

A GENERALIZATION OF THE PRIME RADICAL OF IDEALS IN COMMUTATIVE RINGS

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ABSTRACT. Let R be a commutative ring with identity, and $\phi : \mathcal{I}(R) \rightarrow \mathcal{I}(R) \cup \{\emptyset\}$ be a function where $\mathcal{I}(R)$ is the set of all ideals of R . Following [2], a proper ideal P of R is called a ϕ -prime ideal if $x, y \in R$ with $xy \in P - \phi(P)$ implies $x \in P$ or $y \in P$. For an ideal I of R , we define the ϕ -radical $\sqrt[\phi]{I}$ to be the intersection of all ϕ -prime ideals of R containing I , and show that this notion inherits most of the essential properties of the usual notion of radical of an ideal. We also investigate when the set of all ϕ -prime ideals of R , denoted $\text{Spec}_\phi(R)$, has a Zariski topology analogous to that of the prime spectrum $\text{Spec}(R)$, and show that this topological space is Noetherian if and only if ϕ -radical ideals of R satisfy the ascending chain condition.

1. Introduction

Throughout this paper all rings are commutative with identity and all modules are unitary. Let R be a ring. We denote the set of all ideals (resp. proper ideals) of R by $\mathcal{I}(R)$ (resp. $\mathcal{I}^*(R)$). Anderson and Smith [3], defined a weakly prime ideal, i.e., a proper ideal P of R with the property that for $a, b \in R$, $0 \neq ab \in P$ implies $a \in P$ or $b \in P$. Weakly prime elements were introduced by Galovich in [7], and used by Agargün et al. [1], to study the unique factorization in rings with zero-divisors. In studying unique factorization domains, Bhatwadekar and Sharma [5] defined the notion of almost prime ideal, i.e., an ideal $P \in \mathcal{I}^*(R)$ with the property that if $a, b \in R$, $ab \in P - P^2$, then either $a \in P$ or $b \in P$. Thus a weakly prime ideal is almost prime and any proper idempotent ideal is also almost prime. Moreover, an ideal P of R is almost prime if and only if P/P^2 is a weakly prime ideal of R/P^2 . Anderson and Bataineh in [2], extended these concepts to ϕ -prime ideals. Let $\phi : \mathcal{I}(R) \rightarrow \mathcal{I}(R) \cup \{\emptyset\}$ be a function. A proper ideal P of R is called ϕ -prime if for $x, y \in R$, $xy \in P - \phi(P)$ implies $x \in P$ or $y \in P$. In fact, P is a ϕ -prime ideal of R if and only if $P/\phi(P)$ is a weakly prime ideal of $R/\phi(P)$.

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Since $P - \phi(P) = P - (P \cap \phi(P))$, there is no loss of generality in assuming that $\phi(P) \subseteq P$. We henceforth make this assumption. Given two functions $\psi_1, \psi_2 : \mathcal{I}(R) \rightarrow \mathcal{I}(R) \cup \{\emptyset\}$, we define $\psi_1 \leq \psi_2$ if $\psi_1(I) \subseteq \psi_2(I)$ for each $I \in \mathcal{I}(R)$.

For a ring R , we consider the following functions $\phi_\alpha : \mathcal{I}(R) \rightarrow \mathcal{I}(R) \cup \{\emptyset\}$ and the corresponding ϕ_α -prime ideals which were introduced in [2] and we will refer to these frequently:

ϕ_\emptyset	$\phi(P) = \emptyset$	prime ideal
ϕ_0	$\phi(P) = 0$	weakly prime ideal
ϕ_2	$\phi(P) = P^2$	almost prime ideal
ϕ_n ($n \geq 2$)	$\phi(P) = P^n$	n -almost prime ideal
ϕ_ω	$\phi(P) = \bigcap_{n=1}^{\infty} P^n$	ω -prime ideal
ϕ_1	$\phi(P) = P$	any ideal.

