

CERTAIN NEW WP-BAILEY PAIRS AND BASIC HYPERGEOMETRIC SERIES IDENTITIES

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ABSTRACT. The Bailey lemma has been a powerful tool in the discovery of identities of Rogers-Ramanujan type and also ordinary and basic hypergeometric series identities. The mechanism of Bailey lemma has also led to the concepts of Bailey pair and Bailey chain. In the present work certain new WP-Bailey pairs have been established. We also have deduced a number of basic hypergeometric series identities as an application of new WP-Bailey pairs.

1. Introduction

For $|q| < 1$, we define

$$(a; q)_n = \begin{cases} (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}); n \in \mathbb{N} \\ 1; n = 0 \end{cases}$$

or equivalently,

$$(a; q)_n = \prod_{j=0}^{\infty} \frac{(1-aq^j)}{(1-aq^{n+j})} \equiv \frac{(a; q)_{\infty}}{(aq^n; q)_{\infty}},$$

where a is real or complex. A basic hypergeometric series is defined as

$$\begin{aligned} & {}_r\varphi_s(a_1, a_2, a_3, \dots, a_r; b_1, b_2, b_3, \dots, b_s; q, z) \\ &= \sum_{n=0}^{\infty} \frac{(a_1; q)_n (a_2; q)_n \cdots (a_r; q)_n}{(q; q)_n (b_1; q)_n (b_2; q)_n \cdots (b_s; q)_n} [(-1)^n q^{\frac{n(n-1)}{2}}]^{1+s-r} z^n. \end{aligned}$$

For $0 < |q| < 1$, the series converges absolutely for all z if $r \leq s$ and for $|z| < 1$ if $r = s + 1$. This series also converges absolutely if $|q| > 1$ and $|z| < |b_1 b_2 \cdots b_s| / |a_1 a_2 \cdots a_r|$.

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In 1944, Bailey [7] introduced a very useful and simple identity known as Bailey's lemma. The Bailey's lemma states that, if

$$(1) \quad \beta_n = \sum_{r=0}^n \alpha_r u_{n-r} v_{n+r}$$

and

$$(2) \quad \gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{n+r},$$

then under the suitable convergence conditions and if change in the order of summations is allowed

$$(3) \quad \sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n,$$

where α_r, δ_r, u_r and v_r are functions of r such that β_n and γ_n exist. The proof of the lemma is trivial.

During last seven decades Bailey lemma and its various generalizations have proved to be a powerful tool in the discoveries of Rogers-Ramanujan type of identities, transformations and summations theorems of ordinary and basic hypergeometric series. Slater [17,18] used Bailey lemma to discover the famous list of 130 identities of Rogers-Ramanujan type. Using the same tool more identities of Rogers-Ramanujan type and basic hypergeometric series have been given by Andrew [4], Foda and Quano [11], Denis [9], Denis and Singh [10], Singh [14,15] and Ali and Rizvi [1].

In (1), if we choose $u_r = 1/(q; q)_r$ and $v_r = 1/(aq; q)_r$, we get

$$(4) \quad \beta_n = \sum_{r=0}^n \frac{\alpha_r}{(q; q)_{n-r} (aq; q)_{n+r}}.$$

The pair of sequence (α_n, β_n) that satisfies (4) is called a Bailey pair relative to the parameter a . The concept of Bailey pairs has been generalized in the works of Bressoud [8] and Singh [16]. The most elegant generalizations of Bailey pair have been given by Andrews [5] which is

$$(5) \quad \beta_n = \sum_{r=0}^n \frac{(k/a; q)_{n-r} (k; q)_{n+r}}{(q; q)_{n-r} (aq; q)_{n+r}} \alpha_r.$$

The pair (α_n, β_n) satisfying (5) is termed as WP-Bailey pair. It is easy to see that (5) follows by setting $u_r = \frac{(k/a; q)_r}{(q; q)_r}$ and $v_r = \frac{(k; q)_r}{(aq; q)_r}$ in (1). For $k = 0$ in (5), we get the standard Bailey pair (4).

Andrew et al. [2-4, 6] have exploited very effective mechanism of Bailey lemma in the form of Bailey chain. In fact Andrews [5] described that the process may be iterated to produce a chain of WP-Bailey pairs constructing new WP-bailey pairs from existing initial WP-Bailey pair and introduced the concept of Bailey chain. The aforesaid technique developed by Andrews have

been effectively used to discover new WP-Bailey pairs. The idea of WP-Bailey pairs and chain has further been generalized by Liu and Ma [13] and Spiridonov [19]. For more details on Bailey lemma and its various generalizations and applications one is referred the recent and a very elegant survey of Warnaar [21].

In the present work, we have established some new WP-Bailey pairs using the following results of Andrews [5] and Warnaar [22]. We, further, have used the new WP-Bailey pairs to produce a number of new transformations of basic hypergeometric series.

Theorem 1 ([5]). *If $(\alpha_n(a, k; q), \beta_n(a, k; q))$ is a WP-Bailey pair and satisfies (5), then so is the pair $(\alpha'_n(a, k; q), \beta'_n(a, k; q))$ given by*

$$(6) \quad \alpha'_n(a, k; q) = \frac{(\rho_1, \rho_2; q)_n}{(aq/\rho_1, aq/\rho_2; q)_n} \left(\frac{k}{m}\right)^n \alpha_n(a, m; q),$$

$$(7) \quad \beta'_n(a, k; q) = \frac{(mq/\rho_1, mq/\rho_2; q)_n}{(aq/\rho_1, aq/\rho_2; q)_n} \sum_{r=0}^n \frac{(1 - mq^{2r})(\rho_1, \rho_2; q)_r (k/m; q)_{n-r}}{(1 - m)(mq/\rho_1, mq/\rho_2; q)_r (q; q)_{n-r}} \\ \times \frac{(k; q)_{n+r}}{(mq; q)_{n+r}} \left(\frac{k}{m}\right)^r \beta_r(a, m; q),$$

where $m = k\rho_1\rho_2/aq$.

Theorem 2 ([5]). *If $(\alpha_n(a, k; q), \beta_n(a, k; q))$ is a WP-Bailey pair, then so is the pair $(\alpha'_n(a, k; q), \beta'_n(a, k; q))$ given by*

$$(8) \quad \alpha'_n(a, k; q) = \frac{(m; q)_{2n}}{(k; q)_{2n}} \left(\frac{k}{m}\right)^n \alpha_n(a, m; q),$$

$$(9) \quad \beta'_n(a, k; q) = \sum_{r=0}^n \frac{(k/m; q)_{n-r}}{(q; q)_{n-r}} \left(\frac{k}{m}\right)^r \beta_r(a, m; q),$$

where $m = a^2q/k$.

