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CERTAIN CURVATURE CONDITIONS IN KENMOTSU MANIFOLDS WITH RESPECT TO THE SEMI-SYMMETRIC METRIC CONNECTION

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Abstract. The conharmonic curvature tensor under certain conditions has been studied for Kenmotsu manifolds with respect to the semi-symmetric metric connection.

1. Introduction

In 1969, S. Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension [17]. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, say c. He showed that they can be divided into three classes: (1) homogeneous normal contact Riemannian manifolds with $c > 0$, (2) global Riemannian products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature if $c = 0$ and (3) a warped product space $R \times_f C$ if $c > 0$. It is known that the manifolds of class (1) are characterized by admitting a Sasakian structure. K. Kenmotsu [13] characterized the differential geometric properties of the manifolds of class (3); the structure obtained in this way is now known as Kenmotsu structure. In general, these structures are not Sasakian [13]. Kenmotsu manifolds have been studied by various authors such as N. Asghari and A. Taleshian [3], A. Barman and U. C. De [5], U. C. De and G. Pathak [7], A. Yildiz and U. C. De [20] and many others.

In 1924, the idea of semi-symmetric linear connection on a differentiable manifold was introduced by A. Friedmann and J. A. Schouten [8]. Then in 1932, H. A. Hayden [10] introduced semi-symmetric metric connection in Riemannian manifolds and this was studied systematically by K. Yano [18].

The paper is organized as follows: Section 2 is equipped with some prerequisites about Kenmotsu manifolds. In Section 3, we give the relation between the curvature tensor of Kenmotsu manifolds and related results with respect to the semi-symmetric metric connection and the Levi-Civita connection. In Section

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4, we showed that a conharmonically flat η-Einstein Kenmotsu manifold with respect to the semi-symmetric metric connection is of quasi-constant curvature. Section 5 deals with the study of ϕ-conharmonically semi-symmetric η-Einstein Kenmotsu manifolds with respect to the semi-symmetric metric connection. In Section 6, we study conharmonically flat Kenmotsu manifolds satisfying the curvature condition $\bar{R}(X,Y)\cdot\bar{S}=0$. In the last Section 7, we obtain the non-existence of Kenmotsu manifolds whose curvature tensor of manifold is co-variantly constant with respect to the semi-symmetric metric connection and the manifold is recurrent with respect to the Levi-Civita connection.

2. Kenmotsu manifolds

A smooth manifold (M^n,g) $(n=2m+1)$ is said to be an almost contact metric manifold [13] if it admits a $(1,1)$-tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g which satisfy

$$\phi^2X=-X+\eta(X)\xi, \tag{1}$$

$$g(X,\xi)=\eta(X),\quad \eta(\xi)=1,\quad \phi\xi=0,\quad \eta(\phi X)=0, \tag{2}$$

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y) \tag{3}$$

for all vector fields X,Y on M.

An almost contact metric manifold $M^n(\phi,\xi,\eta,g)$ is said to be a Kenmotsu manifold if the following conditions hold:

$$(\nabla_X\phi)(Y)=g(\phi X,Y)\xi-\eta(Y)\phi X, \tag{4}$$

$$\nabla_X\xi=X-\eta(X)\xi, \tag{5}$$

where ∇ is the Levi-Civita connection.

In a Kenmotsu manifold the following relations hold [12,13,15]:

$$(\nabla_X\eta)Y=g(X,Y)-\eta(X)\eta(Y), \tag{6}$$

$$R(X,Y)\xi=\eta(X)Y-\eta(Y)X, \tag{7}$$

$$R(\xi,X)Y=-R(X,\xi)Y=\eta(Y)X-g(X,Y)\xi, \tag{8}$$

$$\eta(R(X,Y)Z)=g(X,Z)\eta(Y)-g(Y,Z)\eta(X), \tag{9}$$

$$S(X,\xi)=-(n-1)\eta(X),\quad Q\xi=-(n-1)\xi \tag{10}$$

for arbitrary vector fields X,Y,Z on M and R is the Riemannian curvature tensor and S is the Ricci tensor of type (0,2) such that $g(QX,Y)=S(X,Y)$.

