

CHARACTERIZATIONS OF ORDERED INTRA k -REGULAR SEMIRINGS BY ORDERED k -IDEALS

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ABSTRACT. We introduce the notion of ordered intra k -regular semirings, characterize them using their ordered k -ideals and prove that an ordered semiring S is both ordered k -regular and ordered intra k -regular if and only if every ordered quasi k -ideal or every ordered k -bi-ideal of S is ordered k -idempotent.

1. Introduction

The notion of intra-regular semigroups was introduced by Lajos [7] in 1963. Then Shabir, Ali and Batool [14] introduced the notion of intra-regular semirings and gave some of their characterizations by their quasi-ideals.

In 1951, Bourne [4] called a semiring $(S, +, \cdot)$ to be regular if for every element a of S there exist $x, y \in S$ such that $a + axa = aya$. In 1996, Bourne regularity was renamed to be k -regular by Adhikari, Sen and Weinert [1]. Also in [1] k -regular semirings were characterized by their k -ideals. Later, Bhuniya and Jana introduced the notions of k -bi-ideals, quasi k -ideals of semirings and intra k -regular semirings and characterized k -regular semirings and intra k -regular semirings using their k -bi-ideals and quasi k -ideals, see in [3] and [6], respectively.

In 2011, the notion of an ordered semiring $S := (S, +, \cdot, \leq)$ was defined by Gan and Jiang [5] as a semiring $(S, +, \cdot)$ with a partially ordered set (S, \leq) such that \leq is compatible with the operations $+$ and \cdot of S . In this paper, the notion of left (right) ordered ideals and ordered ideals were defined. Then Mandal [8] introduced and studied regular, intra-regular and k -regular ordered semirings. In our previous works [9, 10], in 2016, we introduced the notion of ordered

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quasi-ideals of ordered semirings, studied some of their properties and characterized regular ordered semirings, regular ordered duo-semirings and intra-regular ordered semirings using their ordered quasi-ideals. Later, Patchakhieo and Pibaljomme [12] introduced the notion of ordered k -ideals of ordered semirings, defined the concept of ordered k -regular semirings as the generalization of k -regular ordered semirings in sense of Mandal [8] and characterized them by their ordered k -ideals.

In our previous work [11], we introduced the notion of an ordered quasi k -ideal of an ordered semiring, and characterized ordered k -regular semirings using their ordered quasi k -ideals. As a continuation of our previous work, in this paper, we introduce the notion of ordered intra k -regular semirings and characterize them by their ordered k -ideals. In the last part of this paper, ordered quasi k -ideals are used to characterize an ordered semiring which is both ordered k -regular and ordered intra k -regular.

2. Preliminaries

A *semiring* is a tri-tuple $(S, +, \cdot)$ consisting of a nonempty set S and two binary operations $+$ and \cdot on S such that $(S, +)$ and (S, \cdot) are semigroups and the distributive law holds on S . A semiring S is called *additively commutative* if $x + y = y + x$ for all $x, y \in S$.

A nonempty subset A of a semiring S such that $A + A \subseteq A$ is called a *left (right) ideal* of S if $SA \subseteq A$ ($AS \subseteq A$). We call A an *ideal* of S if A is both a left and a right ideal of S . A subsemiring A of S is called a *bi-ideal (interior ideal)* of S if $ASA \subseteq A$ ($SAS \subseteq A$).

An *ordered semiring* is an algebraic structure $(S, +, \cdot, \leq)$ such that $(S, +, \cdot)$ is a semiring, (S, \leq) is a partially ordered set and the relation \leq is compatible to the operations $+$ and \cdot on S . Mandal [8] called an ordered semiring S to be *regular* if for every $a \in S$, $a \leq axa$ for some $x \in S$. In this paper, we always assume that S is an additively commutative ordered semiring.

