Commun. Korean Math. Soc. **0** (0), No. 0, pp. 1–0
https://doi.org/10.4134/CKMS.c160233
pISSN: 1225-1763 / eISSN: 2234-3024

UNIQUENESS OF SOLUTIONS FOR THE BOUNDARY VALUE PROBLEM OF CERTAIN NONLINEAR ELLIPTIC OPERATORS VIA p-HARMONIC BOUNDARY

Yong Hah Lee

Abstract. We prove the uniqueness of solutions for the boundary value problem of certain nonlinear elliptic operators in the setting: Given any continuous function f on the p-harmonic boundary of a complete Riemannian manifold, there exists a unique solution of certain nonlinear elliptic operators, which is a limit of a sequence of solutions of the operators with finite energy in the sense of supremum norm, on the manifold taking the same boundary value at each p-harmonic boundary as that of f.

1. Introduction

In this paper, we consider the boundary value problem of certain nonlinear elliptic operators on a complete Riemannian manifold. The behavior of energy finite solution of certain nonlinear elliptic operators (of type p) depends on the value of the solution on the p-harmonic boundary of the manifold. This is well understood in the case of the Laplacian which is of type $p = 2$. In [3, Theorem 1], the present author proved that in the case when the p-harmonic boundary of a complete Riemannian manifold has finite cardinality, the behavior of the solution of certain nonlinear elliptic operators is completely determined by the value of the solution on the p-harmonic boundary of the manifold. Later, in general case, he [4, Theorem 1] proved the existence of the solution for the boundary value problem of certain nonlinear elliptic operators via the p-harmonic boundary of a complete Riemannian manifold. In this paper, we will prove the uniqueness of the solution for the boundary value problem in the following setting: Let M be an n-dimensional complete Riemannian manifold and Ω be an open subset of M. Let $W^{1,p}(\Omega)$ be the Sobolev space of all

Received November 7, 2016.
2010 *Mathematics Subject Classification.* 58J05, 31B05.
Key words and phrases. $\mathcal{A}$-harmonic function, p-harmonic boundary, boundary value problem.
This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology(2012006926).

functions u in $L^p(\Omega)$ whose distributional gradient ∇u also belongs to $L^p(\Omega)$, where p is a constant such that $1 < p < \infty$. We equip $W^{1,p}(\Omega)$ with the norm $||u||_{1,p} = ||u||_p + ||\nabla u||_p$. We denote by $W_0^{1,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$.

We consider functionals associated with $\mathbf{F} : T\Omega \to \mathbb{R}$, where

$(A1)$ the mapping $\mathbf{F}_x = \mathbf{F}|_{T_xM} : T_xM \to \mathbb{R}$ is strictly convex and differentiable for all x in Ω, and the mapping $x \mapsto \mathbf{F}_x(\xi)$ is measurable whenever ξ is;

$(A2)$ for a constant $1 < p < \infty$, there exist constants $0 < C_1 \le C_2 < \infty$ such that
$$C_1|\xi|^p \le \mathbf{F}_x(\xi) \le C_2|\xi|^p$$
for all $x \in \Omega$ and $\xi \in T_xM$.

$(A3)$ in case $2 \le p < \infty$,
$$\mathbf{F}_x\Big(\frac{\xi + \xi'}{2}\Big) + \mathbf{F}_x\Big(\frac{\xi - \xi'}{2}\Big) \le \frac{1}{2}(\mathbf{F}_x(\xi) + \mathbf{F}_x(\xi')),$$
in case $1 < p \le 2$,
$$\mathbf{F}_x\Big(\frac{\xi + \xi'}{2}\Big)^{\tilde{p}} + \mathbf{F}_x\Big(\frac{\xi - \xi'}{2}\Big)^{\tilde{p}} \le \Big(\frac{1}{2}(\mathbf{F}_x(\xi) + \mathbf{F}_x(\xi'))\Big)^{\tilde{p}},$$
where $\xi, \xi' \in T_xM$ and $\tilde{p} = 1/(p-1)$;

$(A4)$ for all $\lambda \in \mathbb{R} \setminus \{0\}$,
$$\mathcal{A}_x(\lambda\xi) = \lambda|\lambda|^{p-2}\mathcal{A}_x(\xi),$$
where $\mathcal{A}_x(\xi) = (\mathcal{A}_x^1(\xi), \mathcal{A}_x^2(\xi), \dots, \mathcal{A}_x^n(\xi))$ is defined by $\mathcal{A}_x^i(\xi) = \frac{\partial}{\partial \xi^i} F_x(\xi)$ for $i = 1, 2, \dots, n$.