A linear connection $\bar{\nabla}$ in a Riemannian manifold M is said to be a semi-symmetric connection [9,14] if its torsion tensor T of the connection $\bar{\nabla}$

$$T(X,Y)=\bar{\nabla}_XY-\bar{\nabla}_YX-[X,Y] \tag{11}$$

satisfies

$$T(X,Y) = \eta(Y)X - \eta(X)Y, \tag{12}$$

where η is a 1-form and ξ is a vector field given by

$$g(X,\xi) = \eta(X) \tag{13}$$

for all vector fields $X, Y \in \chi(M)$. Here $\chi(M)$ is the set of all differentiable vector fields on M.

A semi-symmetric connection $\bar{\nabla}$ is called a semi-symmetric metric connection [6, 8] if it further satisfies

$$\bar{\nabla} g = 0. \tag{14}$$

A relation between the semi-symmetric metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ on M has been obtained by K. Yano [18], which is given by

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi, \tag{15}$$

where $\eta(Y) = g(Y,\xi)$.

3. Curvature tensor of a Kenmotsu manifold with respect to the semi-symmetric metric connection

Let R and $\bar{R}$, respectively, be the curvature tensors of the Levi-Civita connection ∇ and the semi-symmetric metric connection $\bar{\nabla}$ in a Kenmotsu manifold M. Then we have [4]

$$\begin{aligned} \bar{R}(X,Y)Z = {} & R(X,Y)Z - 3g(Y,Z)X + 3g(X,Z)Y + 2\eta(Y)\eta(Z)X \\ & - 2\eta(X)\eta(Z)Y + 2g(Y,Z)\eta(X)\xi - 2g(X,Z)\eta(Y)\xi, \end{aligned} \tag{16}$$

$$\eta(\bar{R}(X,Y)Z) = 2g(X,Z)\eta(Y) - 2g(Y,Z)\eta(X), \tag{17}$$

$$\bar{R}(X,Y)\xi = 2\eta(X)Y - 2\eta(Y)X, \tag{18}$$

$$\bar{R}(X,\xi)Y = 2g(X,Y)\xi - 2\eta(Y)X, \tag{19}$$

$$\bar{R}(\xi,X)\xi = 2X - 2\eta(X)\xi, \tag{20}$$

$$\bar{S}(Y,Z) = S(Y,Z) - (3n-5)g(Y,Z) + 2(n-2)\eta(Y)\eta(Z), \tag{21}$$

$$\bar{S}(\phi Y,\phi Z) = \bar{S}(Y,Z) + 2(n-1)\eta(Y)\eta(Z), \tag{22}$$

$$\bar{S}(Y,\xi) = -2(n-1)\eta(Y), \ \ \bar{S}(\xi,\xi) = -2(n-1), \tag{23}$$

$$\bar{Q}Y = QY - (3n-5)Y + 2(n-2)\eta(Y)\xi, \tag{24}$$

$$\bar{Q}\xi = -2(n-1)\xi, \tag{25}$$

$$\bar{r} = r - 3n^2 + 7n - 4, \tag{26}$$

$$\bar{\nabla}_X \xi = 2X - 2\eta(X)\xi \tag{27}$$

for any vector fields X, Y, Z on M.

Definition. A Kenmotsu manifold M is called a manifold of quasi-constant curvature with respect to the semi-symmetric metric connection if the curvature tensor $\bar{R}$ of type $(0,4)$ satisfies the condition

$$\begin{aligned}\bar{R}(X,Y,Z,W) = {} & a[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ & + b[g(X,W)T(Y)T(Z) - g(X,Z)T(Y)T(W) \\ & + g(Y,Z)T(X)T(W) - g(Y,W)T(X)T(Z)],\end{aligned} \tag{28}$$

where $\bar{R}(X,Y,Z,W) = g(\bar{R}(X,Y)Z,W)$, $\bar{R}$ is the curvature tensor of type $(1,3)$; a, b are the scalar functions and ρ is a unit vector field defined by

$$g(X,\rho) = T(X) \tag{29}$$

for any vector fields X, Y, Z, W on M.