Theorem 3 ([5]). *If $(\alpha_n(a, k; q), \beta_n(a, k; q))$ is a WP-Bailey pair, then so is the pair $(\alpha'_n(a, k; q), \beta'_n(a, k; q))$ given by*

$$(10) \quad \alpha'_n(a, k; q) = \frac{(qa^2/k; q)_{2n}}{(k; q)_{2n}} \left(\frac{k^2}{qa^2}\right)^n \alpha_n(a, qa^2/k; q),$$

$$(11) \quad \beta'_n(a, k; q) = \sum_{r=0}^n \frac{(k^2/qa^2; q)_{n-r}}{(q; q)_{n-r}} \left(\frac{k^2}{qa^2}\right)^r \beta_r(a, qa^2/k; q).$$

Theorem 4 ([22]). *If $(\alpha_n(a, k; q), \beta_n(a, k; q))$ is a WP-Bailey pair, then so is the pair $(\alpha'_n(a, k; q), \beta'_n(a, k; q))$ given by*

$$(12) \quad \alpha'_n(a, k; q) = \frac{(1 - \sigma\sqrt{k})(1 + \sigma\sqrt{mq^n})(m; q)_{2n}}{(1 - \sigma\sqrt{kq^n})(1 + \sigma\sqrt{m})(k; q)_{2n}} \left(\frac{k}{m}\right)^n \alpha_n(a, m; q),$$

(13)

$$\beta'_n(a, k; q) = \frac{(1 - \sigma\sqrt{k})}{(1 - \sigma\sqrt{k}q^n)} \sum_{r=0}^n \frac{(1 + \sigma\sqrt{mq^r})(k/m; q)_{n-r}}{(1 + \sigma\sqrt{m})(q; q)_{n-r}} \left(\frac{k}{m}\right)^r \beta_r(a, m; q),$$

where $m = a^2/k$ and $\sigma \in (-1, 1)$.

Theorem 5 ([22]). *If $(\alpha_n(a, k; q), \beta_n(a, k; q))$ is a WP-Bailey pair, then so is the pair $(\alpha'_n(a, k; q), \beta'_n(a, k; q))$ given by*

$$(14) \quad \alpha'_n(a^2, k; q^2) = \alpha_n(a, m; q),$$

(15)

$$\begin{aligned} & \beta'_n(a^2, k; q^2) \\ &= \frac{(-mq; q)_{2n}}{(-aq; q)_{2n}} \sum_{r=0}^n \frac{(1 - mq^{2r})(k/m^2; q^2)_{n-r}(k; q^2)_{n+r}}{(1 - m)(q^2; q^2)_{n-r}(m^2q^2; q^2)_{n+r}} \left(\frac{m}{a}\right)^{n-r} \beta_r(a, m; q), \end{aligned}$$

where $m = k/aq$.

Theorem 6 ([22]). *If $(\alpha_n(a, k; q), \beta_n(a, k; q))$ is a WP-Bailey pair, then so is the pair $(\alpha'_n(a, k; q), \beta'_n(a, k; q))$ given by*

$$(16) \quad \alpha'_n(a^2, k; q^2) = q^{-n} \frac{(1 + aq^{2n})}{(1 + a)} \alpha_n(a, m; q),$$

(17)

$$\begin{aligned} & \beta'_n(a^2, k; q^2) \\ &= \frac{(-mq; q)_{2n} q^{-n}}{(-a; q)_{2n}} \sum_{r=0}^n \frac{(1 - mq^{2r})(k/m^2; q^2)_{n-r}(k; q^2)_{n+r}}{(1 - m)(q^2; q^2)_{n-r}(m^2q^2; q^2)_{n+r}} \left(\frac{m}{a}\right)^{n-r} \beta_r(a, m; q), \end{aligned}$$

where $m = k/a$.

Theorem 7 ([22]). *If $(\alpha_n(a, k; q), \beta_n(a, k; q))$ is a WP-Bailey pair, then so is the pair $(\alpha'_n(a, k; q), \beta'_n(a, k; q))$ given by*

$$(18) \quad \alpha'_{2n}(a, k; q) = \alpha_n(a, m; q^2); \quad \alpha'_{2n+1}(a, k; q) = 0,$$

(19)

$$\begin{aligned} & \beta'_n(a, k; q) \\ &= \frac{(mq; q^2)_n}{(aq; q^2)_n} \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{(1 - mq^{4r})(k/m; q)_{n-2r}(k; q)_{n+2r}}{(1 - m)(q; q)_{n-2r}(mq; q)_{n+2r}} \left(\frac{-k}{a}\right)^{n-2r} \beta_r(a, m; q^2), \end{aligned}$$

where $m = k^2/a$.

In the next section, we shall also require the following identities.

$$(20) \quad {}_4\varphi_3(a, -q\sqrt{a}, b, q^{-n}; -\sqrt{a}, aq/b, aq^{n+1}; q, q^{n+1}a^{1/2}/b) = \frac{(aq, q\sqrt{a}/b; q)_n}{(q\sqrt{a}, aq/b; q)_n},$$

([12]; II.14)

for $b = kq^n$ in (20), we get

$$(21) \quad \begin{aligned} & {}_4\varphi_3(a, -q\sqrt{a}, kq^n, q^{-n}; -\sqrt{a}, aq^{1-n}/k, aq^{n+1}; q, qa^{1/2}/k) \\ &= \frac{(aq, k/\sqrt{a}; q)_n}{(q\sqrt{a}, k/a; q)_n} \left(\frac{1}{\sqrt{a}} \right)^n, \end{aligned}$$

$$(22) \quad \begin{aligned} & {}_4\varphi_3(a, c, aq^{n+1/2}/c, q^{-n}; aq/c, cq^{1/2-n}, aq^{n+1}; q, q^2) \\ &= \frac{(1 + \sqrt{a})(aq, \sqrt{q}, \sqrt{aq}/c, q\sqrt{a}/c; q)_n}{2\sqrt{a}(aq/c, \sqrt{q}/c, \sqrt{aq}, q\sqrt{a}; q)_n} - \frac{(1 - \sqrt{a})(aq, \sqrt{q}, \sqrt{aq}/c, -q\sqrt{a}/c; q)_n}{2\sqrt{a}(aq/c, -\sqrt{q}/c, \sqrt{aq}, -q\sqrt{a}; q)_n}, \end{aligned}$$