For any nonempty subsets A, B of S , and element $a \in S$, we denote

$$\begin{aligned} AB &= \{ab \in S \mid a \in A, b \in B\}, \\ A + B &= \{a + b \in S \mid a \in A, b \in B\}, \\ \Sigma A &= \left\{ \sum_{i=1}^n a_i \in S \mid a_i \in A, n \in \mathbb{N} \right\}, \\ \Sigma AB &= \left\{ \sum_{i=1}^n a_i b_i \in S \mid a_i \in A, b_i \in B, n \in \mathbb{N} \right\}, \\ \Sigma a &= \Sigma\{a\} \text{ and} \\ (A) &= \{x \in S \mid x \leq a \text{ for some } a \in A\}. \end{aligned}$$

Remark 2.1. Let A, B be nonempty subsets of an ordered semiring S . Then the following statements hold:

- (i) $A \subseteq \Sigma A$ and $\Sigma(\Sigma A) = \Sigma A$;
- (ii) if $A \subseteq B$, then $\Sigma A \subseteq \Sigma B$;
- (iii) $A(\Sigma B) \subseteq (\Sigma A)(\Sigma B) \subseteq \Sigma AB$ and $(\Sigma A)B \subseteq (\Sigma A)(\Sigma B) \subseteq \Sigma AB$;
- (iv) $\Sigma(A + B) \subseteq \Sigma A + \Sigma B$;
- (v) $\Sigma(A] \subseteq (\Sigma A]$;
- (vi) $A \subseteq (A]$ and $((A]) = (A]$;
- (vii) if $A \subseteq B$, then $(A] \subseteq (B]$;
- (viii) $A(B] \subseteq (A](B] \subseteq (AB]$ and $(A]B \subseteq (A](B] \subseteq (AB]$;
- (ix) $A + (B] \subseteq (A] + (B] \subseteq (A + B]$;
- (x) $(A \cup B] = (A] \cup (B]$;
- (xi) $(A \cap B] \subseteq (A] \cap (B]$.

Clearly, $A = \Sigma A$ if and only if A is closed under addition. It is easy to check that the equality of Remark 2.1(xi) holds if $(A] = A$ and $(B] = B$ and also true for arbitrary intersections.

Let A be a nonempty subset of S . Then the k -closure [12] of A in S is defined by

$$\bar{A} = \{x \in S \mid x + a \leq b \text{ for some } a, b \in A\}.$$

Remark 2.2. Let A, B be nonempty subsets of an ordered semiring S . Then the following statements hold:

- (i) $\Sigma \bar{A} \subseteq \overline{\Sigma A}$;
- (ii) if $A + A \subseteq A$, then $A \subseteq \bar{A}$ and $\bar{\bar{A}} = \overline{(\bar{A})} = \overline{(\bar{A})}$;
- (iii) if $A \subseteq B$, then $\bar{A} \subseteq \bar{B}$;
- (iv) $A\bar{B} \subseteq \overline{A\bar{B}}$ and $\bar{A}B \subseteq \overline{\bar{A}B}$;
- (v) $\bar{A} + \bar{B} \subseteq \overline{\bar{A} + \bar{B}}$;
- (vi) $\overline{A \cup B} \supseteq \bar{A} \cup \bar{B}$;
- (vii) $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$;
- (viii) if $A + A \subseteq A$, then $A \subseteq (A] \subseteq (\bar{A}) = \bar{A} \subseteq \overline{(\bar{A})}$.

We note that if a nonempty subset A of an ordered semiring S is closed under addition, then $(A]$, \bar{A} and $\overline{(\bar{A})}$ are also closed.

As a consequence of Remark 2.1 and Remark 2.2, we obtain the following remark.

Remark 2.3. Let A, B be nonempty subsets of an ordered semiring S such that A and B are closed under addition. Then the following statements hold:

- (i) $\Sigma(\bar{A}) = \overline{(\Sigma A)}$;
- (ii) $\overline{(\bar{A})} = \overline{(\bar{A})}$;
- (iii) $\Sigma A(\bar{B}) \subseteq \overline{(\Sigma A(\bar{B}))} \subseteq \overline{(\Sigma(\bar{A}) (\bar{B}))} \subseteq \overline{(\Sigma AB)}$ and $\overline{\Sigma(\bar{A})B} \subseteq \overline{(\Sigma(\bar{A})B)} \subseteq \overline{(\Sigma(\bar{A}) (\bar{B}))} \subseteq \overline{(\Sigma AB)}$;
- (iv) $\overline{(\bar{A}) + (\bar{B})} \subseteq \overline{(A + B)}$.