Definition. A Kenmotsu manifold M is said to be an η-Einstein manifold with respect to the semi-symmetric metric connection if its Ricci tensor $\bar{S}$ of type $(0,2)$ satisfies

$$\bar{S}(X,Y) = \alpha g(X,Y) + \beta\eta(X)\eta(Y), \tag{30}$$

where α and β are smooth functions on M. In particular, if $\beta = 0$, then an η-Einstein manifold is an Einstein manifold.

Contracting (30), we have

$$\bar{r} = n\alpha + \beta. \tag{31}$$

On the other hand, putting $X = Y = \xi$ in (30) and using (23), we also have

$$-2(n-1) = \alpha + \beta. \tag{32}$$

Hence it follows from (31) and (32) that

$$\alpha = 2 + \frac{\bar{r}}{n-1}, \ \beta = -2n - \frac{\bar{r}}{n-1}.$$

So the Ricci tensor $\bar{S}$ of an η-Einstein Kenmotsu manifold with respect to the semi-symmetric metric connection is given by

$$\bar{S}(X,Y) = (2 + \frac{\bar{r}}{n-1})g(X,Y) - (2n + \frac{\bar{r}}{n-1})\eta(X)\eta(Y), \tag{33}$$

from which we have

$$\bar{Q}X = (2 + \frac{\bar{r}}{n-1})X - (2n + \frac{\bar{r}}{n-1})\eta(X)\xi. \tag{34}$$

Now, we give an example of Kenmotsu manifold which is an η-Einstein manifold with respect to the semi-symmetric metric connection.

Example. We consider a 7-dimensional Kenmotsu manifold with respect to the semi-symmetric metric connection $M^7=\{(x_1,x_2,x_3,y_1,y_2,y_3,z)\in\mathbb{R}^7 : z>0\}$, where $(x_1,x_2,x_3,y_1,y_2,y_3,z)$ are the standard coordinates in $\mathbb{R}^7$. We choose the vector fields

$$e_1 = e^{-z}\frac{\partial}{\partial x_1},\quad e_2 = e^{-z}\frac{\partial}{\partial x_2},\quad e_3 = e^{-z}\frac{\partial}{\partial x_3},\quad e_4 = e^{-z}\frac{\partial}{\partial y_1},$$
$$e_5 = e^{-z}\frac{\partial}{\partial y_2},\quad e_6 = e^{-z}\frac{\partial}{\partial y_3},\quad e_7 = \frac{\partial}{\partial z}$$

which are linearly independent at each point of M^7. Let g be the Riemannian metric on M^7 defined by

$$g = \sum_{i=1}^{3} e^{2z}(dx_i\otimes dx_i + dy_i\otimes dy_i) + dz\otimes dz.$$

Here it is clear that $g(e_i,e_i)=1$ and $g(e_i,e_j)=0$ for all $i\neq j$, where $i,j=1,2,3,4,5,6,7$. Hence $\{e_1,e_2,e_3,e_4,e_5,e_6,e_7\}$ is an orthonormal base field on M^7.

Let $X=\sum_{i=1}^{3}(X_i\frac{\partial}{\partial x_i}+Y_i\frac{\partial}{\partial y_i})+Z\frac{\partial}{\partial z}$ be a vector field on M^7. We define a $(1,1)$ tensor field ϕ and 1-form η as

$$\phi\left(\sum_{i=1}^{3}(X_i\frac{\partial}{\partial x_i}+Y_i\frac{\partial}{\partial y_i})+Z\frac{\partial}{\partial z}\right)=\sum_{i=1}^{3}(Y_i\frac{\partial}{\partial x_i}-X_i\frac{\partial}{\partial y_i})$$

and $\eta(X)=g(X,e_7)=dz(X)$. Thus we have

$$\phi(e_1)=e_4,\ \phi(e_2)=e_5,\ \phi(e_3)=e_6,\ \phi(e_4)=-e_1,$$
$$\phi(e_5)=-e_2,\ \phi(e_6)=-e_3,\ \phi(e_7)=0.$$

The linearity property of ϕ and g yields that

$$\eta(e_7)=g(e_7,e_7)=1,\phi^2X=-X+\eta(X)e_7,$$
$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y)$$

for any vector fields X,Y on M^7. Let us denote e_7 by ξ, then $M^7(\phi,\xi,\eta,g)$ defines an almost contact metric manifold.

