 2.\end{aligned} \tag{2.24}$$

Noting that

$$\int_0^1 \frac{\ln^{p-1}(1-x)}{1-x}\left(x^{r-1}-x^{k-1}\right)dx = \sum_{i=1}^{k-r}\int_0^1 x^{r+i-2}\ln^{p-1}(1-x)\,dx \tag{2.25}$$

and

$$\int_0^1 \frac{\ln^{p-1}(1-x)}{x}dx = (-1)^{p-1}(p-1)!\zeta(p),\ 2 \leq p \in \mathbb{N}. \tag{2.26}$$

Taking $r = 0$, $k \geq 1$ in (2.24), using (2.14), (2.16), (2.25) and (2.26), we can obtain (2.21). □

Taking $r \geq 1$ in (2.24), then we have

$$(p-1)!\,(k-r)\sum_{n=p-1}^{\infty}\frac{s(n+1,p)}{n!\,(n+r)(n+k)} = \frac{1}{p}\left\{Y_p(k-1) - Y_p(r-1)\right\}. \tag{2.27}$$

Letting $p = 3, 4$ in (2.8), we can give the following well known results (see [26, 34]):

Corollary 2.4. *For $k \in \mathbb{N}$, we have*

$$\sum_{n=1}^{\infty}\frac{H_n^2-\zeta_n(2)}{n(n+k)} = \frac{1}{k}\left\{2\zeta(3) + \frac{H_k^3+3H_k\zeta_k(2)+2\zeta_k(3)}{3} - \frac{H_k^2+\zeta_k(2)}{k}\right\}, \tag{2.28}$$

$$\sum_{n=1}^{\infty}\frac{H_n^3-3H_n\zeta_n(2)+2\zeta_n(3)}{n(n+k)} = \frac{1}{k}\left\{\begin{array}{l}\frac{H_k^4+8H_k\zeta_k(3)+6H_k^2\zeta_k(2)+3\zeta_k^2(2)+6\zeta_k(4)}{4}\\ -\frac{H_k^3+3H_k\zeta_k(2)+2\zeta_k(3)}{k}+6\zeta(4)\end{array}\right\}. \tag{2.29}$$

From (2.1), taking $m = 2$, we have the well known result (see Sofo's paper [25])

$$\sum_{n=1}^{\infty}\frac{\zeta_n(2)}{n(n+k)} = \frac{1}{k}\left\{\zeta(3) + \zeta(2)H_{k-1} - \sum_{i=1}^{k-1}\frac{H_i}{i^2}\right\}. \tag{2.30}$$

Substituting (2.30) into (2.28), we obtain (1.2).

In the same manner, we obtain the following theorem:

Theorem 2.5. *For integer $m,k>0$, we have*

$$\begin{aligned}
(2.31)\qquad &\sum_{n=1}^{\infty}\frac{\zeta_n(m)}{n+k}(-1)^{n-1}\\
&=(-1)^k\left(\sum_{n=1}^{\infty}\frac{\zeta_n(m)}{n}(-1)^{n-1}-\bar{\zeta}(m+1)\right)\\
&\quad+(-1)^{k+m+1}\ln 2\left(\zeta_{k-1}(m)+L_{k-1}(m)\right)\\
&\quad+(-1)^k\sum_{j=1}^{m-1}(-1)^{j-1}\bar{\zeta}(m+1-j)\,L_{k-1}(j)+(-1)^{m+k}\sum_{i=1}^{k-1}\frac{L_i(1)}{i^m},
\end{aligned}$$

$$\begin{aligned}
(2.32)\qquad &\sum_{n=1}^{\infty}\frac{L_n(m)}{n+k}(-1)^{n-1}\\
&=(-1)^{k-1}\left(\zeta(m+1)-\sum_{n=1}^{\infty}\frac{L_n(m)}{n}(-1)^{n-1}+(-1)^m\sum_{i=1}^{k-1}\frac{H_i}{i^m}(-1)^{i-1}\right)\\
&\quad+(-1)^k\sum_{j=1}^{m-1}(-1)^{j-1}\zeta(m+1-j)\,L_{k-1}(j).
\end{aligned}$$

Note that the result (2.31) is given in Sofo's paper [29].

From (2.31) and (2.32), we obtain

(2.33)
$$\sum_{n=1}^{\infty}\frac{H_n}{n+k}(-1)^{n-1}=(-1)^{k-1}\left(\frac{1}{2}\ln^2 2-\ln 2\left(H_{k-1}+L_{k-1}(1)\right)+\sum_{i=1}^{k-1}\frac{L_i(1)}{i}\right),$$

$$(2.34)\qquad \sum_{n=1}^{\infty}\frac{L_n(1)}{n+k}(-1)^{n-1}=(-1)^{k-1}\left(\frac{\zeta(2)-\ln^2 2}{2}-\sum_{i=1}^{k-1}\frac{H_i}{i}(-1)^{i-1}\right).$$

Theorem 2.6. *For positive integers l_1,l_2,m and reals $x,y,z\in[-1,1)$, we have the following relation*

$$\begin{aligned}
(2.35)\qquad &\sum_{n=1}^{\infty}\frac{\zeta_n(l_1,x)\,\zeta_n(l_2,y)}{n^m}z^n+\sum_{n=1}^{\infty}\frac{\zeta_n(l_1,x)\,\zeta_n(m,z)}{n^{l_2}}y^n\\
&\quad+\sum_{n=1}^{\infty}\frac{\zeta_n(l_2,y)\,\zeta_n(m,z)}{n^{l_1}}x^n\\
&=\sum_{n=1}^{\infty}\frac{\zeta_n(m,z)}{n^{l_1+l_2}}(xy)^n+\sum_{n=1}^{\infty}\frac{\zeta_n(l_1,x)}{n^{m+l_2}}(yz)^n+\sum_{n=1}^{\infty}\frac{\zeta_n(l_2,y)}{n^{l_1+m}}(xz)^n\\
&\quad+\mathrm{Li}_m(z)\,\mathrm{Li}_{l_1}(x)\,\mathrm{Li}_{l_2}(y)-\mathrm{Li}_{l_1+l_2+m}(xyz),
\end{aligned}$$

where the partial sum $\zeta_n(l,x)$ of polylogarithm function is defined by $\zeta_n(l,x) := \sum_{k=1}^{n} \frac{x^k}{k^l}$.