Using the Clarkson inequality, the condition $(A3)$ holds in the typical case $\mathbf{F}(\xi) = \frac{1}{p}|\xi|^p$, i.e., the p-harmonic case. (See [2, Section 15].)

A function u in $W_{\text{loc}}^{1,p}(\Omega)$ is a solution of the equation

$$-\text{div}\mathcal{A}_x(\nabla u) = 0 \tag{1}$$

in Ω if
$$\int_\Omega \langle \mathcal{A}_x(\nabla u), \nabla\phi \rangle = 0$$
for any ϕ in $C_0^\infty(\Omega)$. We say that a function u is $\mathcal{A}$-harmonic (of type p) if u is a continuous solution of (1). In a typical case $\mathcal{A}_x(\xi) = \xi|\xi|^{p-2}$, $\mathcal{A}$-harmonic functions are called p-harmonic and, in particular, if $p = 2$, then we obtain harmonic functions. Suppose that E is a measurable set and that $u \in W_{\text{loc}}^{1,p}(\Omega)$ for an open neighborhood Ω of E. Then the variational integral
$$\mathbf{J}(u, E) = \int_E \mathbf{F}_x(\nabla u)$$
is well defined. Given $f \in W^{1,p}(\Omega)$, each $\mathcal{A}$-harmonic function u with $u - f \in W_0^{1,p}(\Omega)$ minimizes the energy functional $\mathbf{J}(v, \Omega)$ on the set $\{v \in W^{1,p}(\Omega) :$

$v - f \in W_0^{1,p}(\Omega)\}$. (See [5, Theorem 2.96].) We say that u is an energy finite $\mathcal{A}$-harmonic function if u is an $\mathcal{A}$-harmonic function with $\mathbf{J}(u, M) < \infty$.

In the above setting, we prove the uniqueness of solutions for the boundary value problem of the nonlinear elliptic operator on a complete Riemannian manifold in terms of the p-harmonic boundary of the manifold as follows:

Theorem 1.1. *Let M be a complete Riemannian manifold and $\mathcal{A}$ be an elliptic operator on M satisfying $(A1)$, $(A2)$, $(A3)$ and $(A4)$. Then for any continuous function f on the p-harmonic boundary Δ_M^p of M, expounded later, there exists a unique $\mathcal{A}$-harmonic function h, which is a limit of a sequence of bounded energy finite $\mathcal{A}$-harmonic functions in the sense of supremum norm, such that*

$$\lim_{x \in M \to \mathbf{x}} h(x) = f(\mathbf{x})$$

for all $\mathbf{x} \in \Delta_M^p$.

2. $\mathcal{A}$-harmonic functions and p-harmonic boundary

We begin with introducing some notations and relevant results which we need in this paper. Let $\mathcal{BD}_p(M)$ denote the set of all bounded continuous functions u on a complete Riemannian manifold M whose distributional gradient ∇u belongs to $L^p(M)$. Then $\mathcal{BD}_p(M)$ forms an algebra over the real numbers with the usual addition and multiplication of functions and scalar multiplication defined pointwise. The function space $\mathcal{BD}_p(M)$ is called the Royden p-algebra of M. (See [6, Chapter 3].) We say that a sequence $\{f_n\}$ of functions in $\mathcal{BD}_p(M)$ converges to a function $f \in \mathcal{BD}_p(M)$ if $\{f_n\}$ is uniformly bounded on M, f_n converges uniformly to f on each compact subset of M and

$$\lim_{n \to \infty} \int_M |\nabla(f_n - f)|^p = 0.$$

Let $\mathcal{BD}_{p,0}(M)$ denote the closure of the set of all compactly supported smooth functions in $\mathcal{BD}_p(M)$. We denote by $\mathcal{HBD}_{\mathcal{A}}(M)$ the subset of all bounded energy finite $\mathcal{A}$-harmonic functions in $\mathcal{BD}_p(M)$, where $\mathcal{A}$ is an elliptic operator on M satisfying condition $(A1)$, $(A2)$, $(A3)$ and $(A4)$. Then one can prove the $\mathcal{A}$-harmonic function version of the Royden decomposition theorem as follows: (See [3, Lemma 3].)