The Koszul's formula for the Riemannian connection ∇ of the Riemannian metric g is given by

$$\begin{aligned}2g(\nabla_X Y,Z) &= Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)\\ &\quad + g([X,Y],Z)-g([Y,Z],X)+g([Z,X],Y)\end{aligned}$$

for any vector fields X,Y,Z on M^7.

Using the Koszul's formula, for any vector fields X,Y,Z on M^7, we have

$$(\nabla_X\phi)Y = g(\phi X,Y)\xi-\eta(Y)\phi X$$

which proves that $M^7(\phi,\xi,\eta,g)$ is a Kenmotsu manifold. Moreover, we have

$$[e_i,\xi]=e_i,\ [e_i,e_j]=0,\ i\neq j,\ i,j=1,2,3,4,5,6.$$

Using the Koszul's formula, we also obtain

$$\nabla_{e_i} e_i = -\xi,\ \nabla_{e_i} e_j = 0 \text{ for } i \neq j, \tag{35}$$
$$\nabla_{e_i}\xi = e_i,\ \nabla_\xi e_i = 0,\ i,j = 1,2,3,4,5,6.$$

Therefore the semi-symmetric metric connection $\bar{\nabla}$ on M^7 is given by

$$\bar{\nabla}_{e_i} e_i = -2\xi,\ \bar{\nabla}_{e_i} e_j = 0 \text{ for } i \neq j, \tag{36}$$
$$\bar{\nabla}_{e_i}\xi = 2e_i,\ \bar{\nabla}_\xi e_i = 0,\ i,j = 1,2,3,4,5,6.$$

With help of results given in (35), it can be easily verified that

$$R(e_i,e_j)e_k = 0 \text{ for } i \neq j \neq k,\ R(e_i,e_j)e_i = e_j \text{ for } i \neq j, \tag{37}$$
$$R(e_i,e_j)e_j = -e_i \text{ for} i \neq j,\ R(e_i,e_i)e_j = 0 \text{ for } i \neq j,$$
$$R(e_i,\xi)e_j = 0 \text{ for } i \neq j,\ R(\xi,e_j)\xi = e_j,\ R(e_i,\xi)e_i = \xi,\ R(\xi,e_i)e_i = -\xi,$$
$$R(e_i,\xi)\xi = -e_i,\ R(e_i,e_j)\xi = 0 \text{ for } i \neq j,\ i,j,k = 1,2,3,4,5,6.$$

From above results in (37) and the equations (16), (18)-(20), we have

$$\bar{R}(e_i,e_j)e_k = 0 \text{ for } i \neq j \neq k,\ \bar{R}(e_i,e_j)e_i = 4e_j \text{ for } i \neq j, \tag{38}$$
$$\bar{R}(e_i,e_j)e_j = -4e_i \text{ for } i \neq j,\ \bar{R}(e_i,e_i)e_j = 0 \text{ for } i \neq j,$$
$$\bar{R}(e_i,\xi)e_j = 0 \text{ for } i \neq j,\ \bar{R}(\xi,e_i)\xi = 2e_i,\ \bar{R}(e_i,\xi)e_i = 2\xi,$$
$$\bar{R}(\xi,e_i)e_i = -2\xi,\ \bar{R}(e_i,\xi)\xi = -2e_i,$$
$$\bar{R}(e_i,e_j)\xi = 0 \text{ for } i \neq j,\ i,j,k = 1,2,3,4,5,6.$$

We know that

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

Let

$$X = \sum_{i=1}^{7} X_i e_i,\ Y = \sum_{i=1}^{7} Y_i e_i, \text{ and } Z = \sum_{i=1}^{7} Z_i e_i, \text{ where } e_7 = \xi.$$