Proof. We construct the function

$$F(x,y,z) := \sum_{n=1}^{\infty} \{\zeta_n(l_1,x)\,\zeta_n(l_2,y) - \zeta_n(l_1+l_2,xy)\}\, z^{n-1},\ \ z \in (-1,1).$$

By the definition of $\zeta_n(l,x)$, we have

$$F(x,y,z) = zF(x,y,z) + \sum_{n=1}^{\infty} \left\{ \frac{\zeta_n(l_1,x)}{(n+1)^{l_2}} y^{n+1} + \frac{\zeta_n(l_2,y)}{(n+1)^{l_1}} x^{n+1} \right\} z^n. \tag{2.36}$$

Moving $zF(x,y,z)$ from right to left and then multiplying $(1-z)^{-1}$ to the equation (2.36) and integrating over the interval $(0,z)$, we obtain

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{\zeta_n(l_1,x)\,\zeta_n(l_2,y) - \zeta_n(l_1+l_2,xy)}{n} z^n \\ &= \sum_{n=1}^{\infty} \left\{ \frac{\zeta_n(l_1,x)}{(n+1)^{l_2}} y^{n+1} + \frac{\zeta_n(l_2,y)}{(n+1)^{l_1}} x^{n+1} \right\} \{\mathrm{Li}_1(z) - \zeta_n(1,z)\}. \end{aligned} \tag{2.37}$$

Furthermore, using integration and the following formula

$$\sum_{n=1}^{\infty} \left\{ \frac{\zeta_n(l_1,x)}{(n+1)^{l_2}} y^{n+1} + \frac{\zeta_n(l_2,y)}{(n+1)^{l_1}} x^{n+1} \right\} = \mathrm{Li}_{l_1}(x)\,\mathrm{Li}_{l_2}(y) - \mathrm{Li}_{l_1+l_2}(xy),$$

we deduce (2.35) to complete the proof of Theorem 2.6. □

Letting $(x,y,z) = (-1,-1,1)$, $(l_1,l_2,m) = (1,1,m)$ and $(x,y,z) = (-1,-1,-1)$, $(l_1,l_2,m) = (1,1,m)$ in (2.35) gives the following:

Corollary 2.7. *For integer $m > 1$ we have that*

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{L_n^2(1)}{n^m} + 2\sum_{n=1}^{\infty} \frac{L_n(1)\,\zeta_n(m)}{n} (-1)^{n-1} \\ &= \sum_{n=1}^{\infty} \frac{\zeta_n(m)}{n^2} + 2\sum_{n=1}^{\infty} \frac{L_n(1)}{n^{m+1}} (-1)^{n-1} + \ln^2 2\zeta(m) - \zeta(m+2),\ \ (m>1) \end{aligned} \tag{2.38}$$

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{L_n^2(1)}{n^m} (-1)^{n-1} + 2\sum_{n=1}^{\infty} \frac{L_n(1)\,L_n(m)}{n} (-1)^{n-1} \\ &= \sum_{n=1}^{\infty} \frac{L_n(m)}{n^2} + 2\sum_{n=1}^{\infty} \frac{L_n(1)}{n^{m+1}} + \ln^2 2\bar{\zeta}(m) - \bar{\zeta}(m+2),\ \ (m>0). \end{aligned} \tag{2.39}$$

In fact, multiplying (2.35) by $(1-z)^{-1}$ and integrating over $(0,z)$ ($z \in (-1,1)$), we can obtain the following corollary.

Corollary 2.8. *For positive integers* $l_1 > 0, l_2 > 0, m > 1$ *and* $x, y, z \in [-1, 1)$, *then we have*

$$
\begin{aligned}
(2.40)\qquad & \sum_{n=1}^{\infty} \frac{\zeta_n(l_1, x)\,\zeta_n(l_2, x)\,\zeta_n(1, z)}{n^m} + \sum_{n=1}^{\infty} \frac{\zeta_n(l_1, x)\left(\sum_{k=1}^{n} \frac{\zeta_k(1,z)}{k^m}\right)}{n^{l_2}} y^n \\
& + \sum_{n=1}^{\infty} \frac{\zeta_n(l_2, y)\left(\sum_{k=1}^{n} \frac{\zeta_k(1,z)}{k^m}\right)}{n^{l_1}} x^n \\
= & \sum_{n=1}^{\infty} \frac{\left(\sum_{k=1}^{n} \frac{\zeta_k(1,z)}{k^m}\right)}{n^{l_1+l_2}} x^n y^n + \sum_{n=1}^{\infty} \frac{\zeta_n(l_1, x)\,\zeta_n(1, z)}{n^{m+l_2}} y^n \\
& + \sum_{n=1}^{\infty} \frac{\zeta_n(l_2, y)\,\zeta_n(1, z)}{n^{m+l_1}} x^n \\
& + \mathrm{Li}_{l_1}(x)\,\mathrm{Li}_{l_2}(y)\left(\sum_{n=1}^{\infty} \frac{\zeta_n(1, z)}{n^m}\right) - \left(\sum_{n=1}^{\infty} \frac{\zeta_n(1, z)}{n^{m+l_1+l_2}} x^n y^n\right).
\end{aligned}
$$

Letting $x, y, z \to 1$ in (2.40), we arrive at the conclusion that, for integers $m, l_1, l_2 > 1$

(2.41)

$$
\begin{aligned}
& \sum_{n=1}^{\infty} \frac{H_n \zeta_n(l_1)\,\zeta_n(l_2)}{n^m} + \sum_{n=1}^{\infty} \frac{\zeta_n(l_1)\left(\sum_{k=1}^{n} \frac{H_k}{k^m}\right)}{n^{l_2}} + \sum_{n=1}^{\infty} \frac{\zeta_n(l_2)\left(\sum_{k=1}^{n} \frac{H_k}{k^m}\right)}{n^{l_1}} \\
= & \sum_{n=1}^{\infty} \frac{\left(\sum_{k=1}^{n} \frac{H_k}{k^m}\right)}{n^{l_1+l_2}} + \sum_{n=1}^{\infty} \frac{H_n \zeta_n(l_1)}{n^{m+l_2}} + \sum_{n=1}^{\infty} \frac{H_n \zeta_n(l_2)}{n^{m+l_1}} + \zeta(l_1)\,\zeta(l_2)\left(\sum_{n=1}^{\infty} \frac{H_n}{n^m}\right) \\
& - \left(\sum_{n=1}^{\infty} \frac{H_n}{n^{m+l_1+l_2}}\right).
\end{aligned}
$$