Lemma 2.1. *For each $f \in \mathcal{BD}_p(M)$, there exist unique $h \in \mathcal{HBD}_{\mathcal{A}}(M)$ and $g \in \mathcal{BD}_{p,0}(M)$ such that $f = h + g$.*

For a complete Riemannian manifold M, there exists a locally compact Hausdorff space $\hat{M}$, called the Royden p-compactification of M, which contains M as an open dense subset. In particular, every function $f \in \mathcal{BD}_p(M)$ can be extended to a continuous function, denoted again by f, on $\hat{M}$ and the class of such extended functions separates points in $\hat{M}$. By the Stone-Weierstrass theorem, $\mathcal{BD}_p(M)$ is dense in the set of all bounded continuous functions on

$\hat{M}$ with respect to the following sense: For any continuous function f on $\hat{M}$, there is a sequence $\{f_n\}$ in $\mathcal{BD}_p(M)$ such that

$$\lim_{n\to\infty}\sup_M |f_n - f| = 0. \tag{2}$$

The Royden p-boundary of $\hat{M}$ is the set $\hat{M}\setminus M$ and will be denoted by $\partial\hat{M}$. An important part of the Royden p-boundary $\partial\hat{M}$ is the p-harmonic boundary Δ_M^p defined by

$$\Delta_M^p = \{\mathbf{x}\in\partial\hat{M} : f(\mathbf{x}) = 0 \text{ for all } f\in\mathcal{BD}_{p,0}(M)\}.$$

In particular, the duality relation between $\mathcal{BD}_{p,0}(M)$ and Δ_M^p holds as follows: (See [3, Lemma 2].)

$$\mathcal{BD}_{p,0}(M) = \{f\in\mathcal{BD}_p(M) : f(\mathbf{x}) = 0 \text{ for all } \mathbf{x}\in\Delta_M^p\}.$$

In fact, the p-harmonic boundary of a complete Riemannian manifold is empty if and only if the manifold is p-parabolic. This case is the trivial one in our problem since every bounded $\mathcal{A}$-harmonic function on the manifold is constant. So, from now on, we assume that the p-harmonic boundary of each manifold M is nonempty unless otherwise specified.

Using the duality relation, we get the comparison principle for $\mathcal{A}$-harmonic functions in terms of the p-harmonic boundary as follows:

Lemma 2.2. *Let h_1 and h_2 be functions in $\mathcal{HBD}_{\mathcal{A}}(M)$ such that $h_1\le h_2$ on Δ_M^p. Then $h_1\le h_2$ on M.*

Proof. Let $g=\max\{h_1-h_2,0\}$. Then $g\ge 0$ on M and $g=0$ on Δ_M^p since $h_1-h_2\le 0$ on Δ_M^p. By the duality relation, $g\in\mathcal{B}D_{p,0}(M)$. Since there exists a sequence of compactly supported continuous functions converging to g in $\mathcal{BD}_p(M)$, we have

$$\int_M\langle\mathcal{A}_x(\nabla h_1),\nabla g\rangle = 0 \quad\text{and}\quad \int_M\langle\mathcal{A}_x(\nabla h_2),\nabla g\rangle = 0,$$

hence

$$\int_M\langle\mathcal{A}_x(\nabla h_1)-\mathcal{A}_x(\nabla h_2),\nabla g\rangle = 0.$$

Let $\Omega=\{x\in M : h_1(x)\ge h_2(x)\}$, then

$$\int_\Omega\langle\mathcal{A}_x(\nabla h_1)-\mathcal{A}_x(\nabla h_2),\nabla(h_1-h_2)\rangle = \int_M\langle\mathcal{A}_x(\nabla h_1)-\mathcal{A}_x(\nabla h_2),\nabla g\rangle = 0.$$