Then using (37), we get

$$R(X,Y)Z = -[g(Y,Z)X - g(X,Z)Y]. \tag{39}$$

So, M^7 is a manifold of constant curvature -1. Contracting X in (39), we get

$$S(Y,Z) = -(n-1)g(Y,Z). \tag{40}$$

For the manifold M^7, we have

$$S(Y,Z) = -6g(Y,Z) \tag{41}$$

which yields

$$QY = -6Y. \tag{42}$$

For the manifold M^7 with respect to the semi-symmetric metric connection, we also have

$$\bar{R}(X,Y)Z = -4[g(Y,Z)X - g(X,Z)Y] + 2\eta(Y)\eta(Z)X - 2\eta(X)\eta(Z)Y \tag{43}$$
$$+2g(Y,Z)\eta(X)\xi - 2g(X,Z)\eta(Y)\xi$$

and

$$\bar{S}(Y,Z) = -22g(Y,Z) + 10\eta(Y)\eta(Z). \tag{44}$$

Thus the manifold is an η-Einstein manifold with respect to the semi-symmetric metric connection.

4. Conharmonically flat η-Einstein Kenmotsu manifolds with respect to the semi-symmetric metric connection

Definition. The conharmonic curvature tensor C of type $(1,3)$ in a Riemannian manifold M of dimension n is defined by [2, 11]

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[&S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY] \end{aligned} \tag{45}$$

for any vector fields $X, Y, Z \in \chi(M)$. In the differential geometry of certain F-structures (for example, complex, almost complex, Kahler, almost Kahler, Hermitian, almost Hermitian structures, etc.), the importance of conharmonic curvature tensor is very well known [19]. While the relativistic significance of this tensor has been explored by Z. Ahsan [1] and S. A. Siddiqui and Z. Ahsan [16].

Analogous to the above definition, we define the conharmonic curvature tensor $\bar{C}$ in a Kenmotsu manifold M with respect to the semi-symmetric metric connection $\bar{\nabla}$ by

$$\begin{aligned} \bar{C}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{(n-2)}[&\bar{S}(Y,Z)X - \bar{S}(X,Z)Y \\ &+ g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y], \end{aligned} \tag{46}$$

where $\bar{R}$, $\bar{S}$ are the curvature tensor, the Ricci tensor respectively on M with respect to the semi-symmetric metric connection and $\bar{Q}$ is the Ricci operator with respect to the semi-symmetric metric connection and is related by $g(\bar{Q}X,Y) = \bar{S}(X,Y)$. If $\bar{C}$ vanishes identically on M, then we say that the manifold is conharmonically flat.

Now, we consider that the manifold M with respect to the semi-symmetric metric connection is conharmonically flat, that is, $\bar{C} = 0$. Then from (46), we have

$$\bar{R}(X,Y)Z = \frac{1}{(n-2)}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y + g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y]. \tag{47}$$

Taking inner product of (47) with W, we have

$$\begin{aligned} g(\bar{R}(X,Y)Z,W) = \frac{1}{(n-2)}[&\bar{S}(Y,Z)g(X,W) - \bar{S}(X,Z)g(Y,W) \\ &+ g(Y,Z)\bar{S}(X,W) - g(X,Z)\bar{S}(Y,W)]. \end{aligned} \tag{48}$$

By using (33), (48) becomes

$$
\begin{aligned}
g(\bar{R}(X,Y)Z,W) = \frac{1}{(n-2)}[(4+\frac{2\bar{r}}{n-1})g(Y,Z)g(X,W) \\
-(2n+\frac{\bar{r}}{n-1})\eta(Y)\eta(Z)g(X,W) \\
-(4+\frac{2\bar{r}}{n-1})g(X,Z)g(Y,W) \\
+(2n+\frac{\bar{r}}{n-1})\eta(X)\eta(Z)g(Y,W) \\
-(2n+\frac{\bar{r}}{n-1})\eta(X)\eta(W)g(Y,Z) \\
+(2n+\frac{\bar{r}}{n-1})\eta(Y)\eta(W)g(X,Z)]
\end{aligned} \tag{49}
$$

or,

$$
\begin{aligned}
g(\bar{R}(X,Y)Z,W) = \frac{1}{(n-2)}(4+\frac{2\bar{r}}{n-1})[g(Y,Z)g(X,W)-g(X,Z)g(Y,W)] \\
-\frac{1}{(n-2)}(2n+\frac{\bar{r}}{n-1})[g(X,W)\eta(Y)\eta(Z) \\
-g(X,Z)\eta(Y)\eta(W)+g(Y,Z)\eta(X)\eta(W) \\
-g(Y,W)\eta(X)\eta(Z)].
\end{aligned} \tag{50}
$$