From [15, 32], we derive the following identity

(2.42)

$$
\sum_{n=1}^{\infty} \frac{1}{n^{p+1}} \left(\sum_{k=1}^{n} \frac{H_k}{k^m}\right) = \sum_{n=1}^{\infty} \frac{H_n}{n^{m+p+1}} + \zeta(p+1) \sum_{n=1}^{\infty} \frac{H_n}{n^m} - \sum_{n=1}^{\infty} \frac{H_n \zeta_n(p+1)}{n^m}.
$$

Therefore, we may rewrite (2.41) as

(2.43)

$$
\begin{aligned}
& \sum_{n=1}^{\infty} \frac{H_n \zeta_n(l_1)\,\zeta_n(l_2)}{n^m} + \sum_{n=1}^{\infty} \frac{\zeta_n(l_1)\left(\sum_{k=1}^{n} \frac{H_k}{k^m}\right)}{n^{l_2}} + \sum_{n=1}^{\infty} \frac{\zeta_n(l_2)\left(\sum_{k=1}^{n} \frac{H_k}{k^m}\right)}{n^{l_1}} \\
= & \sum_{n=1}^{\infty} \left\{ \frac{H_n \zeta_n(l_1)}{n^{m+l_2}} + \frac{H_n \zeta_n(l_2)}{n^{m+l_1}} - \frac{H_n \zeta_n(l_1+l_2)}{n^m} \right\}
\end{aligned}
$$

$$+\zeta(l_1+l_2)\left(\sum_{n=1}^{\infty}\frac{H_n}{n^m}\right)+\zeta(l_1)\zeta(l_2)\left(\sum_{n=1}^{\infty}\frac{H_n}{n^m}\right).$$

Taking $l_1=l_2=m=2l+1$ (l is a positive integer) in (2.43), we conclude that

$$\begin{aligned}(2.44)\qquad &\sum_{n=1}^{\infty}\frac{H_n\zeta_n^2(2l+1)}{n^{2l+1}}+2\sum_{n=1}^{\infty}\frac{\zeta_n(2l+1)\left(\sum_{k=1}^n\frac{H_k}{k^{2l+1}}\right)}{n^{2l+1}}\\ &=\sum_{n=1}^{\infty}\left\{2\frac{H_n\zeta_n(2l+1)}{n^{4l+2}}-\frac{H_n\zeta_n(4l+2)}{n^{2l+1}}\right\}\\ &\quad+\left(\zeta(4l+2)+\zeta^2(2l+1)\right)\left(\sum_{n=1}^{\infty}\frac{H_n}{n^{2l+1}}\right).\end{aligned}$$

3. Closed form of quadratic Euler sums

In this section we evaluate some quadratic Euler sums involving harmonic numbers and alternating harmonic numbers.

Theorem 3.1. *For integers $l>0, m>0, p>1,$ we have*

(3.1)

$$\begin{aligned}&(-1)^{m-1}\sum_{n=1}^{\infty}\frac{\zeta_n(l)}{n^{p+1}}\left(\sum_{k=1}^n\frac{H_k}{k^m}\right)-(-1)^{p+l}\sum_{n=1}^{\infty}\frac{\zeta_n(m)}{n^{p+1}}\left(\sum_{k=1}^n\frac{H_k}{k^l}\right)\\ &=\sum_{i=1}^{p-1}(-1)^{i-1}\left(\sum_{n=1}^{\infty}\frac{\zeta_n(m)}{n^{i+1}}\right)\left(\sum_{n=1}^{\infty}\frac{\zeta_n(l)}{n^{p+1-i}}\right)+(-1)^{p-1}\zeta(l+1)\left(\sum_{n=1}^{\infty}\frac{\zeta_n(m)}{n^{p+1}}\right)\\ &\quad+(-1)^{p-1}\sum_{j=1}^{l-1}(-1)^{j-1}\zeta(l+1-j)\left\{\sum_{n=1}^{\infty}\frac{\zeta_n(m)\zeta_n(j)}{n^{p+1}}-\sum_{n=1}^{\infty}\frac{\zeta_n(m)}{n^{p+j+1}}\right\}\\ &\quad-(-1)^{p+l}\sum_{n=1}^{\infty}\frac{H_n\zeta_n(m)}{n^{p+l+1}}+(-1)^{m-1}\sum_{n=1}^{\infty}\frac{H_n\zeta_n(l)}{n^{p+m+1}}-\zeta(m+1)\left(\sum_{n=1}^{\infty}\frac{\zeta_n(l)}{n^{p+1}}\right)\\ &\quad-\sum_{j=1}^{m-1}(-1)^{j-1}\zeta(m+1-j)\left\{\sum_{n=1}^{\infty}\frac{\zeta_n(l)\zeta_n(j)}{n^{p+1}}-\sum_{n=1}^{\infty}\frac{\zeta_n(l)}{n^{p+j+1}}\right\}.\end{aligned}$$

Proof. Multiplying (2.1) by $\frac{\zeta_k(l)}{k^p}$ and summing with respect to k, we obtain

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{\zeta_k(l)\zeta_n(m)}{k^pn(n+k)}=\sum_{k=1}^{\infty}\frac{\zeta_k(l)}{k^p}\sum_{n=1}^{\infty}\frac{\zeta_n(m)}{n(n+k)}=\sum_{n=1}^{\infty}\frac{\zeta_n(m)}{n}\sum_{k=1}^{\infty}\frac{\zeta_k(l)}{k^p(n+k)}.$$

Then by using (2.1) and the following partial fraction decomposition formula

$$\frac{1}{k^p(n+k)}=\sum_{i=1}^{p-1}\frac{(-1)^{i-1}}{n^i}\frac{1}{k^{p+1-i}}+\frac{(-1)^{p-1}}{n^{p-1}}\frac{1}{k(n+k)},$$

we can obtain (3.1). $\square$

Taking $(p,l)=(2l,m)$ in (3.1), we can find that

(3.2)

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{\zeta_n(m)}{n^{2l+1}}\left(\sum_{k=1}^{n}\frac{H_k}{k^m}\right)\\
&=(-1)^{m-1}\sum_{i=1}^{l}(-1)^{i-1}\left(\sum_{n=1}^{\infty}\frac{\zeta_n(m)}{n^{i+1}}\right)\left(\sum_{n=1}^{\infty}\frac{\zeta_n(m)}{n^{2l+1-i}}\right)\\
&\quad+\frac{(-1)^{m+l-1}}{2}\left(\sum_{n=1}^{\infty}\frac{\zeta_n(m)}{n^{l+1}}\right)^2\\
&\quad-(-1)^{m-1}\zeta(m+1)\left(\sum_{n=1}^{\infty}\frac{\zeta_n(m)}{n^{2l+1}}\right)+\sum_{n=1}^{\infty}\frac{H_n\zeta_n(m)}{n^{2l+m+1}}\\
&\quad-(-1)^{m-1}\sum_{j=1}^{m-1}(-1)^{j-1}\zeta(m+1-j)\left\{\sum_{n=1}^{\infty}\frac{\zeta_n(m)\zeta_n(j)}{n^{2l+1}}-\sum_{n=1}^{\infty}\frac{\zeta_n(m)}{n^{2l+j+1}}\right\}.
\end{aligned}$$