Observe that

$$(*) \quad \phi_\emptyset \leq \phi_0 \leq \phi_\omega \leq \dots \leq \phi_n \leq \dots \leq \phi_2 \leq \phi_1.$$

In the rest of this paper, the set of these functions is denoted by \mathcal{A} , that is $\mathcal{A} = \{\phi_\emptyset, \phi_0, \phi_\omega, \dots, \phi_n, \dots, \phi_2, \phi_1\}$, and $\mathcal{A}^* = \mathcal{A} - \{\phi_1\}$. Let $\phi : \mathcal{I}(R) \rightarrow \mathcal{I}(R) \cup \{\emptyset\}$ be a function. We define the ϕ -radical of an ideal I , denoted by $\sqrt[\phi]{I}$, to be the intersection of all ϕ -prime ideals of R containing I . When $\phi = \phi_\emptyset$, we use \sqrt{I} instead of $\sqrt[\phi_\emptyset]{I}$. It follows from (*) that:

$$\sqrt{I} \supseteq \sqrt[\phi_0]{I} \supseteq \sqrt[\phi_\omega]{I} \supseteq \dots \supseteq \sqrt[\phi_n]{I} \supseteq \sqrt[\phi_2]{I} \supseteq \dots \supseteq \sqrt[\phi_1]{I} = I.$$

If $\phi, \psi : \mathcal{I}(R) \rightarrow \mathcal{I}(R) \cup \{\emptyset\}$ are two functions such that $\phi \leq \psi$, then $\sqrt[\psi]{\sqrt[\phi]{I}} = \sqrt[\psi]{\sqrt[\psi]{I}} = \sqrt[\psi]{I}$ (Theorem 2.5). It is shown that if R is a Noetherian integral domain and $I \in \mathcal{I}^*(R)$, then $\sqrt[\phi]{I} = \sqrt{I}$ for all $\phi : \mathcal{I}(R) \rightarrow \mathcal{I}(R) \cup \{\emptyset\}$ with $\phi \leq \phi_3$ (Proposition 2.12). In particular, if R is a PID, then $\sqrt[\phi]{I} = \sqrt{I}$ for all $\phi : \mathcal{I}(R) \rightarrow \mathcal{I}(R) \cup \{\emptyset\}$ with $\phi \leq \phi_2$ (Proposition 2.14).

The set of all ϕ -prime ideals of R is called the ϕ -prime spectrum of R and denoted by $\text{Spec}_\phi(R)$ or simply X_ϕ . Now, by (*), we have:

$$X_{\phi_\emptyset} \subseteq X_{\phi_0} \subseteq X_{\phi_\omega} \subseteq \dots \subseteq X_{\phi_{n+1}} \subseteq X_{\phi_n} \subseteq \dots \subseteq X_{\phi_2} \subseteq X_{\phi_1} = \mathcal{I}^*(R).$$

In particular, if $\phi = \phi_\emptyset$, then $\text{Spec}_\phi(R) = \text{Spec}(R)$ and if $\phi = \phi_1$, then $\text{Spec}_\phi(R) = \mathcal{I}^*(R)$. For any ideal I of R we define $V_\phi(I)$ to be the set of all ϕ -prime ideals of R containing I . Of course, $V_\phi(R)$ is just the empty set and $V_\phi(0)$ is X_ϕ . Note that for any family of ideals $\{I_\gamma \mid \gamma \in \Gamma\}$ of R , $\bigcap_{\gamma \in \Gamma} V_\phi(I_\gamma) = V_\phi(\sum_{\gamma \in \Gamma} I_\gamma)$. Thus if $\zeta_\phi(R)$ denotes the collections of all subsets $V_\phi(I)$ of X_ϕ ,

then $\zeta_\phi(R)$ contains the empty set and X_ϕ , and $\zeta_\phi(R)$ is closed under arbitrary intersections. We shall say R is a ring with a ϕ -Zariski topology, or a ϕ -top ring for short, if $\zeta_\phi(R)$ is closed under finite union, for in this case $\zeta_\phi(R)$ satisfies the

axioms for the closed sets of a topological space. In this paper, we investigate the behaviour of ϕ -top rings under the idealization of a module and finite direct products. In particular, we study this topological space from the point of view of Noetherian spaces. It is shown that, for a ϕ -top ring R , X_ϕ with ϕ -Zariski topology is a Noetherian space if and only if ascending chain condition holds for ϕ -radical ideals of R (Theorem 3.10).