Thus we can state the following theorem:

Theorem 4.1. *An n-dimensional conharmonically flat η-Einstein Kenmotsu manifold with respect to the semi-symmetric metric connection is of quasi-constant curvature.*

Next, putting $X=\xi$ in (48) and using (2) and (33), we have

$$g(\bar{R}(\xi,Y)Z,W) = \frac{1}{(n-2)}(4+\frac{\bar{r}}{n-1}-2n)[g(Y,Z)\eta(W)-g(Y,W)\eta(Z)] \tag{51}$$

which in view of (19) reduces to

$$\frac{\bar{r}}{n-1}[g(Y,Z)\eta(W)-g(Y,W)\eta(Z)]=0. \tag{52}$$

This shows that either $\bar{r}=0$ or,

$$
\begin{aligned}
& g(Y,Z)\eta(W)=g(Y,W)\eta(Z) \\
\Longrightarrow\ & g(Y,Z)\xi=\eta(Z)Y \ \textit{for all}\ Y,Z\in\chi(M)
\end{aligned} \tag{53}
$$

which is not possible. Thus we can state the following theorem:

Theorem 4.2. *In an n-dimensional conharmonically flat η-Einstein Kenmotsu manifold with respect to the semi-symmetric metric connection the scalar curvature vanishes.*

5. ϕ-conharmonically semisymmetric η-Einstein Kenmotsu manifolds with respect to the semi-symmetric metric connection

Definition. An η-Einstein Kenmotsu manifold (M^n, g), $n > 1$ is said to be ϕ-conharmonically semisymmetric with respect to the semi-symmetric metric connection if

$$\bar{C}(X,Y)\cdot\phi = 0 \tag{54}$$

for all $X, Y \in \chi(M)$.

Now, let M be an n-dimensional ϕ-conharmonically semisymmetric η-Einstein Kenmotsu manifold with respect to the semi-symmetric metric connection. Therefore $\bar{C}(X,Y).\phi = 0$ turns into

$$(\bar{C}(X,Y)\cdot\phi)Z = \bar{C}(X,Y)\phi Z - \phi\bar{C}(X,Y)Z = 0 \tag{55}$$

for any vector fields X, Y, Z on M. From (46), we have

$$\begin{aligned}\bar{C}(X,Y)\phi Z = \bar{R}(X,Y)\phi Z &- \frac{1}{(n-2)}[\bar{S}(Y,\phi Z)X - \bar{S}(X,\phi Z)Y \\ &+ g(Y,\phi Z)\bar{Q}X - g(X,\phi Z)\bar{Q}Y].\end{aligned} \tag{56}$$

By using (16), (33) and (34), (56) takes the form

$$\begin{aligned}\bar{C}(X,Y)\phi Z = R(X,Y)\phi Z &- 3g(Y,\phi Z)X + 3g(X,\phi Z)Y \\ &+ 2g(Y,\phi Z)\eta(X)\xi - 2g(X,\phi Z)\eta(Y)\xi \\ &- \frac{1}{(n-2)}[(4+\frac{2\bar{r}}{n-1})g(Y,\phi Z)X - (4+\frac{2\bar{r}}{n-1})g(X,\phi Z)Y \\ &- (2n+\frac{\bar{r}}{n-1})g(Y,\phi Z)\eta(X)\xi + (2n+\frac{\bar{r}}{n-1})g(X,\phi Z)\eta(Y)\xi].\end{aligned} \tag{57}$$