Putting $m=2l+1$ in (3.2) and combining (2.44), we obtain

(3.3)

$$\begin{aligned}
&S_{1(2l+1)^2,0,(2l+1)}\\
&=\sum_{n=1}^{\infty}\frac{H_n\zeta_n^2(2l+1)}{n^{2l+1}}\\
&=2\zeta(2l+2)\left(\sum_{n=1}^{\infty}\frac{\zeta_n(2l+1)}{n^{2l+1}}\right)\\
&\quad+\left(\zeta(4l+2)+\zeta^2(2l+1)\right)\left(\sum_{n=1}^{\infty}\frac{H_n}{n^{2l+1}}\right)\\
&\quad-(-1)^{l}\left(\sum_{n=1}^{\infty}\frac{\zeta_n(2l+1)}{n^{l+1}}\right)^2-\sum_{n=1}^{\infty}\frac{H_n\zeta_n(4l+2)}{n^{2l+1}}\\
&\quad-2\sum_{i=1}^{l}(-1)^{i}\left(\sum_{n=1}^{\infty}\frac{\zeta_n(2l+1)}{n^{i+1}}\right)\left(\sum_{n=1}^{\infty}\frac{\zeta_n(2l+1)}{n^{2l+1-i}}\right)\\
&\quad+2\sum_{j=1}^{2l}(-1)^{j-1}\zeta(2l+2-j)\left\{\sum_{n=1}^{\infty}\frac{\zeta_n(2l+1)\zeta_n(j)}{n^{2l+1}}-\sum_{n=1}^{\infty}\frac{\zeta_n(2l+1)}{n^{2l+j+1}}\right\}.
\end{aligned}$$

Letting $l=1, p=m-1$ in (3.1) and using the formula

$$\sum_{n=1}^{\infty}\frac{H_n}{n^m}\left(\sum_{k=1}^{n}\frac{H_k}{k^m}\right)=\frac{1}{2}\left\{\left(\sum_{n=1}^{\infty}\frac{H_n}{n^m}\right)^2+\sum_{n=1}^{\infty}\frac{H_n^2}{n^{2m}}\right\},\ 2\le m\in\mathbb{Z},$$

we can get the following result

$$
\begin{aligned}
S_{1^2m,0,m} =& \sum_{n=1}^{\infty} \frac{H_n^2 \zeta_n(m)}{n^m} \\
=& 2(-1)^{m-1} \sum_{i=1}^{m-2} (-1)^{i-1} \left(\sum_{n=1}^{\infty} \frac{\zeta_n(m)}{n^{i+1}} \right) \left(\sum_{n=1}^{\infty} \frac{H_n}{n^{m-i}} \right) \\
& - \zeta(2) \left(\zeta^2(m) + \zeta(2m) \right) - 2(-1)^{m-1} \zeta(m+1) \left(\sum_{n=1}^{\infty} \frac{H_n}{n^m} \right) \\
& - 2(-1)^{m-1} \sum_{j=1}^{m-1} (-1)^{j-1} \zeta(m+1-j) \left\{ \sum_{n=1}^{\infty} \frac{H_n \zeta_n(j)}{n^m} - \sum_{n=1}^{\infty} \frac{H_n}{n^{m+j}} \right\} \\
& + 2 \sum_{n=1}^{\infty} \frac{H_n \zeta_n(m)}{n^{m+1}} + \sum_{n=1}^{\infty} \frac{H_n^2}{n^{2m}} - \left(\sum_{n=1}^{\infty} \frac{H_n}{n^m} \right)^2 - \sum_{n=1}^{\infty} \frac{\zeta_n(2) \zeta_n(m)}{n^m}.
\end{aligned} \tag{3.4}
$$

In [15], Flajolet and Salvy gave the following conclusion: If $p_1 + p_2 + q$ is even, and $p_1 > 1, p_2 > 1, q > 1$, the quadratic sums

$$S_{p_1 p_2,0,q} = \sum_{n=1}^{\infty} \frac{\zeta_n(p_1) \zeta_n(p_2)}{n^q}$$

are reducible to linear sums. In [32, 33], we showed that all quadratic Euler sums of the form

$$S_{1m,0,p} = \sum_{n=1}^{\infty} \frac{H_n \zeta_n(m)}{n^p} \quad (m + p \le 9)$$

are reducible to polynomials in zeta values and to linear sums. Hence, from (3.4), we know that the cubic sums $S_{1^2m,0,m}$ are reducible to polynomials in zeta values and to linear sums when $m = 2, 3, 4, 5$. For example

$$
\begin{aligned}
S_{1^2 2,0,2} &= \sum_{n=1}^{\infty} \frac{H_n^2 \zeta_n(2)}{n^2} = \frac{41}{12} \zeta(6) + 2\zeta^2(3), \\
S_{1^2 3,0,3} &= \sum_{n=1}^{\infty} \frac{H_n^2 \zeta_n(3)}{n^3} = \frac{9}{2} \zeta(3) \zeta(5) + \frac{3}{2} \zeta(2) \zeta^2(3) - \frac{443}{288} \zeta(8) - \frac{23}{4} S_{2,0,6},
\end{aligned}
$$

where $S_{2,0,6} = \sum_{n=1}^{\infty} \frac{\zeta_n(2)}{n^6}$.

Noting that, from Theorem 4.2 in the reference [15], we deduce that

$$S_{23,0,3} = \sum_{n=1}^{\infty} \frac{\zeta_n(2) \zeta_n(3)}{n^3} = \frac{45}{2} \zeta(3) \zeta(5) - \frac{827}{48} \zeta(8) - \frac{3}{2} \zeta(2) \zeta^2(3) - \frac{23}{4} S_{2,0,6}.$$

In the same way, we can obtain the following theorems.