By the assumptions $(A1)$ and $(A2)$, h_1-h_2 must be constant on Ω. (See [5, Theorem 2.98].) From the continuity of h_1 and h_2 and the assumption $h_1\le h_2$ on Δ_M^p, we have $h_1-h_2=0$ on Ω. Consequently, $h_1\le h_2$ on M. □

We are now ready to prove our main result:

Theorem 2.3. *Let M be a complete Riemannian manifold and $\mathcal{A}$ be an elliptic operator on M satisfying (A1), (A2), (A3) and (A4). Then for any continuous function f on the p-harmonic boundary Δ_M^p of M, there exists a unique $\mathcal{A}$-harmonic function h, which is a limit of a sequence $\{h_n\}$ of functions in $\mathcal{HBD}_{\mathcal{A}}(M)$ with respect to the topology (2), such that*

$$\lim_{x\in M\to \mathbf{x}} h(x) = f(\mathbf{x}) \tag{3}$$

for all $\mathbf{x} \in \Delta_M^p$.

Proof. Let $f : \Delta_M^p \to \mathbf{R}$ be a continuous function. Then there is a continuous extension $\hat{f}$ on $\hat{M}$ of f such that $\hat{f}|_{\Delta_M^p} \equiv f$. By the Stone-Weierstrass theorem, there is a sequence $\{f_n\}$ in $\mathcal{BD}_p(M)$ such that

$$\lim_{n\to\infty} \sup_M |f_n - \hat{f}| = 0.$$

For any given $\epsilon > 0$, there is an $N \in \mathbf{N}$ such that for all $n \geq N$,

$$\sup_M |f_n - \hat{f}| < \epsilon.$$

By the proof of Theorem 1 in [4], there exist a sequence $\{h_n\}$ in $\mathcal{HBD}_{\mathcal{A}}(M)$ and an $\mathcal{A}$-harmonic function h on M such that $h_n = f_n$ on Δ_M^p and for all $n \geq N$,

$$\sup_M |h_n - h| < \epsilon, \tag{4}$$

furthermore, h satisfies the equation (3).

Suppose that h' is another $\mathcal{A}$-harmonic function on M satisfying (3) and there is a sequence $\{k_n\}$ in $\mathcal{HBD}_{\mathcal{A}}(M)$ converging to h' with respect to the topology (2). Then for any given $\epsilon > 0$, we can choose $N \in \mathbf{N}$ such that for all $n \geq N$,

$$\sup_M | k_n - h'| < \epsilon. \tag{5}$$

From this together with (3), we get

$$\sup_{\Delta_M^p} | k_n - f| \leq \epsilon \quad \text{and} \quad \sup_{\Delta_M^p} | h_n - f| \leq \epsilon.$$

Since $|k_n - h_n| \leq 2\epsilon$ on Δ_M^p, by Lemma 2.2,

$$\sup_M | k_n - h_n| \leq 2\epsilon.$$

From this together with (4) and (5), we have

$$\sup_M | h' - h| < 4\epsilon.$$

Since $\epsilon > 0$ is arbitrarily chosen, $h' \equiv h$ on M. □

We denote by $\mathbf{n}(r)$ the number of unbounded components of $M \setminus B_r(o)$, where o is a fixed point of M, and we call each of the components an end of M corresponding to $B_r(o)$. In particular, $\mathbf{n}(r)$ is nondecreasing in $r > 0$. Let $\lim_{r\to\infty} \mathbf{n}(r) = k$, where k may be infinity, then we say that the number of ends of M is k. In fact, if an end E is p-nonparabolic, then the closure $\hat{E}$ of the end E in $\hat{M}$ has at least one point of the p-harmonic boundary, and otherwise, it has no point of the p-harmonic boundary. In particular, if every $\mathcal{A}$-harmonic function on M is asymptotically constant at infinity of the p-nonparabolic end E, then $\hat{E}$ contains only one point of the p-harmonic boundary. (See [3, Section 4].)