2. ϕ -radical of ideals

Let R be a ring and I be an ideal of R . It is well-known that the radical of I is the intersection of all prime ideals of R containing I and characterized as the set of all $a \in R$ for which $a^n \in I$ for some positive integer n . A natural generalization of this notion is the following.

Definition. Let R be a ring, I an ideal of R and $\phi : \mathcal{S}(R) \rightarrow \mathcal{S}(R) \cup \{\emptyset\}$ be a function. The ϕ -radical of I , denoted by ${}^\phi\sqrt{I}$, is defined to be the intersection of all ϕ -prime ideals of R containing I . In other words,

$${}^\phi\sqrt{I} = \bigcap \{P \in \text{Spec}_\phi(R) : P \supseteq I\}.$$

Moreover, I is called a ϕ -radical ideal if ${}^\phi\sqrt{I} = I$.

We note that, by definition, an ideal I of R is a ϕ_\emptyset -radical ideal if and only if I is a radical ideal.

Example 2.1. In this example we compute and compare the ϕ -radical of some ideals for some $\phi \in \mathcal{A}$.

- (1) Let $R = \mathbb{Z}_4$, $0 = (\bar{0})$, and $P = (\bar{2})$. Then

$$\sqrt{0} = {}^\phi\sqrt{0} = P \supseteq 0 = {}^\phi\sqrt{0}.$$

This also shows that 0 is a ϕ_0 -radical ideal of R which is not radical.

- (2) Let $R = \mathbb{Z}_{12}$, $0 = (\bar{0})$, $P_1 = (\bar{2})$, $P_2 = (\bar{3})$, $I = (\bar{4})$, and $J = (\bar{6})$. Then $\mathcal{S}^*(R) = \{0, I, J, P_1, P_2\}$, $X = X_{\phi_\emptyset} = \{P_1, P_2\}$, $X_{\phi_0} = \{0, P_1, P_2\}$ and $X_{\phi_2} = \{0, P_1, P_2, I\}$. Hence we have

$${}^{\phi_2}\sqrt{0} = {}^\phi\sqrt{0} = 0 \subsetneq \sqrt{0} = P_1 \cap P_2 = J, \quad {}^{\phi_2}\sqrt{I} = I \subsetneq {}^\phi\sqrt{I} = \sqrt{I} = P_1$$

and

$${}^{\phi_2}\sqrt{J} = {}^\phi\sqrt{J} = \sqrt{J} = P_1 \cap P_2 = J.$$

- (3) Let (R, \mathcal{M}) be a quasilocal ring with $\mathcal{M}^2 = 0$. Let $I \subset \mathcal{M}$. Then by [3, Example 12], the $\mathcal{M}[X]$ -primary ideal $I[X]$ of $R[X]$ is weakly prime. Hence we have

$$\mathcal{M}[X] = \sqrt{I[X]} \supset I[X] = {}^\phi\sqrt{I[X]}.$$

Thus $I[X]$ is a ϕ -radical ideal of $R[X]$ for all $\phi \in \mathcal{A} - \{\phi_\emptyset\}$.

- (4) Let S be a ring such that $\sqrt{0} \neq 0$, T a ring and $R = S \times T$. Let $I = 0 \times T$. Then ${}^\phi\sqrt{I} \subsetneq {}^\phi\sqrt{I}$. In fact, the weakly prime ideals of R containing I are exactly the prime ideals of R containing I [3, Theorem 7]. Hence ${}^\phi\sqrt{I} = \sqrt{I} = \sqrt{0} \times T \supsetneq 0 \times T = I$. On the other hand, since

0 is a weakly prime ideal, $I = 0 \times T$ is a ϕ_ω -prime ideal of R , by [2, Theorem 8] and hence ${}^\phi\sqrt{I} = I$.