Theorem 3.2. *For integers $p_1 > 1, p_2 > 1, m > 0$, we have*

(3.5)
$$\begin{aligned}
&\frac{(p_1-1)!}{p_2}\sum_{n=p_1-1}^{\infty}\frac{s(n+1,p_1)\,Y_{p_2}(n)}{n^{m+1}n!}\\
&\quad-(-1)^{m-1}\frac{(p_2-1)!}{p_1}\sum_{n=p_2-1}^{\infty}\frac{s(n+1,p_2)\,Y_{p_1}(n)}{n^{m+1}n!}\\
=\ &(p_1-1)!\sum_{n=p_1-1}^{\infty}\frac{s(n+1,p_1)\,Y_{p_2-1}(n)}{n^{m+2}n!}\\
&\quad-(-1)^{m-1}(p_2-1)!\sum_{n=p_2-1}^{\infty}\frac{s(n+1,p_2)\,Y_{p_1-1}(n)}{n^{m+2}n!}\\
&\quad+(p_1-1)!\,(p_2-1)!\sum_{i=1}^{m-1}(-1)^{i-1}\left(\sum_{n=p_1-1}^{\infty}\frac{s(n+1,p_1)}{n^{m+1-i}n!}\right)\left(\sum_{n=p_2-1}^{\infty}\frac{s(n+1,p_2)}{n^{i+1}n!}\right)\\
&\quad+(-1)^{m-1}(p_1-1)!\,(p_2-1)!\zeta(p_1)\left(\sum_{n=p_2-1}^{\infty}\frac{s(n+1,p_2)}{n^{m+1}n!}\right)\\
&\quad-(p_1-1)!\,(p_2-1)!\zeta(p_2)\left(\sum_{n=p_1-1}^{\infty}\frac{s(n+1,p_1)}{n^{m+1}n!}\right).
\end{aligned}$$

Proof. Replacing p by p_2 in (2.21), we get

(3.6)
$$(p_2-1)!\sum_{n=p_2-1}^{\infty}\frac{s(n+1,p_2)}{n!n(n+k)}=\frac{1}{k}\left\{(p_2-1)!\zeta(p_2)+\frac{Y_{p_2}(k)}{p_2}-\frac{Y_{p_2-1}(k)}{k}\right\}.$$

Multiplying (3.6) by $(p_1-1)!\frac{s(k+1,p_1)}{k!k^m}$ and summing with respect to k, we obtain

$$\begin{aligned}
&(p_1-1)!\,(p_2-1)!\sum_{k=p_1-1}^{\infty}\sum_{n=p_2-1}^{\infty}\frac{s(k+1,p_1)\,s(n+1,p_2)}{k!k^m n!n(n+k)}\\
=\ &(p_1-1)!\sum_{k=p_1-1}^{\infty}\frac{s(k+1,p_1)}{k!k^m}(p_2-1)!\sum_{n=p_2-1}^{\infty}\frac{s(n+1,p_2)}{n!n(n+k)}\\
=\ &(p_2-1)!\sum_{n=p_2-1}^{\infty}\frac{s(n+1,p_2)}{n!n}(p_1-1)!\sum_{k=p_1-1}^{\infty}\frac{s(k+1,p_1)}{k!k^m(n+k)}.
\end{aligned}$$

Then with the help of formula (2.21) we may easily deduce the result. □

Theorem 3.3. *For integer $m>0$, then we have*

$$\left(\frac{1}{2}+(-1)^m\right)\sum_{n=1}^{\infty}\frac{H_n^2}{n^{m+1}} \tag{3.7}$$
$$=\sum_{j=0}^{m-2}(-1)^j\zeta(m-j)\sum_{n=1}^{\infty}\frac{H_n}{n^{j+2}}-\zeta(2)\zeta(m+1)+\sum_{n=1}^{\infty}\frac{H_n}{n^{m+2}}-\frac{1}{2}\sum_{n=1}^{\infty}\frac{\zeta_n(2)}{n^{m+1}},$$

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_n^2}{n^{m+1}}(-1)^{n-1}+(-1)^{m-1}\sum_{n=1}^{\infty}\frac{H_nL_n(1)}{n^{m+1}}(-1)^{n-1} \tag{3.8}$$
$$\begin{aligned}=&\sum_{j=1}^{m-1}(-1)^{j-1}\bar{\zeta}(m-j+1)\sum_{n=1}^{\infty}\frac{H_n}{n^{j+1}}+(-1)^{m-1}\ln 2\sum_{n=1}^{\infty}\frac{H_n}{n^{m+1}}\\&+\sum_{n=1}^{\infty}\frac{H_n}{n^{m+2}}(-1)^{n-1}+(-1)^{m-1}\ln 2\sum_{n=1}^{\infty}\frac{H_n}{n^{m+1}}(-1)^{n-1}\\&-\frac{1}{2}\sum_{n=1}^{\infty}\frac{\zeta_n(2)}{n^{m+1}}(-1)^{n-1}-\zeta(2)\bar{\zeta}(m+1),\end{aligned}$$

$$\left(\frac{1}{2}+(-1)^m\right)\sum_{n=1}^{\infty}\frac{L_n^2(1)}{n^{m+1}}(-1)^{n-1} \tag{3.9}$$
$$\begin{aligned}=&\bar{\zeta}(2)\bar{\zeta}(m+1)+\ln 2\sum_{n=1}^{\infty}\frac{H_n+L_n(1)}{n^{m+1}}(-1)^{n-1}\\&+(-1)^m\ln 2\sum_{n=1}^{\infty}\frac{L_n(1)}{n^{m+1}}\left(1+(-1)^{n-1}\right)\\&+\sum_{n=1}^{\infty}\frac{L_n(1)}{n^{m+2}}-\ln 2\left(\bar{\zeta}(m+2)+\zeta(m+2)\right)-\frac{1}{2}\sum_{n=1}^{\infty}\frac{\zeta_n(2)}{n^{m+1}}(-1)^{n-1}\\&-\sum_{j=1}^{m-1}(-1)^{j-1}\bar{\zeta}(m-j+1)\sum_{n=1}^{\infty}\frac{L_n(1)}{n^{j+1}},\end{aligned}$$

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{L_n^2(1)}{n^{m+1}}+(-1)^{m-1}\sum_{n=1}^{\infty}\frac{H_nL_n(1)}{n^{m+1}} \tag{3.10}$$
$$\begin{aligned}=&\bar{\zeta}(2)\zeta(m+1)+\ln 2\sum_{n=1}^{\infty}\frac{H_n+L_n(1)}{n^{m+1}}\\&-\ln 2\left(\bar{\zeta}(m+2)+\zeta(m+2)\right)-\frac{1}{2}\sum_{n=1}^{\infty}\frac{\zeta_n(2)}{n^{m+1}}+\sum_{n=1}^{\infty}\frac{L_n(1)}{n^{m+2}}(-1)^{n-1}\end{aligned}$$

$$-\sum_{j=1}^{m-1}(-1)^{j-1}\zeta(m-j+1)\sum_{n=1}^{\infty}\frac{L_n(1)}{n^{j+1}}.$$