If the number of p-nonparabolic ends of a complete Riemannian manifold M is finite and every $\mathcal{A}$-harmonic function on the manifold M is asymptotically constant at infinity of each p-nonparabolic end, then the p-harmonic boundary of the manifold M has finite cardinality. In the case, every continuous function on $\hat{M}$ is in $\mathcal{BD}_p(M)$. Hence, as a corollary of Theorem 2.3, we have the following result:

Corollary 2.4. *Let M be a complete Riemannian manifold with p-nonparabolic ends $E_1, E_2, \ldots, E_l$, and $\mathcal{A}$ be an elliptic operator on M satisfying (A1), (A2), (A3) and (A4). Suppose that every $\mathcal{A}$-harmonic function on M is asymptotically constant at infinity of each end of M. Then for given real numbers $a_1, a_2, \cdots, a_l$, there exists a unique $\mathcal{A}$-harmonic function h in $\mathcal{HBD}_{\mathcal{A}}(M)$ such that*

$$\lim_{x\in E_i\to\infty} h(x) = a_i, \;\; i = 1, 2, \ldots, l.$$

In particular, if an end of a complete Riemannian manifold satisfies the volume doubling condition, the Poincaré inequality, and the finite covering condition at infinity, then every $\mathcal{A}$-harmonic function on the manifold is asymptotically constant at infinity of the end. (See [3, Section 4].) In fact, if a complete Riemannian manifold with nonnegative Ricci curvature outside a compact set and finite first Betti number, then the three conditions holds on each end of the manifold. Hence every $\mathcal{A}$-harmonic function on the manifold is also asymptotically constant at infinity of each end. As a simpler case, if a complete Riemannian manifold has nonnegative Ricci curvature everywhere, then by the splitting theorem of Cheeger and Gromoll [1], the manifold has at most two ends. In particular, in the case that the manifold is p-nonparabolic, the number of ends of the manifold is one. Therefore, as a corollary of Theorem 2.3, we have a generalization of the result of [3] and of [4], in which the finiteness of connected sum is necessary, as follows:

Corollary 2.5. *Let M_i, $i = 1, 2, \ldots$, be complete Riemannian manifolds with nonnegative Ricci curvature. Let M be a connected sum $\sharp_{i=1}^{\infty} M_i$ and $\mathcal{A}$ be an elliptic operator on M satisfying (A1), (A2), (A3) and (A4). Suppose that P is a subset of $\mathbf{N}$ such that M_j is p-nonparabolic for each $j \in P$. Then for given real numbers a_j, $j \in P$, there exists a unique $\mathcal{A}$-harmonic function h, which* *is a limit of a sequence of functions in $\mathcal{HBD}_{\mathcal{A}}(M)$ with respect to the topology* (2), *such that*

$$\lim_{x\in M_j\to\infty} h(x) = a_j, \;\; j \in P.$$

In fact, if an end of a complete Riemannian manifold is roughly isometric to an end satisfying the volume doubling condition, the Poincaré inequality and the finite covering condition at infinity, then every $\mathcal{A}$-harmonic function on the manifold is asymptotically constant at infinity of the end. Furthermore, the number of ends and the p-nonparabolicity of ends are roughly isometric invariants. (See [3] and [4].) Therefore, our results can be extended to the class being roughly isometric to the complete Riemannian manifolds given in Theorem 2.3, Corollary 2.4 and Corollary 2.5, respectively.

References

[1] J. Cheeger and D. Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Differential Geometry **6** (1971), 119–128.

[2] E. Hewitt and K. Stormberg, *Real and Abstract Analysis*, Springer-Verlag, New York, Heidelberg, Berlin, 1965.

[3] Y. H. Lee, *Rough isometry and energy finite solutions of elliptic equations on Riemannian manifolds*, Math. Ann. **318** (2000), no. 1, 181–204.

[4] ______, *Rough isometry and p-harmonic boundaries of complete Riemannian manifolds*, Potential Anal. **23** (2005), no.1, 83–97.

[5] J. Malý and W. P. Ziemer, *Fine regularity of solutions of elliptic partial differential equations*, Mathematical Surveys and Monographs, 51. American Mathematical Society, Providence, RI, 1997.

[6] L. Sario and M. Nakai, *Classification theory of Riemann surfaces*, Springer Verlag, Berlin, Heidelberg, New York, 1970.

Yong Hah Lee
Department of Mathematics Education
Ewha Womans University
Seoul 120-750, Korea
E-mail address: yonghah@ewha.ac.kr