Lemma 2.2. *Let R be a ring, $\phi : \mathcal{I}(R) \rightarrow \mathcal{I}(R) \cup \{\emptyset\}$ a function and $I, J \in \mathcal{I}(R)$. Then*

- (1) *If $I \subseteq J$, then ${}^\phi\sqrt{I} \subseteq {}^\phi\sqrt{J}$.*
- (2) *${}^\phi\sqrt{\bigcap_{\gamma \in \Gamma} I_\gamma} \subseteq \bigcap_{\gamma \in \Gamma} {}^\phi\sqrt{I_\gamma}$ of ideals of R .*
- (3) *${}^\phi\sqrt{I_1 I_2 \cdots I_n} \subseteq {}^\phi\sqrt{I_1 \cap I_2 \cap \cdots \cap I_n} \subseteq {}^\phi\sqrt{I_1} \cap {}^\phi\sqrt{I_2} \cap \cdots \cap {}^\phi\sqrt{I_n}$ for each finite set $\{I_1, \dots, I_n\}$ of ideals of R , and the equality holds if $I_1 I_2 \cdots I_n \not\subseteq \phi(P)$ for all ϕ -prime ideals P of R containing $I_1 I_2 \cdots I_n$.*
- (4) *${}^\phi\sqrt{{}^\phi\sqrt{I}} = {}^\phi\sqrt{I}$.*

Proof. (1) It is clear, since every ϕ -prime ideal P of R containing J contains also I . (2) It is a direct result of (1). (3) Given inclusions are clear by (1). Let P be a ϕ -prime ideal of R containing $I_1 I_2 \cdots I_n$. By assumption, $I_1 I_2 \cdots I_n \not\subseteq \phi(P)$ and hence $I_{i_1} I_{i_2} \cdots I_{i_j} \not\subseteq \phi(P)$ for all $1 \leq j \leq n$. Now [2, Theorem 13] gives the result. (4) Since $I \subseteq {}^\phi\sqrt{I}$, by (1), ${}^\phi\sqrt{I} \subseteq {}^\phi\sqrt{{}^\phi\sqrt{I}}$. The reverse containment follows from the fact that every ϕ -prime ideal of R containing I contains also ${}^\phi\sqrt{I}$. \square

Corollary 2.3. *Let R be a ring and $\phi : \mathcal{I}(R) \rightarrow \mathcal{I}(R) \cup \{\emptyset\}$ be a function. Then I is a ϕ -radical ideal of R if and only if I is an intersection of ϕ -prime ideals of R .*

Proof. (\Rightarrow) It follows from definition.

(\Leftarrow) Use Lemma 2.2 (2). \square

Corollary 2.4. *Let R be a ring, n a positive integer and $\phi : \mathcal{I}(R) \rightarrow \mathcal{I}(R) \cup \{\emptyset\}$ a function. If I is an ideal of R such that for every ϕ -prime ideal P of R containing I , $I^n \not\subseteq \phi(P)$, then ${}^\phi\sqrt{I^n} = {}^\phi\sqrt{I}$. In particular, ${}^\phi\sqrt{I^n} = 0$ or ${}^\phi\sqrt{I^n} = {}^\phi\sqrt{I}$.*

Proof. The first part follows from Lemma 2.2. For the ‘‘in particular’’ part, it is clear that if $I^n = 0$, then ${}^\phi\sqrt{I^n} = 0$, and if $I^n \neq 0$, then, by the first part, ${}^\phi\sqrt{I^n} = {}^\phi\sqrt{I}$. \square