([20]; 44)

for $c = a\sqrt{q}/k$ in (22), we get

$$(23) \quad \begin{aligned} & {}_4\varphi_3(a, a\sqrt{q}/k, kq^n, q^{-n}; k\sqrt{q}, aq^{1-n}/k, aq^{n+1}; q, q^2) \\ &= \frac{(1 + \sqrt{a})(aq, \sqrt{q}, k/\sqrt{a}, k\sqrt{q}/\sqrt{a}; q)_n}{2\sqrt{a}(k\sqrt{q}, k/a, \sqrt{aq}, q\sqrt{a}; q)_n} - \frac{(1 - \sqrt{a})(aq, \sqrt{q}, k/\sqrt{a}, -k\sqrt{q}/\sqrt{a}; q)_n}{2\sqrt{a}(k\sqrt{q}, -k/a, \sqrt{aq}, -q\sqrt{a}; q)_n}. \end{aligned}$$

2. New WP-Bailey pairs

If $(\alpha_n(a, k; q), \beta_n(a, k; q))$ is a WP-Bailey pair, then so is the pair $(\alpha'_n(a, k; q), \beta'_n(a, k; q))$ given by

$$(24) \quad \alpha'_n(a, k; q) = \frac{(a, \rho_1, \rho_2, -q\sqrt{a}; q)_n}{(q, -\sqrt{a}, aq/\rho_1, aq/\rho_2; q)_n} \left(\frac{k}{m\sqrt{a}} \right)^n,$$

$$(25) \quad \begin{aligned} \beta'_n(a, k; q) &= \frac{(mq/\rho_1, mq/\rho_2, k/m, k; q)_n}{(q, aq/\rho_1, aq/\rho_2, mq; q)_n} \\ &\quad \times \sum_{r=0}^n \frac{(q\sqrt{m}, -q\sqrt{m}, \rho_1, \rho_2, q^{-n}, kq^n, m, m/\sqrt{a}; q)_r}{(q, q\sqrt{a}, \sqrt{m}, -\sqrt{m}, mq/\rho_1, mq/\rho_2, mq^{1-n}/k, mq^{n+1}; q)_r} \left(\frac{q}{\sqrt{a}} \right)^r, \end{aligned}$$

where $m = k\rho_1\rho_2/aq$.

Proof of (24)-(25). Let us choose

$$(26) \quad \alpha_r(a, k; q) = \frac{(a, -q\sqrt{a}; q)_r}{(q, -\sqrt{a}; q)_r} \left(\frac{1}{\sqrt{a}} \right)^r,$$

in (5), we obtain

$$\beta_n(a, k; q) = \frac{(k, k/a; q)_n}{(q, aq; q)_n} \sum_{r=0}^n \frac{(kq^n, q^{-n}, a, -q\sqrt{a}; q)_r}{(q, aq^{1-n}/k, aq^{n+1}, -\sqrt{a}; q)_r} \left(\frac{q\sqrt{a}}{k} \right)^r,$$

by making the use (21), we have

$$(27) \quad \beta_n(a, k; q) = \frac{(k, k/\sqrt{a}; q)_n}{(q, q\sqrt{a}; q)_n} \left(\frac{1}{\sqrt{a}} \right)^n.$$

We obtain new WP-Bailey pair (26), (27). Now using WP-Bailey pair (26) and (27) in (6) and (7), we get (24), (25).

$$(28) \quad \alpha'_n(a, k; q) = \frac{(m; q)_{2n}(a, -q\sqrt{a}; q)_n}{(k; q)_{2n}(q, -\sqrt{a}; q)_n} \left(\frac{k}{m\sqrt{a}} \right)^n,$$

$$(29) \quad \beta'_n(a, k; q) = \frac{(k/m; q)_n}{(q; q)_n} \sum_{r=0}^n \frac{(m, q^{-n}, m/\sqrt{a}; q)_r}{(q, q\sqrt{a}, mq^{1-n}/k; q)_r} \left(\frac{q}{\sqrt{a}} \right)^r,$$

where $m = qa^2/k$. \square

Proof of (28)-(29). Using WP-Bailey pair (26), (27) in (8) and (9), we obtain (28) and (29).

$$(30) \quad \alpha'_n(a, k; q) = \frac{(qa^2/k; q)_{2n}(a, -q\sqrt{a}; q)_n}{(k; q)_{2n}(q, -\sqrt{a}; q)_n} \left(\frac{k^2}{qa^{5/2}} \right)^n,$$

$$(31) \quad \beta'_n(a, k; q) = \frac{(k^2/qa^2; q)_n}{(q; q)_n} \sum_{r=0}^n \frac{(q^{-n}, qa^2/k, a^{3/2}q/k; q)_r}{(q, q\sqrt{a}, a^2q^{2-n}/k^2; q)_r} \left(\frac{q}{\sqrt{a}} \right)^r. \quad \square$$

Proof of (30)-(31). By using WP-Bailey pair (26), (27) in (10) and (11), we get (30) and (31).

$$(32) \quad \alpha'_n(a, k; q) = \frac{(\sigma\sqrt{k}, -\sigma q\sqrt{m}, a, -q\sqrt{a}; q)_n (m; q)_{2n}}{(q, -\sqrt{a}, \sigma q\sqrt{k}, -\sigma\sqrt{m}; q)_n (k; q)_{2n}} \left(\frac{k}{m\sqrt{a}} \right)^n,$$

$$(33) \quad \beta'_n(a, k; q) = \frac{(k/m, \sigma\sqrt{k}; q)_n}{(q, q\sigma\sqrt{k}; q)_n} \sum_{r=0}^n \frac{(m, q^{-n}, m/\sqrt{a}, -q\sigma\sqrt{m}; q)_r}{(q, q\sqrt{a}, mq^{1-n}/k, -\sigma\sqrt{m}; q)_r} \left(\frac{q}{\sqrt{a}} \right)^r,$$

where $m = a^2/k, \sigma \in (-1, 1)$. \square

Proof of (32)-(33). By making the use of WP-Bailey pair (26), (27) in (12) and (13), we have (32), (33).

$$(34) \quad \alpha'_n(a^2, k; q^2) = \frac{(a, -q\sqrt{a}; q)_n}{(q, -\sqrt{a}; q)_n} \left(\frac{1}{\sqrt{a}} \right)^n,$$

$$(35) \quad \beta'_n(a^2, k; q^2) = \frac{(-mq; q)_{2n}(k, k/m^2; q^2)_n}{(-aq; q)_{2n}(q^2, m^2q^2; q^2)_n} \left(\frac{m}{a} \right)^n \\ \times \sum_{r=0}^n \frac{(m, m/\sqrt{a}; q)_r (q^{-2n}, kq^{2n}, mq^2; q^2)_r}{(q, q\sqrt{a}; q)_r (m, m^2q^{2+2n}, m^2q^{2-2n}/k; q^2)_r} \left(\frac{mq^2\sqrt{a}}{k} \right)^r,$$

where $m = k/aq$. \square

Proof of (34)-(35). Now using WP-Bailey pair (26), (27) in (14) and (15), we get (34) and (35).