We also have

$$\begin{aligned}\phi\bar{C}(X,Y)Z = \phi R(X,Y)Z &- 3g(Y,Z)\phi X + 3g(X,Z)\phi Y \\ &+ 2\eta(Y)\eta(Z)\phi X - 2\eta(X)\eta(Z)\phi Y \\ &- \frac{1}{(n-2)}[(4+\frac{2\bar{r}}{n-1})g(Y,Z)\phi X - (4+\frac{2\bar{r}}{n-1})g(X,Z)\phi Y \\ &+ (2n+\frac{\bar{r}}{n-1})\eta(X)\eta(Z)\phi Y - (2n+\frac{\bar{r}}{n-1})\eta(Y)\eta(Z)\phi X].\end{aligned} \tag{58}$$

From (55), (57) and (58), we have

$$\begin{aligned}&(\frac{2\bar{r}}{n-1}+4n-4)g(X,\phi Z)Y - (\frac{2\bar{r}}{n-1}+4n-4)g(Y,\phi Z)X \\ &+ (\frac{\bar{r}}{n-1}+4n-4)g(Y,\phi Z)\eta(X)\xi - (\frac{\bar{r}}{n-1}+4n-4)g(X,\phi Z)\eta(Y)\xi \\ &- (\frac{2\bar{r}}{n-1}+4n-4)g(X,Z)\phi Y + (\frac{2\bar{r}}{n-1}+4n-4)g(Y,Z)\phi X \\ &- (\frac{\bar{r}}{n-1}+4n-4)\eta(Y)\eta(Z)\phi X + (\frac{\bar{r}}{n-1}+4n-4)\eta(X)\eta(Z)\phi Y = 0\end{aligned} \tag{59}$$

which by taking $Y = \xi$ and then using (2) reduces to

$$\frac{\bar{r}}{n-1}(g(X,\phi Z)\xi + \eta(Z)\phi X) = 0. \tag{60}$$

By operating ϕ on (60) and using (2), we get

$$\frac{\bar{r}}{n-1}\phi^2 X = 0.$$

Since $\phi^2 X = 0$ is not possible for all X. So we get $\bar{r} = 0$. Thus we can state the following theorem:

Theorem 5.1. *In an n-dimensional ϕ-conharmonically semisymmetric η-Einstein Kenmotsu manifold with respect to the semi-symmetric metric connection, the scalar curvature vanishes.*

6. Conharmonically flat Kenmotsu manifolds with respect to the semi-symmetric metric connection satisfying the curvature condition $\bar{R} \cdot \bar{S} = 0$

In this section we consider the conharmonically flat Kenmotsu manifolds with respect to the semi-symmetric metric connection $\bar{\nabla}$ satisfying the curvature condition

$$\bar{R}(X,Y) \cdot \bar{S} = 0. \tag{61}$$

Then we have

$$\bar{S}(\bar{R}(X,Y)Z,W) + \bar{S}(Z,\bar{R}(X,Y)W) = 0 \tag{62}$$

for all vector fields $X,Y,Z,W \in \chi(M)$.

Using (47) in (62), we have

$$\begin{aligned} &g(Y,Z)\bar{S}(\bar{Q}X,W) - g(X,Z)\bar{S}(\bar{Q}Y,W) \\ &+ g(Y,W)\bar{S}(\bar{Q}X,Z) - g(X,W)\bar{S}(\bar{Q}Y,Z) = 0 \end{aligned} \tag{63}$$

which by replacing $Y = Z = \xi$ and then using (2), (23) and (25) takes the form

$$\bar{S}(\bar{Q}X,W) - 4(n-1)^2\eta(X)\eta(W) + \eta(W)\bar{S}(\bar{Q}X,\xi) - 4(n-1)^2 g(X,W) = 0. \tag{64}$$

Let λ be the eigenvalue of the endomorphism $\bar{Q}$ corresponding to an eigenvector X. Then

$$\bar{Q}X = \lambda X. \tag{65}$$

Using (65) in (64), we have

$$\lambda\bar{S}(X,W) - [4(n-1)^2 + 2(n-1)\lambda]\eta(X)\eta(W) - 4(n-1)^2 g(X,W) = 0. \tag{66}$$