We recall the notions of some types of ordered k -ideals occurring in [9, 11–13] as follows. A nonempty subset A of an ordered semiring S is called a *left ordered*

k -ideal (resp. *right ordered k -ideal*, *ordered k -ideal*, *ordered k -bi-ideal*, *ordered k -interior ideal*) if A is a left ideal (resp. right ideal, ideal, bi-ideal, interior ideal) of S and $A = \overline{A}$. A nonempty subset Q of an ordered semiring S such that $Q+Q \subseteq Q$ is said to be an *ordered quasi k -ideal* of S if $(\overline{\Sigma SQ}) \cap (\overline{\Sigma QS}) \subseteq Q$ and $Q = \overline{Q}$.

Remark 2.4. Let S be an ordered semiring. Then the following statements hold:

- (i) every left (right) ordered k -ideal of S is an ordered quasi k -ideal of S ;
- (ii) every ordered quasi k -ideal of S is an ordered k -bi-ideal of S ;
- (iii) the intersection of a left ordered k -ideal and a right ordered k -ideal of S is an ordered quasi k -ideal of S ;
- (iv) every ordered k -ideal of S is an ordered k -interior ideal of S .

The converses of Remark 2.4(i)-(iv) are not generally true as examples in [11] for (i)-(iii) and as the following example for (iv).

Example 2.5. Let $S = \{a, b, c, d, e\}$. Define binary operations $+$ and \cdot on S by the following tables:

$+$	a	b	c	d	e
a	a	b	c	d	e
b	b	b	d	d	d
c	c	d	d	d	d
d	d	d	d	d	d
e	e	d	d	d	d

and

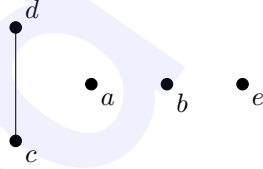
\cdot	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	a
c	a	a	a	a	a
d	a	a	a	a	a
e	a	c	a	a	a

Define a binary relation \leq on S by

$$\leq := \{(a, a), (b, b), (c, c), (d, d), (e, e), (c, d)\}.$$

We give the covering relation “ \prec ” and the figure of S :

$$\prec := \{(c, d)\}.$$



Then $(S, +, \cdot, \leq)$ is an additively commutative ordered semiring. Let $I = \{a, b\}$. Clearly, I is a subsemiring of S . We have $SIS = \{a\} \subseteq I$ and $I = \overline{I}$. Hence, I is an ordered k -interior ideal of S , but not an ordered left k -ideal of S , since $SI = \{a, c\} \not\subseteq I$.

If an ordered semiring S has an identity (i.e., $\exists e \in S, ea = a = ae, \forall a \in S$), then the converse of Remark 2.4(iv) is true as follows.

Theorem 2.6. *If an ordered semiring S has an identity, then their ordered k -ideals and ordered k -interior ideals coincide.*

Proof. Let I be an ordered k -interior ideal and e be an identity of S . Then $IS = eIS \subseteq SIS \subseteq I$ and $SI = SIE \subseteq SIS \subseteq I$. Hence, I is an ordered k -ideal of S . \square

For any nonempty subset A of an ordered semiring S , we denote $L_k(A)$, $R_k(A)$, $J_k(A)$, $Q_k(A)$ and $B_k(A)$ as the smallest left ordered k -ideal, right ordered k -ideal, ordered k -ideal, ordered quasi k -ideal and ordered k -bi-ideal of S containing A , respectively. Now we recall their constructions occurring in [11, 12] as the following lemma.

Lemma 2.7. *Let A be a nonempty subset of an ordered semiring S . Then the following statements hold:*

- (i) $L_k(A) = \overline{(\Sigma A + \Sigma SA)}$;
- (ii) $R_k(A) = \overline{(\Sigma A + \Sigma AS)}$;
- (iii) $J_k(A) = \overline{(\Sigma A + \Sigma SA + \Sigma AS + \Sigma SAS)}$;
- (iv) $Q_k(A) = \overline{(\Sigma A + ((\Sigma SA) \cap (\Sigma AS)))}$;
- (v) $B_k(A) = \overline{(\Sigma A + \Sigma A^2 + \Sigma ASA)}$.