Proof. Similarly as in the proofs of Theorems 3.1 and 3.2, we consider the following sums

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{H_n}{k^m n(n+k)},\ \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{H_n}{k^m n(n+k)}(-1)^{k-1},$$
$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{L_n(1)}{k^m n(n+k)},\ \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{L_n(1)}{k^m n(n+k)}(-1)^{k-1}.$$

Then using identities (2.12), (2.13) with the help of the following formula

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^m(n+k)}=\sum_{j=1}^{m-1}\frac{(-1)^{j-1}}{k^j}\bar{\zeta}(m-j+1)+\frac{(-1)^{m-1}}{k^m}\ln 2$$
$$+\frac{(-1)^{m+k}}{k^m}\ln 2-\frac{(-1)^{m+k}}{k^m}L_k(1),$$

we deduce Theorem 3.3 holds. □

Theorem 3.4. *For positive integer m, we have*

$$\left(\frac{1}{3}+(-1)^m\right)\sum_{n=1}^{\infty}\frac{H_n^3}{n^{m+1}}+\left(1+(-1)^{m-1}\right)\sum_{n=1}^{\infty}\frac{H_n\zeta_n(2)}{n^{m+1}} \tag{3.11}$$
$$=\sum_{j=0}^{m-2}(-1)^j\zeta(m-j)\sum_{n=1}^{\infty}\frac{H_n^2-\zeta_n(2)}{n^{j+2}}+\sum_{n=1}^{\infty}\frac{H_n^2+\zeta_n(2)}{n^{m+2}}$$
$$-\frac{2}{3}\sum_{n=1}^{\infty}\frac{\zeta_n(3)}{n^{m+1}}-2\zeta(3)\zeta(m+1),$$

$$\sum_{n=1}^{\infty}\frac{L_n^3(1)+L_n(1)\zeta_n(2)}{n^{2m+1}} \tag{3.12}$$
$$=2\bar{\zeta}(2)\left(\sum_{n=1}^{\infty}\frac{L_n(1)}{n^{2m+1}}\right)+2\ln 2\sum_{n=1}^{\infty}\frac{H_nL_n(1)+L_n^2(1)}{n^{2m+1}}$$
$$-2\ln 2\sum_{n=1}^{\infty}\frac{L_n(1)}{n^{2m+2}}\left(1+(-1)^{n-1}\right)+2\sum_{n=1}^{\infty}\frac{L_n^2(1)}{n^{2m+2}}(-1)^{n-1}$$
$$-2\sum_{i=1}^{m}(-1)^{i-1}\left(\sum_{n=1}^{\infty}\frac{L_n(1)}{n^{i+1}}\right)\left(\sum_{n=1}^{\infty}\frac{L_n(1)}{n^{2m+1-i}}\right)$$
$$+(-1)^{m-1}\left(\sum_{n=1}^{\infty}\frac{L_n(1)}{n^{m+1}}\right)^2,$$

$$\frac{1}{2}\sum_{n=1}^{\infty}\frac{H_nL_n^2\left(1\right)+H_n\zeta_n\left(2\right)}{n^{m+1}}+\frac{\left(-1\right)^{m-1}}{2}\sum_{n=1}^{\infty}\frac{H_n^2L_n\left(1\right)+L_n\left(1\right)\zeta_n\left(2\right)}{n^{m+1}} \tag{3.13}$$

$$\begin{aligned}
&=\bar{\zeta}\left(2\right)\left(\sum_{n=1}^{\infty}\frac{H_n}{n^{m+1}}\right)+\ln 2\sum_{n=1}^{\infty}\frac{H_n^2+H_nL_n\left(1\right)}{n^{m+1}}\\
&\quad-\ln 2\sum_{n=1}^{\infty}\frac{H_n}{n^{m+2}}\left(1+\left(-1\right)^{n-1}\right)\\
&\quad-\sum_{i=1}^{m-1}\left(-1\right)^{i-1}\left(\sum_{n=1}^{\infty}\frac{L_n\left(1\right)}{n^{i+1}}\right)\left(\sum_{n=1}^{\infty}\frac{H_n}{n^{m+1-i}}\right)\\
&\quad-\left(-1\right)^{m-1}\zeta\left(2\right)\left(\sum_{n=1}^{\infty}\frac{L_n\left(1\right)}{n^{m+1}}\right)\\
&\quad+\left(-1\right)^{m-1}\sum_{n=1}^{\infty}\frac{H_nL_n\left(1\right)}{n^{m+2}}+\sum_{n=1}^{\infty}\frac{H_nL_n\left(1\right)}{n^{m+2}}\left(-1\right)^{n-1}.
\end{aligned}$$

Proof. Similarly as in the proof of Theorems 3.1-3.3, we consider the following sums

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{H_n^2-\zeta_n\left(2\right)}{k^mn\left(n+k\right)},\ \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{L_k\left(1\right)L_n\left(1\right)}{k^{2m}n\left(n+k\right)},\ \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{H_kL_n\left(1\right)}{k^mn\left(n+k\right)}.$$

Then using (2.12), (2.13) and (2.28), by a simple calculation, we obtain the desired results. □

Letting $p_1=p_2=p$, $m=2k$ in Theorem 3.2, we can get the following Corollary

Corollary 3.5. *For integers $k>0$, $p>1$, we have*

(3.14)

$$\begin{aligned}
&\frac{(p-1)!}{p}\sum_{n=p-1}^{\infty}\frac{s\left(n+1,p\right)Y_p\left(n\right)}{n^{2k+1}n!}\\
=\ &(p-1)!\sum_{n=p-1}^{\infty}\frac{s\left(n+1,p\right)Y_{p-1}\left(n\right)}{n^{2k+2}n!}-\left[(p-1)!\right]^2\zeta\left(p\right)\left(\sum_{n=p-1}^{\infty}\frac{s\left(n+1,p\right)}{n^{2k+1}n!}\right)\\
&+\left[(p-1)!\right]^2\sum_{i=1}^{k}\left(-1\right)^{i-1}\left(\sum_{n=p-1}^{\infty}\frac{s\left(n+1,p\right)}{n^{2k+1-i}n!}\right)\left(\sum_{n=p-1}^{\infty}\frac{s\left(n+1,p\right)}{n^{i+1}n!}\right)\\
&-\frac{\left[(p-1)!\right]^2}{2}\left(-1\right)^{k-1}\left(\sum_{n=p-1}^{\infty}\frac{s\left(n+1,p\right)}{n^{k+1}n!}\right)^2.
\end{aligned}$$