$$(36) \quad \alpha'_n(a^2, k; q^2) = \frac{(1 + aq^{2n})(a, -q\sqrt{a}; q)_n}{(1 + a)(q, -\sqrt{a}; q)_n} \left(\frac{1}{q\sqrt{a}} \right)^n,$$

$$(37) \quad \beta'_n(a^2, k; q^2) = \frac{(-mq; q)_{2n}(k, k/m^2; q^2)_n}{(-a; q)_{2n}(q^2, m^2q^2; q^2)_n} \left(\frac{m}{aq} \right)^n \\ \times \sum_{r=0}^n \frac{(m, m/\sqrt{a}; q)_r (q^{-2n}, kq^{2n}, mq^2; q^2)_r}{(q, q\sqrt{a}; q)_r (m^2q^{2+2n}, m^2q^{2-2n}/k, m; q^2)_r} \left(\frac{mq^2\sqrt{a}}{k} \right)^r,$$

where $m = k/a$. \square

Proof of (36)-(37). Using WP-Bailey pair (26), (27) in (16) and (17), we obtain (36) and (37).

$$(38) \quad \alpha'_{2n}(a, k; q) = \frac{(a, -q^2\sqrt{a}; q^2)_n}{(-\sqrt{a}, q^2; q^2)_n} \left(\frac{1}{\sqrt{a}} \right)^n; \quad \alpha'_{2n+1}(a, k; q) = 0,$$

$$(39) \quad \beta'_n(a, k; q) = \frac{(\sqrt{mq}, -\sqrt{mq}, k, k/m; q)_n}{(\sqrt{aq}, -\sqrt{aq}, q, mq; q)_n} \left(\frac{-k}{a} \right)^n \\ \times \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{(1 - mq^{4r})(q^{-n}, q^{-n+1}, kq^n, kq^{n+1}, m, m/\sqrt{a}; q^2)_r}{(1 - m)(mq^{1-n}/k, mq^{2-n}/k, mq^{n+1}, mq^{n+2}, q^2, q^2\sqrt{a}; q^2)_r} \left(\frac{m^2a^{3/2}q^2}{k^4} \right)^r,$$

where $m = k^2/a$. \square

Proof of (38)-(39). By using WP-Bailey pair (26), (27) in (18) and (19), we get (38), (39).

$$(40) \quad \alpha'_n(a, k; q) = \frac{(\rho_1, \rho_2, a, a\sqrt{q}/m; q)_n}{(q, m\sqrt{q}, aq/\rho_1, aq/\rho_2; q)_n} \left(\frac{kq}{a} \right)^n,$$

$$(41) \quad \beta'_n(a, k; q) = \frac{(1 + \sqrt{a})(k, k/m, mq/\rho_1, mq/\rho_2; q)_n}{2\sqrt{a} (q, mq, aq/\rho_1, aq/\rho_2; q)_n} \\ \times \sum_{r=0}^n \frac{(q\sqrt{m}, -q\sqrt{m}, \rho_1, \rho_2, q^{-n}, kq^n, m, \sqrt{q}, m/\sqrt{a}, m\sqrt{q}/\sqrt{a}; q)_r}{(\sqrt{m}, -\sqrt{m}, mq/\rho_1, mq/\rho_2, mq^{1-n}/k, mq^{n+1}, q, m\sqrt{q}, \sqrt{aq}, q\sqrt{a}; q)_r} q^r \\ - \frac{(1 - \sqrt{a})(k, k/m, mq/\rho_1, mq/\rho_2; q)_n}{2\sqrt{a} (q, mq, aq/\rho_1, aq/\rho_2; q)_n} \\ \times \sum_{r=0}^n \frac{(q\sqrt{m}, -q\sqrt{m}, \rho_1, \rho_2, q^{-n}, kq^n, m, m/a, \sqrt{q}, m/\sqrt{a}, -m\sqrt{q}/\sqrt{a}; q)_r}{(\sqrt{m}, -\sqrt{m}, mq/\rho_1, mq/\rho_2, mq^{1-n}/k, mq^{n+1}, q, m\sqrt{q}, -m/a, \sqrt{aq}, -q\sqrt{a}; q)_r} q^r,$$

where $m = k\rho_1\rho_2/aq$. \square

Proof of (40)-(41). Let us choose

$$(42) \quad \alpha_r(a, k; q) = \frac{(a, a\sqrt{q}/k; q)_r}{(q, k\sqrt{q}; q)_r} \left(\frac{kq}{a}\right)^r$$

in (5) and by using (23), we obtain

$$(43) \quad \beta_n(a, k; q) = \frac{(1 + \sqrt{a})}{2\sqrt{a}} \frac{(k, \sqrt{q}, k/\sqrt{a}, k\sqrt{q}/\sqrt{a}; q)_n}{(q, k\sqrt{q}, \sqrt{aq}, q\sqrt{a}; q)_n} \\ - \frac{(1 - \sqrt{a})}{2\sqrt{a}} \frac{(k, k/a, \sqrt{q}, k/\sqrt{a}, -k\sqrt{q}/\sqrt{a}; q)_n}{(q, k\sqrt{q}, -k/a, \sqrt{aq}, -q\sqrt{a}; q)_n}.$$

We obtain new WP-Bailey pair (42) and (43), use (42), (43) in (6) and (7), we get (40) and (41).

$$(44) \quad \alpha'_n(a, k; q) = \frac{(a, a\sqrt{q}/m; q)_n (m; q)_{2n}}{(q, m\sqrt{q}; q)_n (k; q)_{2n}} \left(\frac{kq}{a}\right)^n,$$

(45)

$$\beta'_n(a, k; q) \\ = \frac{(1 + \sqrt{a})(k/m; q)_n}{2\sqrt{a}} \sum_{r=0}^n \frac{(m, \sqrt{q}, m/\sqrt{a}, m\sqrt{q}/\sqrt{a}, q^{-n}; q)_r}{(q, m\sqrt{q}, \sqrt{aq}, q\sqrt{a}, mq^{1-n}/k; q)_r} q^r \\ - \frac{(1 - \sqrt{a})(k/m; q)_n}{2\sqrt{a}} \sum_{r=0}^n \frac{(m, m/a, \sqrt{q}, m/\sqrt{a}, -m\sqrt{q}/\sqrt{a}, q^{-n}; q)_r}{(q, m\sqrt{q}, -m/a, \sqrt{aq}, -q\sqrt{a}, mq^{1-n}/k; q)_r} q^r,$$

where $m = qa^2/k$. \square

Proof of (44)-(45). By making the use (42), (43) in (8) and (9), we obtain (44), (45).

$$(46) \quad \alpha'_n(a, k; q) = \frac{(qa^2/k; q)_{2n}}{(k; q)_{2n}} \frac{(a, k/a\sqrt{q}; q)_n}{(q, a^2q^{3/2}/k; q)_n} \left(\frac{kq}{a}\right)^n,$$