3. Ordered k -regular semirings

Here, we recall the definition of ordered k -regular semirings and show that their ordered k -interior ideals and their ordered k -ideals coincide.

Definition. Let S be an ordered semiring.

- (i) An element $a \in S$ is called k -regular [8] if $a + axa \leq aya$ for some $x, y \in S$.
- (ii) An element $a \in S$ is called *ordered k -regular* [12] if $a \in \overline{(aSa)}$.

An ordered semiring S is said to be k -regular (resp. *ordered k -regular*) if every element $a \in S$ is k -regular (resp. *ordered k -regular*).

We note that every k -regular ordered semiring is an ordered k -regular semiring but the converse is not true (see Example 3.1 in [12]).

In ordered k -regular semirings, the converse of Remark 2.4(iv) is true as the following theorem shows.

Theorem 3.1. *Let S be an ordered semiring. If S is ordered k -regular, then ordered k -ideals and ordered k -interior ideals coincide in S .*

Proof. Let I be an ordered k -interior ideal of S . If $x \in IS$, then $x \in \overline{(xSx)} \subseteq \overline{(ISSIS)} \subseteq \overline{(ISI)} \subseteq \overline{(I)} = I$. Similarly, we can show that $SI \subseteq I$. Therefore, I is an ordered k -ideal of S . \square

Now, we recall some properties of ordered k -regular semirings that will be used in the later sections.

Lemma 3.2 ([12]). *Let S be an ordered semiring. Then S is ordered k -regular if and only if $R \cap L = \overline{(RL)}$ for every right ordered k -ideal R and left ordered k -ideal L of S .*

Corollary 3.3 ([11]). *Let S be an ordered semiring. Then S is ordered k -regular if and only if $A \subseteq \overline{(\Sigma R_k(A)L_k(A))}$ for each $A \subseteq S$.*

4. Ordered intra k -regular semirings

In [2], Ahsan, Mordeson and Shabir gave a general definition of intra-regular semirings, namely a semiring S is said to be *intra-regular* if every element a of S , $a = \sum_{i=1}^n x_i a^2 y_i$ for some $x_i, y_i \in S$ and for some $n \in \mathbb{N}$. In this section, we introduce the notion of ordered intra k -regular semirings as a generalization of the intra-regular semiring in sense of Ahsan, Mordeson and Shabir, show that their ordered k -ideals and ordered k -interior ideals coincide and characterize them using their ordered k -ideals.

Definition. Let S be an ordered semiring.

- (i) An element $a \in S$ is called *intra k -regular* if $a + \sum_{i=1}^n x_i a^2 y_i \leq \sum_{i=1}^m x'_i a^2 y'_i$ for some $x_i, y_i, x'_i, y'_i \in S, n, m \in \mathbb{N}$.
- (ii) An element $a \in S$ is called *ordered intra k -regular* if $a \in \overline{(\Sigma S a^2 S)}$.

If every $a \in S$ is intra k -regular (resp. ordered intra k -regular), then we call S an *intra k -regular ordered semiring* (resp. *ordered intra k -regular semiring*).

Lemma 4.1. *Let S be an ordered semiring. Then S is ordered intra k -regular if and only if $A \subseteq \overline{(\Sigma S A^2 S)}$ for every $A \subseteq S$.*

It is easy to check that if an ordered semiring S is intra k -regular, then S is ordered intra k -regular, but the converse is not true that is the concept of ordered intra k -regular semirings is a generalization of the concept of intra k -regular semirings. The following example shows that there exists an ordered intra k -regular semiring which is not intra k -regular.