Taking $p=2$ in (3.14), we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{H_n^3+H_n\zeta_n(2)}{n^{2k+1}} \qquad (3.15)\\
&=2\sum_{n=1}^{\infty}\frac{H_n^2}{n^{2k+2}}+2\sum_{i=1}^{k}(-1)^{i-1}\left(\sum_{n=1}^{\infty}\frac{H_n}{n^{2k+1-i}}\right)\left(\sum_{n=1}^{\infty}\frac{H_n}{n^{i+1}}\right)\\
&\quad-2\zeta(2)\left(\sum_{n=1}^{\infty}\frac{H_n}{n^{2k+1}}\right)-(-1)^{k-1}\left(\sum_{n=1}^{\infty}\frac{H_n}{n^{k+1}}\right)^2.
\end{aligned}$$

Letting $m=2k-1$ in (3.11), we get

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{H_n^3}{n^{2k+1}}&=\frac{3}{4}\sum_{j=0}^{2k-2}(-1)^j\zeta(2k-j)\sum_{n=1}^{\infty}\frac{H_n^2-\zeta_n(2)}{n^{j+2}}+\frac{3}{4}\sum_{n=1}^{\infty}\frac{H_n^2+\zeta_n(2)}{n^{2k+2}}\\
&\quad-\frac{1}{2}\sum_{n=1}^{\infty}\frac{\zeta_n(3)}{n^{2k+1}}-\frac{3}{2}\zeta(3)\zeta(2k+1). \qquad (3.16)
\end{aligned}$$

Substituting (3.16) into (3.15), we arrive at the conclusion that

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{H_n\zeta_n(2)}{n^{2k+1}}&=2\sum_{n=1}^{\infty}\frac{H_n^2}{n^{2k+2}}+2\sum_{i=1}^{k}(-1)^{i-1}\left(\sum_{n=1}^{\infty}\frac{H_n}{n^{2k+1-i}}\right)\left(\sum_{n=1}^{\infty}\frac{H_n}{n^{i+1}}\right)\\
&\quad+\frac{1}{2}\sum_{n=1}^{\infty}\frac{\zeta_n(3)}{n^{2k+1}}+\frac{3}{2}\zeta(3)\zeta(2k+1)\\
&\quad-2\zeta(2)\left(\sum_{n=1}^{\infty}\frac{H_n}{n^{2k+1}}\right)-(-1)^{k-1}\left(\sum_{n=1}^{\infty}\frac{H_n}{n^{k+1}}\right)^2\\
&\quad-\frac{3}{4}\sum_{j=0}^{2k-2}(-1)^j\zeta(2k-j)\sum_{n=1}^{\infty}\frac{H_n^2-\zeta_n(2)}{n^{j+2}}\\
&\quad-\frac{3}{4}\sum_{n=1}^{\infty}\frac{H_n^2+\zeta_n(2)}{n^{2k+2}}. \qquad (3.17)
\end{aligned}$$

Similarly, taking $(p_1,p_2)=(2,3),\,(1,4)$ in Theorem 3.2, we deduce that

$$\begin{aligned}
&\left(\frac{1}{3}-\frac{(-1)^{m-1}}{2}\right)\sum_{n=1}^{\infty}\frac{H_n^4}{n^{m+1}}+\sum_{n=1}^{\infty}\frac{H_n^2\zeta_n(2)}{n^{m+1}} \qquad (3.18)\\
&+\frac{2}{3}\sum_{n=1}^{\infty}\frac{H_n\zeta_n(3)}{n^{m+1}}+\frac{(-1)^{m-1}}{2}\sum_{n=1}^{\infty}\frac{\zeta_n^2(2)}{n^{m+1}})\\
&=\left(1-(-1)^{m-1}\right)\sum_{n=1}^{\infty}\frac{H_n^3}{n^{m+2}}+\left(1+(-1)^{m-1}\right)\sum_{n=1}^{\infty}\frac{H_n\zeta_n(2)}{n^{m+2}}
\end{aligned}$$

$$+(-1)^{m-1}\zeta(2)\sum_{n=1}^{\infty}\frac{H_n^2-\zeta_n(2)}{n^{m+1}})-2\zeta(3)\sum_{n=1}^{\infty}\frac{H_n}{n^{m+1}}$$
$$+\sum_{i=1}^{m-1}(-1)^{i-1}\left(\sum_{n=1}^{\infty}\frac{H_n}{n^{m+1-i}}\right)\left(\sum_{n=1}^{\infty}\frac{H_n^2-\zeta_n(2)}{n^{i+1}}\right),$$

$$\begin{aligned}(3.19)\qquad &\left(\frac{1}{4}+(-1)^m\right)\sum_{n=1}^{\infty}\frac{H_n^4}{n^{m+1}}+3\left(\frac{1}{2}+(-1)^{m-1}\right)\sum_{n=1}^{\infty}\frac{H_n^2\zeta_n(2)}{n^{m+1}}\\
&+2\left(1+(-1)^m\right)\sum_{n=1}^{\infty}\frac{H_n\zeta_n(3)}{n^{m+1}}+\frac{3}{4}\sum_{n=1}^{\infty}\frac{\zeta_n^2(2)}{n^{m+1}}\\
&=\sum_{n=1}^{\infty}\frac{H_n^3+3H_n\zeta_n(2)+2\zeta_n(3)}{n^{m+2}}\\
&+\sum_{i=1}^{m-1}(-1)^{i-1}\zeta(m+1-i)\left(\sum_{n=1}^{\infty}\frac{H_n^3-3H_n\zeta_n(2)+2\zeta_n(3)}{n^{i+1}}\right)\\
&-\frac{3}{2}\sum_{n=1}^{\infty}\frac{\zeta_n(4)}{n^{m+1}}-6\zeta(4)\zeta(m+1).\end{aligned}$$

Proceeding in a similar fashion to evaluation of the Theorem 3.1-3.4, it is possible to evaluate other Euler sums involving harmonic numbers and alternating harmonic numbers. For instance, multiplying (2.21) by $\frac{(-1)^{k-1}}{k^m}$, $\frac{L_k(1)}{k^m}$ and summing with respect to k, we obtain