(47)

$$\beta'_n(a, k; q) = \frac{(1 + \sqrt{a})(k^2/qa^2; q)_n}{2\sqrt{a}} \sum_{r=0}^n \frac{(q^{-n}, \sqrt{q}, qa^2/k, a^{3/2}q/k, a^{3/2}q^{3/2}/k; q)_r}{(q, q\sqrt{a}, a^2q^{3/2}/k, \sqrt{aq}, a^2q^{2-n}/k^2; q)_r} q^r \\ - \frac{(1 - \sqrt{a})(k^2/qa^2; q)_n}{2\sqrt{a}} \sum_{r=0}^n \frac{(q^{-n}, \sqrt{q}, aq/k, qa^2/k, a^{3/2}q/k, -a^{3/2}q^{3/2}/k; q)_r}{(q, -q\sqrt{a}, -aq/k, a^2q^{3/2}/k, \sqrt{aq}, a^2q^{2-n}/k^2; q)_r} q^r.$$

\square

Proof of (46)-(47). By using (42), (43) in (10) and (11), we deduce (46) and (47).

$$(48) \quad \alpha'_n(a, k; q) = \frac{(1 - \sigma\sqrt{k})(1 + \sigma\sqrt{m}q^n)(a, a\sqrt{q}/m; q)_n (m; q)_{2n}}{(1 - \sigma q^n \sqrt{k})(1 + \sigma\sqrt{m})(q, m\sqrt{q}; q)_n (k; q)_{2n}} \left(\frac{kq}{a}\right)^n,$$

(49)

$$\begin{aligned} \beta'_n(a, k; q) &= \frac{(1 + \sqrt{a})(\sigma\sqrt{k}, k/m; q)_n}{2\sqrt{a}(\sigma q\sqrt{k}, q; q)_n} \\ &\times \sum_{r=0}^n \frac{(m, \sqrt{q}, m/\sqrt{a}, m\sqrt{q}/\sqrt{a}, -\sigma q\sqrt{m}, q^{-n}; q)_r q^r}{(q, m\sqrt{q}, \sqrt{aq}, q\sqrt{a}, -\sigma\sqrt{m}, mq^{1-n}/k; q)_r} \\ &- \frac{(1 - \sigma\sqrt{k})(1 - \sqrt{a})(k/m; q)_n}{2\sqrt{a}(1 - \sigma q^n\sqrt{k})(q; q)_n} \\ &\times \sum_{r=0}^n \frac{(m, m/a, \sqrt{q}, m/\sqrt{a}, -m\sqrt{q}/\sqrt{a}, -\sigma q\sqrt{m}, q^{-n}; q)_r q^r}{(q, m\sqrt{q}, -m/a, \sqrt{aq}, -q\sqrt{a}, -\sigma\sqrt{m}, mq^{1-n}/k; q)_r}, \end{aligned}$$

where $m = a^2/k, \sigma \in (-1, 1)$. \square

Proof of (48)-(49). Making the use (42), (43) in (12) and (13), we obtain (48) and (49).

$$(50) \quad \alpha'_n(a^2, k; q^2) = \frac{(a, a\sqrt{q}/m; q)_n}{(q, m\sqrt{q}; q)_n} \left(\frac{mq}{a}\right)^n,$$

(51)

$$\begin{aligned} \beta'_n(a^2, k; q^2) &= \frac{(1 + \sqrt{a})(-mq, -mq^2, k/m^2, k; q^2)_n}{2\sqrt{a}(-aq, -aq^2, q^2, m^2q^2; q^2)_n} \left(\frac{m}{a}\right)^n \\ &\times \sum_{r=0}^n \frac{(m, \sqrt{q}, m/\sqrt{a}, m\sqrt{q}/\sqrt{a}, q\sqrt{m}, -q\sqrt{m}, q^{-n}, -q^{-n}, q^n\sqrt{k}, -q^n\sqrt{k}; q)_r}{(q, m\sqrt{q}, \sqrt{aq}, q\sqrt{a}, \sqrt{m}, -\sqrt{m}, mq^{1-n}/\sqrt{k}, -mq^{1-n}/\sqrt{k}, mq^{1+n}, -mq^{1+n}; q)_r} \\ &\times \left(\frac{amq^2}{k}\right)^r - \frac{(1 - \sqrt{a})(-mq, -mq^2, k/m^2, k; q^2)_n}{2\sqrt{a}(-aq, -aq^2, q^2, m^2q^2; q^2)_n} \left(\frac{m}{a}\right)^n \\ &\times \sum_{r=0}^n \frac{(m, m/a, m/\sqrt{a}, \sqrt{q}, -m\sqrt{q}/\sqrt{a}, q\sqrt{m}, -q\sqrt{m}, q^{-n}, -q^{-n}, q^n\sqrt{k}, -q^n\sqrt{k}; q)_r}{(q, m\sqrt{q}, -m/a, \sqrt{aq}, -q\sqrt{a}, \sqrt{m}, -\sqrt{m}, mq^{1-n}/\sqrt{k}, -mq^{1-n}/\sqrt{k}, mq^{1+n}, -mq^{1+n}; q)_r} \left(\frac{amq^2}{k}\right)^r, \end{aligned}$$

where $m = k/aq$. \square

Proof of (50)-(51). Using (42), (43) in (14) and (15), we get (50), (51).

$$(52) \quad \alpha'_n(a^2, k; q^2) = \frac{(a, a\sqrt{q}/m, iq\sqrt{a}, -iq\sqrt{a}; q)_n}{(q, m\sqrt{q}, i\sqrt{a}, -i\sqrt{a}; q)_n} \left(\frac{m}{a}\right)^n,$$