Example 4.2. Let $S = \{a, b, c\}$. Define binary operations $+$ and \cdot on S by the following tables:

$+$	a	b	c	and	\cdot	a	b	c
	a	a	a			a	b	b
	b	a	b			b	b	b
	c	a	c			c	b	b

Define a binary relation \leq on S by

$$\leq := \{(a, a), (b, b), (c, c), (a, b), (a, c), (b, c)\}.$$

We give the covering relation “ \prec ” and the figure of S :

$$\prec := \{(a, b), (b, c)\}.$$



Then $(S, +, \cdot, \leq)$ is an additively commutative ordered semiring. We have $x \in \overline{(\Sigma Sx^2S)}$ for all $x \in S$. This means S is ordered intra k -regular. However, S is not intra k -regular, since the inequality $c + \sum_{i=1}^n x_i c^2 y_i \leq \sum_{i=1}^n x'_i c^2 y'_i$ has no a solution.

The converse of Remark 2.4(iv) is true in ordered intra k -regular semirings as the following theorem.

Theorem 4.3. *Let S be an ordered semiring. If S is ordered intra k -regular, then ordered k -ideals and ordered k -interior ideals coincide in S .*

Proof. Let I be an ordered k -interior ideal of S . If $x \in SI$, then

$$x \in \overline{(\Sigma Sx^2S)} \subseteq \overline{(\Sigma SSSISIS)} \subseteq \overline{(\Sigma SIIIS)} \subseteq \overline{(\Sigma SIS)} \subseteq \overline{(\Sigma I)} = I.$$

Similarly, we can show that $IS \subseteq I$. Therefore, I is an ordered k -ideal of S . \square

Now, we give some characterizations of ordered intra k -regular semirings by their ordered k -ideals.

Theorem 4.4. *An ordered semiring S is ordered intra k -regular if and only if $L \cap R \subseteq \overline{(\Sigma LR)}$ for every left ordered k -ideal L and right ordered k -ideal R of S .*

Proof. Assume that S is ordered intra k -regular. Let L and R be a left and a right ordered k -ideal of S , respectively. If $x \in L \cap R$, then $x \in \overline{(\Sigma Sx^2S)} \subseteq \overline{(\Sigma SLRS)} \subseteq \overline{(\Sigma LR)}$.

Conversely, let $A \subseteq S$. By assumption and using Remark 2.3(iii) and Lemma 2.7, we obtain

$$\begin{aligned} A \subseteq L_k(A) \cap R_k(A) &\subseteq \overline{(\Sigma L_k(A)R_k(A))} \\ &= \overline{(\Sigma(\Sigma A + \Sigma SA) (\Sigma A + \Sigma AS))} \\ &\subseteq \overline{(\Sigma(\Sigma A + \Sigma SA)(\Sigma A + \Sigma AS))} \\ (1) \quad &\subseteq \overline{(\Sigma A^2 + \Sigma A^2S + \Sigma SA^2 + \Sigma SA^2S)}. \end{aligned}$$

Using (1) and Remark 2.3(iii), we have

$$\begin{aligned} \Sigma A^2 = \Sigma AA &\subseteq \Sigma A \overline{(\Sigma A^2 + \Sigma A^2S + \Sigma SA^2 + \Sigma SA^2S)} \\ &\subseteq \overline{(\Sigma A^3 + \Sigma A^3S + \Sigma ASA^2 + \Sigma ASA^2S)} \\ (2) \quad &\subseteq \overline{(\Sigma SA^2 + \Sigma SA^2S)}. \end{aligned}$$

Using (1) and Remark 2.3(iii) again, we have

$$\begin{aligned}
 \Sigma A^2 &= \Sigma AA \subseteq \Sigma(\overline{\Sigma A^2 + \Sigma A^2 S + \Sigma S A^2 + \Sigma S A^2 S})A \\
 &\subseteq \overline{(\Sigma A^3 + \Sigma A^2 S A + \Sigma S A^3 + \Sigma S A^2 S A)} \\
 (3) \quad &\subseteq \overline{(\Sigma A^2 S + \Sigma S A^2 S)}.
 \end{aligned}$$

Using (2) and Remark 2.3(iii), we have

$$(4) \quad \Sigma A^2 S \subseteq \Sigma(\overline{\Sigma S A^2 + \Sigma S A^2 S})S \subseteq \overline{(\Sigma(\overline{\Sigma S A^2 S}))} = \overline{(\Sigma S A^2 S)}.$$