Theorem 2.5. *Let R be a ring, $\phi, \psi : \mathcal{I}(R) \rightarrow \mathcal{I}(R) \cup \{\emptyset\}$ two functions such that $\phi \leq \psi$ and $I \in \mathcal{I}(R)$. Then*

- (1) *${}^\psi\sqrt{I} \subseteq {}^\phi\sqrt{I}$.*
- (2) *${}^\psi\sqrt{{}^\phi\sqrt{I}} = {}^\phi\sqrt{{}^\psi\sqrt{I}} = {}^\phi\sqrt{I}$. In particular, $\sqrt{{}^\psi\sqrt{I}} = {}^\psi\sqrt{\sqrt{I}} = \sqrt{I}$.*
- (3) *$\sqrt[\lambda_n]{\sqrt[\lambda_2]{\sqrt[\lambda_1]{I}}} = {}^\phi\sqrt{I}$ for $\lambda_1, \lambda_2, \dots, \lambda_n \in \{\phi, \psi\}$ and for $n \geq 2$.*
- (4) *If I is a ϕ -radical ideal of R , then it is ψ -radical.*

Proof. (1) Since $\phi \leq \psi$, every ϕ -prime ideal is a ψ -prime ideal. Thus the desired result is clear. (2) By (1), $\sqrt[\phi]{I} \subseteq \sqrt[\phi]{\sqrt[\psi]{I}} \subseteq \sqrt[\psi]{\sqrt[\phi]{I}}$. Hence, by Lemma 2.2(4), we have $\sqrt[\phi]{\sqrt[\psi]{I}} = \sqrt[\psi]{\sqrt[\phi]{I}}$. On the other hand, by (1) and Lemma 2.2(4), $\sqrt[\psi]{\sqrt[\phi]{I}} \subseteq \sqrt[\psi]{\sqrt[\psi]{I}} = \sqrt[\psi]{I} \subseteq \sqrt[\psi]{\sqrt[\phi]{I}}$. Thus we have the asserted equality. For “in particular” part, put $\phi = \phi_\emptyset$ in the first part. (3) It follows inductively by (2) and Lemma 2.2. (4) It follows directly from (2). \square

Corollary 2.6. *Let R be a ring, I and J ideals of R , and $\phi : \mathcal{S}(R) \rightarrow \mathcal{S}(R) \cup \{\emptyset\}$ a function. Then*

- (1) $\sqrt[\phi]{I} = R$ if and only if $I = R$.
- (2) $\sqrt[\phi]{I+J} = \sqrt[\phi]{\sqrt[\phi]{I} + \sqrt[\phi]{J}}$. In particular, $I+J = R$ if and only if $\sqrt[\phi]{I} + \sqrt[\phi]{J} = R$.

Proof. (1) (\Rightarrow) Since $\sqrt[\phi]{I} \subseteq \sqrt{I}$, $\sqrt[\phi]{I} = R$ implies that $\sqrt{I} = R$ and this gives that $I = R$. (\Leftarrow) Clear.

(2) Clearly $I+J \subseteq \sqrt[\phi]{I} + \sqrt[\phi]{J} \subseteq \sqrt[\phi]{I+J}$. Now taking ϕ -radical and using Lemma 2.2 give the result. \square

Let R be a ring and M be an R -module. Then $R_{(+)}M$ with coordinate-wise addition, and multiplication $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$ is a commutative ring with identity called the idealization of M . The prime ideals of $R_{(+)}M$ have the form $P_{(+)}M$ where P is a prime ideal of R [4, Theorem 3.2]. The homogeneous ideals of (the graded ring) $R_{(+)}M$ have the form $I_{(+)}N$, where I is an ideal of R , N is a submodule of M , and $IM \subseteq N$ [4, Theorem 3.3]. A ring $R_{(+)}M$ is called a homogeneous ring if every ideal of $R_{(+)}M$ is homogeneous.