(53)

$$\begin{aligned} \beta'_n(a^2, k; q^2) &= \frac{(1 + \sqrt{a})(-mq, -mq^2, k/m^2, k; q^2)_n}{2\sqrt{a}(-a, -aq, q^2, m^2q^2; q^2)_n} \left(\frac{m}{aq}\right)^n \\ &\times \sum_{r=0}^n \frac{(m, \sqrt{q}, m/\sqrt{a}, m\sqrt{q}/\sqrt{a}, q\sqrt{m}, -q\sqrt{m}, q^{-n}, -q^{-n}, q^n\sqrt{k}, -q^n\sqrt{k}; q)_r}{(q, m\sqrt{q}, \sqrt{aq}, q\sqrt{a}, \sqrt{m}, -\sqrt{m}, mq^{1-n}/\sqrt{k}, -mq^{1-n}/\sqrt{k}, mq^{1+n}, -mq^{1+n}; q)_r} \left(\frac{amq^2}{k}\right)^r \\ &- \frac{(1 - \sqrt{a})(-mq, -mq^2, k/m^2, k; q^2)_n}{2\sqrt{a}(-a, -aq, q^2, m^2q^2; q^2)_n} \left(\frac{m}{aq}\right)^n \\ &\times \sum_{r=0}^n \frac{(m, m/a, \sqrt{q}, m/\sqrt{a}, -m\sqrt{q}/\sqrt{a}, q\sqrt{m}, -q\sqrt{m}, q^{-n}, -q^{-n}, q^n\sqrt{k}, -q^n\sqrt{k}; q)_r}{(q, -m/a, m\sqrt{q}, \sqrt{aq}, -q\sqrt{a}, \sqrt{m}, -\sqrt{m}, mq^{1-n}/\sqrt{k}, -mq^{1-n}/\sqrt{k}, mq^{1+n}, -mq^{1+n}; q)_r} \left(\frac{amq^2}{k}\right)^r, \end{aligned}$$

where $m = k/a$. \square

Proof of (52)-(53). By using (42), (43) in (16) and (17), we obtain (52) and (53).

$$(54) \quad \alpha'_{2n}(a, k; q) = \frac{(a, aq/m; q^2)_n}{(q^2, mq; q^2)_n} \left(\frac{mq^2}{a}\right)^n; \quad \alpha'_{2n+1}(a, k; q) = 0,$$

(55)

$$\begin{aligned} \beta'_n(a, k; q) &= \frac{(1 + \sqrt{a})(mq; q^2)_n}{2\sqrt{a}(aq; q^2)_n} \frac{(k, k/m; q)_n}{(q, mq; q)_n} \left(\frac{-k}{a}\right)^n \\ &\quad \times \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{(1 - mq^{4r})(q^{-n}, q^{-n+1}, kq^n, kq^{n+1}, m, q, m/\sqrt{a}, mq/\sqrt{a}; q^2)_r}{(1 - m)(mq^{1-n}/k, mq^{2-n}/k, mq^{n+1}, mq^{n+2}, q^2, mq, q\sqrt{a}, q^2\sqrt{a}; q^2)_r} \left(\frac{amq}{k^2}\right)^{2r} \\ &\quad - \frac{(1 - \sqrt{a})(mq; q^2)_n}{2\sqrt{a}(aq; q^2)_n} \frac{(k, k/m; q)_n}{(q, mq; q)_n} \left(\frac{-k}{a}\right)^n \sum_{r=0}^{\lfloor n/2 \rfloor} \frac{(1 - mq^{4r})}{(1 - m)} \\ &\quad \times \frac{(q^{-n}, q^{-n+1}, kq^n, kq^{n+1}, m, m/a, q, m/\sqrt{a}, -mq/\sqrt{a}; q^2)_r}{(mq^{1-n}/k, mq^{2-n}/k, mq^{n+1}, mq^{n+2}, q^2, mq, -m/a, q\sqrt{a}, -q^2\sqrt{a}; q^2)_r} \left(\frac{amq}{k^2}\right)^{2r}, \end{aligned}$$

where $m = k^2/a$. \square

Proof of (54)-(55). By making the use (42), (43) in (18) and (19), we get (54) and (55). \square

3. Applications

As an application of the new WP-Bailey pairs established in the previous section, we obtain a number of basic hypergeometric series identities in this section. If we use the WP-Bailey pairs (24)-(55) in (5), after some simplification, we obtain the following presumably new transformations.

$$\begin{aligned} (56) \quad & {}_8\varphi_7(q\sqrt{m}, -q\sqrt{m}, \rho_1, \rho_2, q^{-n}, kq^n, m, m/\sqrt{a}; \sqrt{m}, -\sqrt{m}, mq/\rho_1, mq/\rho_2, \\ & \quad mq^{1-n}/k, mq^{n+1}, q\sqrt{a}; q, q/\sqrt{a}) \\ &= \frac{(k/a, mq, aq/\rho_1, aq/\rho_2; q)_n}{(aq, mq/\rho_1, mq/\rho_2, k/m; q)_n} {}_6\varphi_5(a, -q\sqrt{a}, \rho_1, \rho_2, kq^n, q^{-n}; -\sqrt{a}, \\ & \quad aq/\rho_1, aq/\rho_2, aq^{1-n}/k, aq^{n+1}; q, q\sqrt{a}/m), \end{aligned}$$

where $m = k\rho_1\rho_2/aq$, $|q/\sqrt{a}| < 1$ and $|q\sqrt{a}/m| < 1$.

$$\begin{aligned} (57) \quad & {}_8\varphi_7(a, -q\sqrt{a}, \sqrt{m}, -\sqrt{m}, \sqrt{mq}, -\sqrt{mq}, kq^n, q^{-n}; -\sqrt{a}, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, \\ & \quad -\sqrt{kq}, aq^{1-n}/k, aq^{1+n}; q, q\sqrt{a}/m) \\ &= \frac{(aq, k/m; q)_n}{(k/a, k; q)_n} {}_3\varphi_2(m, m/\sqrt{a}, q^{-n}; q\sqrt{a}, mq^{1-n}/k; q, q/\sqrt{a}), \end{aligned}$$

where $m = qa^2/k$, $|q\sqrt{a}/m| < 1$ and $|q/\sqrt{a}| < 1$.

$$(58) \quad {}_8\varphi_7(a, -q\sqrt{a}, q^{-n}, kq^n, a\sqrt{q/k}, -a\sqrt{q/k}, aq/\sqrt{k}, -aq/\sqrt{k}; -\sqrt{a}, aq^{n+1}, \\ aq^{1-n}/k, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}; q, k/a^{3/2})$$

$$= \frac{(aq, k^2/qa^2; q)_n}{(k/a, k; q)_n} {}_3\varphi_2(qa^2/k, qa^{3/2}/k, q^{-n}; q\sqrt{a}, a^2q^{2-n}/k^2; q, q/\sqrt{a}),$$

$|k/a^{3/2}| < 1$ and $|q/\sqrt{a}| < 1$.

(59)

$$\begin{aligned} & {}_{10}\varphi_9(\sigma\sqrt{k}, -q\sigma\sqrt{m}, \sqrt{m}, -\sqrt{m}, \sqrt{mq}, -\sqrt{mq}, a, -q\sqrt{a}, kq^n, q^{-n}; q\sigma\sqrt{k}, \\ & \quad -\sigma\sqrt{m}, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}, -\sqrt{a}, aq^{1-n}/k, aq^{n+1}; q, q\sqrt{a}/m) \\ &= \frac{(aq, k/m, \sigma\sqrt{k}; q)_n}{(k/a, k, \sigma q\sqrt{k}; q)_n} \\ & \quad \times {}_4\varphi_3(m, m/\sqrt{a}, q^{-n}, -\sigma q\sqrt{m}; q\sqrt{a}, mq^{1-n}/k, -\sigma\sqrt{m}; q, q/\sqrt{a}), \end{aligned}$$

where $m = a^2/k$, $\sigma \in (-1, 1)$, $|q\sqrt{a}/m| < 1$ and $|q/\sqrt{a}| < 1$.