(3.20)
$$\begin{aligned}&\frac{1}{p}\sum_{n=1}^{\infty}\frac{Y_p(n)}{n^{m+1}}(-1)^{n-1}+(-1)^{m-1}(p-1)!\sum_{n=p-1}^{\infty}\frac{s(n+1,p)L_n(1)}{n!n^{m+1}}(-1)^{n-1}\\
&=\sum_{n=1}^{\infty}\frac{Y_{p-1}(n)}{n^{m+2}}(-1)^{n-1}\\
&+(p-1)!\sum_{i=1}^{m-1}(-1)^{i-1}\bar{\zeta}(m+1-i)\left(\sum_{n=p-1}^{\infty}\frac{s(n+1,p)}{n!n^{i+1}}\right)\\
&+(-1)^{m-1}(p-1)!\ln 2\sum_{n=p-1}^{\infty}\frac{s(n+1,p)}{n!n^{m+1}}\left(1+(-1)^{n-1}\right)\\
&-(p-1)!\zeta(p)\bar{\zeta}(m+1),\end{aligned}$$

and

(3.21)
$$\frac{1}{p}\sum_{n=1}^{\infty}\frac{Y_p(n)L_n(1)}{n^{m+1}}+\frac{(-1)^{m-1}}{2}(p-1)!\sum_{n=p-1}^{\infty}\frac{s(n+1,p)\left(L_n^2(1)+\zeta_n(2)\right)}{n!n^{m+1}}$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{Y_{p-1}(n) L_n(1)}{n^{m+2}} \\
&\quad + (p-1)! \sum_{i=1}^{m-1} (-1)^{i-1} \left(\sum_{n=1}^{\infty} \frac{L_n(1)}{n^{m+1-i}} \right) \left(\sum_{n=p-1}^{\infty} \frac{s(n+1,p)}{n! n^{i+1}} \right) \\
&\quad + (-1)^{m-1} (p-1)! \bar{\zeta}(2) \sum_{n=p-1}^{\infty} \frac{s(n+1,p)}{n! n^{m+1}} \\
&\quad + (-1)^{m-1} (p-1)! \sum_{n=p-1}^{\infty} \frac{s(n+1,p) L_n(1)}{n! n^{m+2}} \\
&\quad - (-1)^{m-1} (p-1)! \ln 2 \sum_{n=p-1}^{\infty} \frac{s(n+1,p)}{n! n^{m+2}} \left(1 + (-1)^{n-1}\right) \\
&\quad + (-1)^{m-1} (p-1)! \ln 2 \sum_{n=p-1}^{\infty} \frac{s(n+1,p)(H_n + L_n(1))}{n! n^{m+1}} \\
&\quad - (p-1)! \zeta(p) \sum_{n=1}^{\infty} \frac{L_n(1)}{n^{m+1}}.
\end{aligned}
$$

4. Some examples

Now, we give some examples.

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{L_n^2(1)}{n^2} (-1)^{n-1} = &- \frac{41}{16} \zeta(4) + 2\zeta(2) \ln^2 2 + \frac{1}{6} \ln^4 2 + \frac{7}{4} \zeta(3) \ln 2 \\
&+ 4 \mathrm{Li}_4 \left(\frac{1}{2} \right),
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{L_n^2(1)}{n^3} (-1)^{n-1} = &- 4 \mathrm{Li}_4 \left(\frac{1}{2} \right) \ln 2 + \frac{19}{8} \zeta(4) \ln 2 + \zeta(2) \ln^3 2 - \frac{1}{6} \ln^5 2 \\
&+ \frac{3}{8} \zeta(2) \zeta(3) - \frac{19}{32} \zeta(5),
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=1}^{\infty} \frac{L_n^2(1)}{n^4} (-1)^{n-1} = &\frac{15}{4} \ln^2 2 \zeta(4) + \frac{9}{4} \zeta(2) \zeta(3) \ln 2 - \frac{93}{16} \zeta(5) \ln 2 \\
&+ \frac{35}{64} \zeta(6) - \frac{15}{16} \zeta^2(3) + \sum_{n=1}^{\infty} \frac{\zeta_n(2)}{n^4} (-1)^{n-1},
\end{aligned}
$$

$$
\sum_{n=1}^{\infty} \frac{L_n(1) L_n(2)}{n} (-1)^{n-1} = \frac{61}{16} \zeta(4) - \frac{7}{8} \zeta(3) \ln 2 - \frac{1}{4} \zeta(2) \ln^2 2
$$

$$-\frac{1}{6}\ln^4 2-4\mathrm{Li}_4\left(\frac{1}{2}\right),$$

$$\sum_{n=1}^{\infty}\frac{L_n(1)L_n(3)}{n}(-1)^{n-1}=2\ln 2\mathrm{Li}_4\left(\frac{1}{2}\right)+\frac{1}{12}\ln^5 2+\frac{3}{8}\zeta(3)\ln^2 2-\frac{19}{32}\zeta(5)$$
$$-\frac{1}{2}\zeta(2)\ln^3 2+\frac{11}{16}\zeta(4)\ln 2+\frac{1}{4}\zeta(2)\zeta(3),$$

$$\sum_{n=1}^{\infty}\frac{L_n(1)L_n(4)}{n}(-1)^{n-1}=-\frac{35}{128}\zeta(6)+\frac{3}{4}\zeta^2(3)-\frac{9}{8}\zeta(2)\zeta(3)\ln 2$$
$$+\frac{155}{32}\zeta(5)\ln 2-\frac{23}{16}\zeta(4)\ln^2 2$$
$$-\sum_{n=1}^{\infty}\frac{\zeta_n(2)}{n^4}(-1)^{n-1},$$

$$\sum_{n=1}^{\infty}\frac{H_n^3}{n^5}=\frac{469}{32}\zeta(8)-16\zeta(3)\zeta(5)+\frac{3}{2}\zeta(2)\zeta^2(3)+\frac{11}{4}\sum_{n=1}^{\infty}\frac{\zeta_n(2)}{n^6},$$

$$\sum_{n=1}^{\infty}\frac{H_n\zeta_n(2)}{n^5}=-\frac{343}{48}\zeta(8)+12\zeta(3)\zeta(5)-\frac{5}{2}\zeta(2)\zeta^2(3)-\frac{3}{4}\sum_{n=1}^{\infty}\frac{\zeta_n(2)}{n^6},$$

$$\sum_{n=1}^{\infty}\frac{H_n^2\zeta_n(3)}{n^3}=\frac{9}{2}\zeta(3)\zeta(5)+\frac{3}{2}\zeta(2)\zeta^2(3)-\frac{443}{288}\zeta(8)-\frac{23}{4}\sum_{n=1}^{\infty}\frac{\zeta_n(2)}{n^6}.$$

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