Using (3) and Remark 2.3(iii), we have

$$(5) \quad \Sigma S A^2 \subseteq \Sigma S(\overline{\Sigma A^2 S + \Sigma S A^2 S}) \subseteq \overline{(\Sigma(\overline{\Sigma S A^2 S}))} = \overline{(\Sigma S A^2 S)}.$$

Using (3), (5) and Remark 2.3(iv), we have

$$(6) \quad \Sigma A^2 \subseteq \overline{(\Sigma A^2 S + \Sigma S A^2 S)} \subseteq \overline{((\overline{\Sigma S A^2 S}) + \Sigma S A^2 S)} \subseteq \overline{(\Sigma S A^2 S)}.$$

By (1), (4), (5), (6) and using Remark 2.3(iv), we obtain

$$\begin{aligned}
 A &\subseteq \overline{(\Sigma A^2 + \Sigma A^2 S + \Sigma S A^2 + \Sigma S A^2 S)} \\
 &\subseteq \overline{((\overline{\Sigma S A^2 S}) + (\overline{\Sigma S A^2 S}) + (\overline{\Sigma S A^2 S}) + \Sigma S A^2 S)} \\
 &\subseteq \overline{(\Sigma S A^2 S)}.
 \end{aligned}$$

By Lemma 4.1, S is ordered intra k -regular. \square

Corollary 4.5. *An ordered semiring S is ordered intra k -regular if and only if $A \subseteq \overline{(\Sigma L_k(A)R_k(A))}$ for each $A \subseteq S$.*

Theorem 4.6. *Let S be an ordered semiring. Then the following statements are equivalent:*

- (i) S is ordered intra k -regular;
- (ii) $L \cap B \subseteq \overline{(\Sigma L B S)}$ for every left ordered k -ideal L and ordered k -bi-ideal B of S ;
- (iii) $L \cap Q \subseteq \overline{(\Sigma L Q S)}$ for every left ordered k -ideal L and ordered quasi k -ideal Q of S ;
- (iv) $B \cap R \subseteq \overline{(\Sigma S B R)}$ for every right ordered k -ideal R and ordered k -bi-ideal B of S ;
- (v) $Q \cap R \subseteq \overline{(\Sigma S Q R)}$ for every right ordered k -ideal R and ordered quasi k -ideal Q of S .

Proof. (i) \Rightarrow (ii): Let L and B be a left ordered k -ideal and an ordered k -bi-ideal of S , respectively. If $x \in L \cap B$, then $x \in \overline{(\Sigma S x^2 S)} \subseteq \overline{(\Sigma S L B S)} \subseteq \overline{(\Sigma L B S)}$.

(ii) \Rightarrow (iii): It follows from Remark 2.4(ii).

(iii) \Rightarrow (i): Assume that (iii) holds. Let $A \subseteq S$. By assumption, we obtain

$$A \subseteq L_k(A) \cap Q_k(A) \subseteq \overline{(\Sigma L_k(A)Q_k(A)S)} \subseteq \overline{(\Sigma L_k(A)R_k(A)S)} \subseteq \overline{(\Sigma L_k(A)R_k(A))}.$$

By Corollary 4.5, S is ordered intra k -regular.

(i) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (i) can be prove in analogous way of (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i). \square

Theorem 4.7. *Let S be an ordered semiring. Then the following statements are equivalent:*

- (i) S is ordered intra k -regular;
- (ii) $B \cap Q \subseteq \overline{(\Sigma SBQS)}$ for every ordered k -bi-ideal B and ordered quasi k -ideal Q of S ;
- (iii) $B \cap Q \subseteq \overline{(\Sigma SQBS)}$ for every ordered k -bi-ideal B and ordered quasi k -ideal Q of S .

Proof. (i) \Leftrightarrow (ii): Assume that S is ordered intra k -regular. Let B and Q be an ordered k -bi-ideal and an ordered quasi k -ideal of S , respectively. If $x \in B \cap Q$, then $x \in \overline{(\Sigma Sx^2S)} \subseteq \overline{(\Sigma SBQS)}$.