By putting $W = \xi$ in (66), we get

$$[\lambda^2 - 2(n-1)\lambda - 8(n-1)^2]\eta(X) = 0. \tag{67}$$

This gives

$$\lambda^2 - 2(n-1)\lambda - 8(n-1)^2 = 0 \quad \text{as} \quad \eta(X) \neq 0. \tag{68}$$

Thus we can state the following theorem:

Theorem 6.1. *If an n-dimensional $(n \geq 3)$ conharmonically flat Kenmotsu manifold with respect to the semi-symmetric metric connection with non zero Ricci tensor $\bar{S}$ satisfying the curvature condition $\bar{R}(X,Y) \cdot \bar{S} = 0$, then the symmetric endomorphism $\bar{Q}$ of the tangent space corresponding to $\bar{S}$ has two different non-zero eigenvalues, namely, $4(n-1)$ and $-2(n-1)$.*

7. Non-existence of Kenmotsu manifolds whose curvature tensor of manifold is covariantly constant with respect to the semi-symmetric metric connection and M is recurrent with respect to the Levi-Civita connection

Definition. A Kenmotsu manifold M with respect to the Levi-Civita connection is called the recurrent, if its curvature tensor R satisfies the condition

$$(\nabla_W R)(X,Y)Z = A(W)R(X,Y)Z, \tag{69}$$

where A is the 1-form.

From (15), we can write

$$\begin{aligned}(\bar{\nabla}_W R)(X,Y)Z &= \bar{\nabla}_W R(X,Y)Z - R(\bar{\nabla}_W X,Y)Z - R(X,\bar{\nabla}_W Y)Z,\\ -R(X,Y)\bar{\nabla}_W Z &= (\nabla_W R)(X,Y)Z + \eta(R(X,Y)Z)W\\ &\quad - g(W,R(X,Y)Z)\xi - \eta(X)R(W,Y)Z\\ &\quad - \eta(Y)R(X,W)Z - \eta(Z)R(X,Y)W\\ &\quad + g(X,W)R(\xi,Y)Z + g(Y,W)R(X,\xi)Z\\ &\quad + R(W,Z)R(X,Y)\xi\end{aligned} \tag{70}$$

which on using (7)-(9) reduces to

$$\begin{aligned}(\bar{\nabla}_W R)(X,Y)Z &= (\nabla_W R)(X,Y)Z + \eta(X)[g(W,Y)Z - g(Z,Y)W\\ &\quad - g(W,Z)Y] + \eta(Y)[g(W,Z)X - g(W,X)Z + g(Z,X)W].\end{aligned} \tag{71}$$

Let $(\bar{\nabla}_W R)(X,Y)Z = 0$, then from (71), it follows that

$$\begin{aligned}&(\nabla_W R)(X,Y)Z + \eta(X)[g(W,Y)Z - g(Z,Y)W - g(W,Z)Y]\\ &+ \eta(Y)[g(W,Z)X - g(W,X)Z + g(Z,X)W] = 0.\end{aligned} \tag{72}$$

Now using (69) in (72), we have

$$\begin{aligned}&A(W)R(X,Y)Z + \eta(X)[g(W,Y)Z - g(Z,Y)W - g(W,Z)Y]\\ &+ \eta(Y)[g(W,Z)X - g(W,X)Z + g(Z,X)W] = 0.\end{aligned} \tag{73}$$

Contracting X in (73), we get

$$A(W)S(Y,Z) + g(W,Y)\eta(Z) - g(Z,Y)\eta(W) + (n-1)g(W,Z)\eta(Y) = 0. \tag{74}$$

Now considering $Y = \xi$ in (74) and using (2), we get

$$A(W)\eta(Z) - g(W,Z) = 0. \tag{75}$$

Suppose the associated 1-form A is equal to the associated 1-form η, then from (75), we have

$$g(W,Z) = \eta(W)\eta(Z). \tag{76}$$

Therefore in view of (3), we get $g(\phi W, \phi Z) = 0$, which is not possible.

Thus we can state the following theorem:

Theorem 7.1. *There is no Kenmotsu manifold whose curvature tensor of manifold is covariantly constant with respect to the semi-symmetric metric connection and the manifold is recurrrent with respect to the Levi-Civita connection and the associated 1-form A is equal to the associated 1-form η.*

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