(60)

$$\begin{aligned} & {}_8\varphi_7(q\sqrt{m}, -q\sqrt{m}, q^{-n}, -q^{-n}, q^n\sqrt{k}, -q^n\sqrt{k}, m, m/\sqrt{a}; \sqrt{m}, -\sqrt{m}, q\sqrt{a}, \\ & \quad mq^{1-n}/\sqrt{k}, -mq^{1-n}/\sqrt{k}, mq^{1+n}, -mq^{1+n}; q, mq^2\sqrt{a}/k) \\ &= \frac{(m^2q^2, -aq, -aq^2, q^2; q^2)_n (k, k/a; q)_n}{(k, -mq, -mq^2, k/m^2; q^2)_n (q, aq; q)_n} \left(\frac{a}{m}\right)^n {}_6\varphi_5(a, -q\sqrt{a}, q^{-n}, -q^{-n}, \\ & \quad q^n\sqrt{k}, -q^n\sqrt{k}; -\sqrt{a}, aq^{1-n}/\sqrt{k}, -aq^{1-n}/\sqrt{k}, aq^{n+1}, -aq^{n+1}; q, q^2a^{3/2}/k), \end{aligned}$$

where $m = k/aq$, $|mq^2\sqrt{a}/k| < 1$ and $|q^2a^{3/2}/k| < 1$.

(61)

$$\begin{aligned} & {}_8\varphi_7(q\sqrt{m}, -q\sqrt{m}, m, m/\sqrt{a}, q^{-n}, -q^{-n}, q^n\sqrt{k}, -q^n\sqrt{k}; \sqrt{m}, -\sqrt{m}, \\ & \quad mq^{1-n}/\sqrt{k}, -mq^{1-n}/\sqrt{k}, mq^{1+n}, -mq^{1+n}, q\sqrt{a}; q, mq^2\sqrt{a}/k) \\ &= \frac{(m^2q^2, -aq, -a, q^2; q^2)_n (k, k/a; q)_n}{(k, -mq, -mq^2, k/m^2; q^2)_n (q, aq; q)_n} \left(\frac{aq}{m}\right)^n {}_8\varphi_7(iq\sqrt{a}, -iq\sqrt{a}, a, -q\sqrt{a}, \\ & \quad q^{-n}, -q^{-n}, q^n\sqrt{k}, -q^n\sqrt{k}; i\sqrt{a}, -i\sqrt{a}, -\sqrt{a}, aq^{1-n}/\sqrt{k}, -aq^{1-n}/\sqrt{k}, \\ & \quad aq^{n+1}, -aq^{n+1}; q, a^{3/2}q/k), \end{aligned}$$

where $m = k/a$, $|mq^2\sqrt{a}/k| < 1$ and $|a^{3/2}q/k| < 1$.

(62)

$$\begin{aligned} & \frac{(1+\sqrt{a})}{2\sqrt{a}} {}_{10}\varphi_9(q\sqrt{m}, -q\sqrt{m}, \rho_1, \rho_2, q^{-n}, kq^n, \sqrt{q}, m, m/\sqrt{a}, m\sqrt{q}/\sqrt{a}; \\ & \quad \sqrt{m}, -\sqrt{m}, mq/\rho_1, mq/\rho_2, mq^{1-n}/k, mq^{n+1}, m\sqrt{q}, \sqrt{aq}, q\sqrt{a}; q, q) \\ & \quad - \frac{(1-\sqrt{a})}{2\sqrt{a}} {}_{11}\varphi_{10}(q\sqrt{m}, -q\sqrt{m}, \rho_1, \rho_2, q^{-n}, kq^n, \sqrt{q}, m, m/a, m/\sqrt{a}, \\ & \quad -m\sqrt{q}/\sqrt{a}; \sqrt{m}, -\sqrt{m}, mq/\rho_1, mq/\rho_2, mq^{1-n}/k, mq^{n+1}, m\sqrt{q}, \sqrt{aq}), \end{aligned}$$

$$\begin{aligned}
& -m/a, -q\sqrt{a}; q; q) \\
&= \frac{(aq/\rho_1, aq/\rho_2, mq, k/a; q)_n}{(mq/\rho_1, mq/\rho_2, k/m, aq; q)_n} \\
& \quad \times {}_6\varphi_5(\rho_1, \rho_2, a, a\sqrt{q}/m, q^{-n}, kq^n; aq/\rho_1, aq/\rho_2, m\sqrt{q}, aq^{1-n}/k, aq^{n+1}; q, q^2),
\end{aligned}$$

where $m = k\sigma_1\sigma_2/aq$.

$$\begin{aligned}
(63) \quad & \frac{(1+\sqrt{a})}{2\sqrt{a}} {}_5\varphi_4(m, \sqrt{q}, q^{-n}, m/\sqrt{a}, m\sqrt{q}/\sqrt{a}; m\sqrt{q}, \sqrt{aq}, q\sqrt{a}, mq^{1-n}/k; q, q) \\
& - \frac{(1-\sqrt{a})}{2\sqrt{a}} {}_6\varphi_5(m, \sqrt{q}, m/a, q^{-n}, m/\sqrt{a}, -m\sqrt{q}/\sqrt{a}; m\sqrt{q}, -m/a, \\
& \quad \sqrt{aq}, -q\sqrt{a}, mq^{1-n}/k; q, q) \\
&= \frac{(k, k/a; q)_n}{(aq, k/m; q)_n} {}_8\varphi_7(\sqrt{m}, -\sqrt{m}, \sqrt{mq}, -\sqrt{mq}, a, a\sqrt{q}/m, kq^n, q^{-n}; \\
& \quad \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}, m\sqrt{q}, aq^{1+n}, aq^{1-n}/k; q, q^2),
\end{aligned}$$

where $m = a^2q/k$.