Conversely, assume that (ii) holds. Let $A \subseteq S$. By assumption, we obtain

$$\begin{aligned} A \subseteq B_k(A) \cap Q_k(A) &\subseteq \overline{(\Sigma SB_k(A)Q_k(A)S)} \\ &\subseteq \overline{(\Sigma SL_k(A)R_k(A)S)} \\ &\subseteq \overline{(\Sigma L_k(A)R_k(A))}. \end{aligned}$$

By Corollary 4.5, S is ordered intra k -regular.

(i) \Leftrightarrow (iii) can be prove similar to (i) \Leftrightarrow (ii). □

5. Ordered k -regular and ordered intra k -regular semirings

In this section, we give some characterizations of an ordered semiring which is both ordered k -regular and ordered intra k -regular and show that its ordered k -bi-ideals and ordered quasi k -ideals are ordered k -idempotent.

First, we give an example of an ordered semiring which is both ordered k -regular and ordered intra k -regular.

Example 5.1. Let $S = \{a, b, c, d\}$. Define binary operations $+$ and \cdot by the following tables:

$+$	a	b	c	d
a	a	b	a	d
b	b	b	b	b
c	a	b	c	d
d	d	b	d	d

and

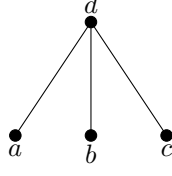
\cdot	a	b	c	d
a	a	d	a	d
b	a	b	a	d
c	a	d	a	d
d	a	d	a	d

Define a binary relation \leq on S by

$$\leq := \{(a, a), (b, b), (c, c), (d, d), (a, d), (b, d), (c, d)\}.$$

We give the covering relation “ \prec ” and the figure of S :

$$\prec := \{(a, d), (b, d), (c, d)\}.$$



Then $(S, +, \cdot, \leq)$ is an additively commutative ordered semiring. Clearly, a, b, d are ordered k -regular and ordered intra k -regular. We consider $c \in \overline{(cSc)} = \overline{\{a\}} = \{a, c\}$ and $c \in \overline{(\Sigma Sc^2 S)} = S$. Therefore, S is ordered k -regular and ordered intra k -regular.

Lemma 5.2. *Let S be an ordered semiring. Then the following statements are equivalent:*

- (i) S is ordered k -regular and ordered intra k -regular;
- (ii) $A \subseteq \overline{(\Sigma ASA^2 SA)}$ for each $A \subseteq S$;
- (iii) $a \in \overline{(aSa^2 Sa)}$ for each $a \in S$.

Proof. (i) \Rightarrow (ii): Assume that (i) holds. Let $A \subseteq S$. Then $A \subseteq \overline{(ASA)}$ and $A \subseteq \overline{(\Sigma SA^2 S)}$. By Remark 2.3(iii), it follows that

$$\begin{aligned} A \subseteq \overline{(\Sigma ASA)} &\subseteq \overline{(\Sigma AS(\Sigma ASA))} \subseteq \overline{(\Sigma ASASA)} \subseteq \overline{(\Sigma AS(\Sigma SA^2 S)SA)} \\ &\subseteq \overline{(\Sigma ASSA^2 SSA)} \subseteq \overline{(\Sigma ASA^2 SA)}. \end{aligned}$$

(ii) \Rightarrow (iii) and (iii) \Rightarrow (i) are obvious. \square

Theorem 5.3. *Let S be an ordered semiring. Then S is ordered k -regular and ordered intra k -regular if and only if $R \cap L = \overline{((RL)^2)}$ for every right ordered k -ideal R and left ordered k -ideal L of S .*

Proof. Assume that S is ordered k -regular and ordered intra k -regular. Let R and L be a right and a left ordered k -ideal of S , respectively. Then

$$\overline{((RL)^2)} \subseteq \overline{((RS)^2)} \subseteq \overline{(R^2)} \subseteq \overline{(R)} = R \text{ and } \overline{((RL)^2)} \subseteq \overline{((SL)^2)} \subseteq \overline{(L^2)} \subseteq \overline{(L)} = L.$$

Thus $\overline{((RL)^2)} \subseteq R \cap L$. On the other hand, let $x \in R \cap L$. By Lemma 5.2, we get $x \in \overline{(xSx^2Sx)} \subseteq \overline{(RSLRSL)} \subseteq \overline{(RLRL)} = \overline{((RL)^2)}$. Hence, $\overline{((RL)^2)} = R \cap L$.