Proposition 2.7. *Let R be a ring, M an R -module, and $R_{(+)}M$ a homogeneous ring. Let $\phi : \mathcal{S}(R) \rightarrow \mathcal{S}(R) \cup \{\emptyset\}$ and $\psi : \mathcal{S}(R_{(+)}M) \rightarrow \mathcal{S}(R_{(+)}M) \cup \{\emptyset\}$ be two functions such that $\psi(I_{(+)}N) = \phi(I)_{(+)}N$. Then*

- (1) If $Q = P_{(+)}N$ is a ψ -prime ideal of $R_{(+)}M$, then P is a ϕ -prime ideal of R .
- (2) P is a ϕ -prime ideal of R if and only if $P_{(+)}M$ is a ψ -prime ideal of $R_{(+)}M$.
- (3) $\sqrt[\psi]{I_{(+)}M} = \sqrt[\phi]{I}_{(+)}M$.

Proof. (1) Let $r_1r_2 \in P - \phi(P)$ for $r_1, r_2 \in R$. Then $(r_1, 0)(r_2, 0) \in Q - \psi(Q)$. Since Q is ψ -prime, $(r_1, 0) \in Q$ or $(r_2, 0) \in Q$ which implies that $r_1 \in P$ or $r_2 \in P$. Thus P is a ϕ prime ideal of R . (2) Let P be a ϕ -prime ideal of R , $Q = P_{(+)}M$ and let $(r_1, m_1)(r_2, m_2) \in Q - \psi(Q)$ for $r_1, r_2 \in R$ and $m_1, m_2 \in M$. Then $r_1r_2 \in P - \phi(P)$. Since P is ϕ -prime, $r_1 \in P$ or $r_2 \in P$. Thus $(r_1, m_1) \in Q$ or $(r_2, m_2) \in Q$. Hence Q is a ψ -prime ideal of $R_{(+)}M$. The converse follows from (1). (3) Let Q be a ψ -prime ideal of $R_{(+)}M$ containing $I_{(+)}M$. Since Q contains $0_{(+)}M$, $Q = P_{(+)}M$ where P is a ϕ -prime ideal of R containing I by (2). Hence $\sqrt[\psi]{I_{(+)}M} \subseteq \sqrt[\phi]{I}_{(+)}M$. Also, if P is a ϕ -prime

ideal of R containing I , then $P_{(+)}M$ is a ψ -prime ideal containing $I_{(+)}M$. This follows $\sqrt{I_{(+)}M} \subseteq \sqrt{I_{(+)}M}$. \square

Proposition 2.8. *Let R be a ring and $I \in \mathcal{I}^*(R)$. Then either $\sqrt{I} = \sqrt{I}$ or $(\sqrt{I})^2 \subseteq \phi(P)$ for some ϕ -prime ideal P of R containing I .*

Proof. If every ϕ -prime ideal of R containing I is prime, then clearly $\sqrt{I} = \sqrt{I}$. Now let P be a ϕ -prime ideal of R containing I which is not prime and let $x, y \in \sqrt{I}$. Then $x, y \in P$ and hence $xy \in P^2 \subseteq \phi(P)$, by [2, Theorem 5]. Thus $(\sqrt{I})^2 \subseteq \phi(P)$. \square

Corollary 2.9. *Let R be a ring, $I \in \mathcal{I}^*(R)$ and $\phi \in \mathcal{A}^*$. Then either $\sqrt{I} = \sqrt{I}$ or $(\sqrt{I})^2 \subseteq P^2$ for some ϕ_2 -prime ideal P of R containing I .*

Proof. Let $\sqrt{I} \neq \sqrt{I}$. Then there exists a ϕ -prime ideal P of R containing I which is not prime. Since $\phi \in \mathcal{A}^*$, P is a ϕ_2 -prime ideal, and hence by Proposition 2.8, we have

$$(\sqrt{I})^2 \subseteq \phi(P) \subseteq \phi_2(P) = P^2. \quad \square$$

Corollary 2.10. *Let R be a ring, and $I \in \mathcal{I}^*(R)$. Then either $\sqrt{I} = \sqrt{I}$ or $(\sqrt{I})^2 = 0$.*