$$\begin{aligned}
(64) \quad & {}_8\varphi_7(a, q^{-n}, kq^n, k/a\sqrt{q}, a\sqrt{q}/k, -a\sqrt{q}/k, aq/\sqrt{k}, -aq/\sqrt{k}; aq^{1-n}/k, \\
& \quad aq^{1+n}, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}, a^2q^{3/2}/k; q, q^2) \\
&= \frac{(aq, k^2/qa^2; q)_n(1+\sqrt{a})}{(k, k/a; q)_n(2\sqrt{a})} {}_5\varphi_4(\sqrt{q}, qa^2/k, qa^{3/2}/k, q^{3/2}a^{3/2}/k, \\
& \quad q^{-n}; \sqrt{aq}, q\sqrt{a}, a^2q^{2-n}/k^2, a^2q^{3/2}/k; q, q) \\
& \quad - \frac{(aq, k^2/qa^2; q)_n(1-\sqrt{a})}{(k, k/a; q)_n(2\sqrt{a})} {}_6\varphi_5(\sqrt{q}, qa^2/k, aq/k, -q^{3/2}a^{3/2}/k, \\
& \quad qa^{3/2}/k, q^{-n}; -aq/k, \sqrt{aq}, -q\sqrt{a}, a^2q^{2-n}/k^2, a^2q^{3/2}/k; q, q),
\end{aligned}$$

$$\begin{aligned}
(65) \quad & \frac{(1+\sqrt{a})}{2\sqrt{a}} {}_6\varphi_5(-\sigma q\sqrt{m}, q^{-n}, m, \sqrt{q}, m/\sqrt{a}, m\sqrt{q}/\sqrt{a}; -\sigma\sqrt{m}, m\sqrt{q}, \\
& \quad \sqrt{aq}, q\sqrt{a}, mq^{1-n}/k; q, q) \\
& - \frac{(1-\sqrt{a})}{2\sqrt{a}} {}_7\varphi_6(-\sigma q\sqrt{m}, q^{-n}, m, m/a, \sqrt{q}, m/\sqrt{a}, -m\sqrt{q}/\sqrt{a}; -\sigma\sqrt{m}, \\
& \quad m\sqrt{q}, -m/a, \sqrt{aq}, -q\sqrt{a}, mq^{1-n}/k; q, q) \\
&= \frac{(\sigma q\sqrt{k}, k/a, k; q)_n}{(\sigma\sqrt{k}, k/m, aq; q)_n} {}_{10}\varphi_9(\sigma\sqrt{k}, -\sigma q\sqrt{m}, \sqrt{m}, -\sqrt{m}, \sqrt{mq}, -\sqrt{mq}, \\
& \quad a, a\sqrt{q}/m, q^{-n}, kq^n; \sigma q\sqrt{k}, -\sigma\sqrt{m}, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}, m\sqrt{q}, \\
& \quad aq^{1-n}/k, aq^{n+1}; q, q^2),
\end{aligned}$$

where $m = a^2/k, \sigma \in (-1, 1)$.

(66)

$$\begin{aligned} & \frac{(1 + \sqrt{a})}{2\sqrt{a}} {}_{10}\varphi_9(q\sqrt{m}, -q\sqrt{m}, q^{-n}, -q^n, q^n\sqrt{k}, -q^n\sqrt{k}, m, \sqrt{q}, m/\sqrt{a}, \\ & m\sqrt{q}/\sqrt{a}; \sqrt{m}, -\sqrt{m}, -mq^{1-n}/\sqrt{k}, mq^{1-n}/\sqrt{k}, mq^{n+1}, -mq^{n+1}, m\sqrt{q}, \\ & \sqrt{aq}, q\sqrt{a}; q, amq^2/k) \\ & - \frac{(1 - \sqrt{a})}{2\sqrt{a}} {}_{11}\varphi_{10}(q\sqrt{m}, -q\sqrt{m}, q^{-n}, -q^n, q^n\sqrt{k}, -q^n\sqrt{k}, m, m/a, m/\sqrt{a}, \\ & \sqrt{q}, -m\sqrt{q}/\sqrt{a}; \sqrt{m}, -\sqrt{m}, -mq^{1-n}/\sqrt{k}, mq^{1-n}/\sqrt{k}, mq^{n+1}, -mq^{n+1}, \\ & -m/a, m\sqrt{q}, \sqrt{aq}, -q\sqrt{a}; q, amq^2/k) \\ & = \frac{(-aq, -aq^2, q^2, m^2q^2; q^2)_n (k, k/a; q)_n}{(-mq, -mq^2, k/m^2, k; q^2)_n (q, aq; q)_n} \left(\frac{a}{m}\right)^n {}_6\varphi_5(a, a\sqrt{q}/m, q^{-n}, -q^{-n}, \\ & q^n\sqrt{k}, -q^n\sqrt{k}; m\sqrt{q}, aq^{1-n}/\sqrt{k}, -aq^{1-n}/\sqrt{k}, \\ & aq^{n+1}, -aq^{n+1}; q, amq^3/k), \end{aligned}$$

where $m = k/aq$ and $|amq^2/k| < 1$.

(67)

$$\begin{aligned} & \frac{(1 + \sqrt{a})}{2\sqrt{a}} {}_{10}\varphi_9(q\sqrt{m}, -q\sqrt{m}, q^{-n}, -q^n, q^n\sqrt{k}, -q^n\sqrt{k}, m, \sqrt{q}, m/\sqrt{a}, \\ & \sqrt{q}/\sqrt{a}; \sqrt{m}, -\sqrt{m}, -mq^{1-n}/\sqrt{k}, mq^{1-n}/\sqrt{k}, mq^{n+1}, -mq^{n+1}, m\sqrt{q}, \sqrt{aq}, \\ & q\sqrt{a}; q, amq^2/k) \\ & - \frac{(1 - \sqrt{a})}{2\sqrt{a}} {}_{11}\varphi_{10}(q\sqrt{m}, -q\sqrt{m}, q^{-n}, -q^n, q^n\sqrt{k}, -q^n\sqrt{k}, m, \sqrt{q}, m/a, \\ & m/\sqrt{a}, -m\sqrt{q}/\sqrt{a}; \sqrt{m}, -\sqrt{m}, -mq^{1-n}/\sqrt{k}, mq^{1-n}/\sqrt{k}, mq^{n+1}, -mq^{n+1}, \\ & -m/a, m\sqrt{q}, \sqrt{aq}, -q\sqrt{a}; q, amq^2/k) \\ & = \frac{(-a, -aq, q^2, m^2q^2; q^2)_n (k, k/a; q)_n}{(-mq, -mq^2, k/m^2, k; q^2)_n (q, aq; q)_n} \left(\frac{aq}{m}\right)^n {}_8\varphi_7(iq\sqrt{a}, -iq\sqrt{a}, a, a\sqrt{q}/m, \\ & q^{-n}, -q^{-n}, q^n\sqrt{k}, -q^n\sqrt{k}; i\sqrt{a}, -i\sqrt{a}, m\sqrt{q}, \\ & aq^{1-n}/\sqrt{k}, -aq^{1-n}/\sqrt{k}, aq^{n+1}, -aq^{n+1}; q, amq^2/k), \end{aligned}$$

where $m = k/a$ and $|amq^2/k| < 1$.

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