Conversely, assume that $\overline{((RL)^2)} = R \cap L$ for every right ordered k -ideal R and left ordered k -ideal L of S . Then $R \cap L = \overline{(RLRL)} \subseteq \overline{(RL)} \subseteq R \cap L$, i.e., $R \cap L = \overline{(RL)}$ and $R \cap L = \overline{(RLRL)} \subseteq \overline{(LR)} \subseteq \overline{(\Sigma LR)}$. By Lemma 3.2 and Theorem 4.4, we obtain that S is ordered k -regular and ordered intra k -regular. \square

Theorem 5.4. *Let S be an ordered semiring. Then the following statements are equivalent:*

- (i) S is ordered k -regular and ordered intra k -regular;

- (ii) $B \cap L \cap R \subseteq \overline{(BLRB]}$ for every ordered k -bi-ideal B , left ordered k -ideal L and right ordered k -ideal R of S ;
 (iii) $Q \cap L \cap R \subseteq \overline{(QLRQ]}$ for every ordered quasi k -ideal Q , left ordered k -ideal L and right ordered k -ideal R of S .

Proof. (i) \Rightarrow (ii): Assume that (i) holds. Let B , L and R be ordered k -bi-ideal, left ordered k -ideal and right ordered k -ideal of S , respectively. Let $x \in B \cap L \cap R$. By Lemma 5.2, we get $x \in \overline{(xSx^2Sx]} \subseteq \overline{(BSLRSB]} \subseteq \overline{(BLRB]}$.

(ii) \Rightarrow (iii): It follows from Remark 2.4(ii).

(iii) \Rightarrow (i): Assume that (iii) holds. Let R and L be a right and a left ordered k -ideal of S , respectively. By Remark 2.4(iii), we have that $R \cap L$ is an ordered quasi k -ideal of S . By assumption, we obtain

$$\begin{aligned} R \cap L &= (R \cap L) \cap L \cap R \subseteq \overline{((R \cap L)LR(R \cap L))} \\ &\subseteq \overline{(RLRL]} = \overline{((RL)^2]} \subseteq R \cap L. \end{aligned}$$

Thus, $R \cap L = \overline{((RL)^2]}$. By Theorem 5.3, we have that S is ordered k -regular and ordered intra k -regular. \square

Let I be an ordered k -bi-ideal (ordered quasi k -ideal) of an ordered semiring S . Then we call I *ordered k -idempotent* if $I = \overline{(\Sigma I^2]}$.

Theorem 5.5. *Let S be an ordered semiring. Then the following statements are equivalent:*

- (i) S is ordered k -regular and ordered intra k -regular;
 (ii) every ordered k -bi-ideal of S is ordered k -idempotent;
 (iii) every ordered quasi k -ideal of S is ordered k -idempotent.

Proof. (i) \Rightarrow (ii): Assume that (i) holds. Clearly, $\overline{(\Sigma B^2]} \subseteq \overline{(\Sigma B]} = B$. Let $x \in B$. Using Lemma 5.2, we obtain $x \in \overline{(xSx^2Sx]} \subseteq \overline{(BSBBSB]} \subseteq \overline{(BB]} = \overline{(B^2]} \subseteq \overline{(\Sigma B^2]}$. Now, $B = \overline{(\Sigma B^2]}$.

(ii) \Rightarrow (iii): It follows from Remark 2.4(ii).

(iii) \Rightarrow (i): Assume that (iii) holds. Let $A \subseteq S$. By assumption, we obtain

$$\begin{aligned} A &\subseteq Q_k(A) = \overline{(\Sigma Q_k(A)Q_k(A))} \subseteq \overline{(\Sigma R_k(A)L_k(A))} \text{ and} \\ A &\subseteq Q_k(A) = \overline{(\Sigma Q_k(A)Q_k(A))} \subseteq \overline{(\Sigma L_k(A)R_k(A))}. \end{aligned}$$

By Corollary 3.3 and 4.5, we have that S is ordered k -regular and ordered intra k -regular. \square

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