Proof. It is obtained by considering $\phi = \phi_0$ in Proposition 2.8. \square

Proposition 2.11. *Let R be a ring, $I \in \mathcal{I}^*(R)$ and $\phi : \mathcal{I}(R) \rightarrow \mathcal{I}(R) \cup \{\emptyset\}$ a function such that $\phi_\omega \leq \phi \leq \phi_3$. Then $\sqrt{I} = \sqrt{I}$. In particular, if I is a ϕ -radical ideal, then I is ϕ_ω -radical.*

Proof. Since $\phi_\omega \leq \phi$, by Theorem 2.5, $\sqrt{I} \subseteq \sqrt{I}$. Let P be a ϕ -prime ideal of R containing I . Since $\phi \leq \phi_3$, by [2, Corollary 6], P is a ϕ_ω -prime ideal and so $\sqrt{I} \subseteq \sqrt{I}$. The ‘‘In particular’’ part follows immediately from the first part. \square

The next three propositions, which are easily obtained from some assertions of [2], provide a good supply of examples of ϕ -radical ideals for some $\phi \in \mathcal{A}$.

Proposition 2.12. *Let R be a Noetherian integral domain, $I \in \mathcal{I}^*(R)$ and $\phi : \mathcal{I}(R) \rightarrow \mathcal{I}(R) \cup \{\emptyset\}$ a function such that $\phi \leq \phi_3$. Then $\sqrt{I} = \sqrt{I}$. In particular, I is a radical ideal of R if and only if I is a ϕ -radical ideal of R .*

Proof. By [2, Corollary 10], P is a prime ideal of R if and only if P is a ϕ_3 -prime ideal of R . Hence we have $\sqrt{I} = \sqrt{I}$. Thus $\sqrt{I} = \sqrt{I}$ for all functions $\phi : \mathcal{I}(R) \rightarrow \mathcal{I}(R) \cup \{\emptyset\}$ with $\phi \leq \phi_3$, which also yields the ‘‘in particular’’ part. \square

Example 2.13. Let K be a field, $R = K[[X^3, X^4, X^5]]$, and $\mathcal{M} = (X^3, X^4, X^5)$. By [5, Example, p. 47], $I = (X^3, X^4)$ is an \mathcal{M} -primary ideal of R which is ϕ_2 -prime. On the other hand, as we show in Proposition 2.12, $\sqrt{I} = \sqrt{I}$ for all $\phi \in \mathcal{A} - \{\phi_1, \phi_2\}$. It follows that $\sqrt{I} = \mathcal{M} \supseteq I = \sqrt{I}$.

Proposition 2.14. *Let R be a PID, $I \in \mathcal{I}^*(R)$ and $\phi : \mathcal{I}(R) \rightarrow \mathcal{I}(R) \cup \{\emptyset\}$ a function such that $\phi \leq \phi_2$. Then $\sqrt[\phi]{I} = \sqrt{I}$. In particular, I is a radical ideal of R if and only if I is a ϕ -radical ideal of R .*

Proof. By [2, Theorem 12], P is a prime ideal of R if and only if P is ϕ -prime ideal of R , where $\phi \leq \phi_2$. Hence the result follows clearly. \square

Proposition 2.15. *Let R be a ring, $I \in \mathcal{I}^*(R)$ and $\phi : \mathcal{I}(R) \rightarrow \mathcal{I}(R) \cup \{\emptyset\}$ a function. If R is von Neumann regular or (R, \mathcal{M}) is quasilocal with $\mathcal{M}^2 = 0$, then I is a ϕ -radical ideal for each $\phi_\omega \leq \phi \leq \phi_2$.*

Proof. By [2, Corollary 18]. \square

3. ϕ -top rings

Let R be a ring and $\phi : \mathcal{I}(R) \rightarrow \mathcal{I}(R) \cup \{\emptyset\}$ a function. Recall that the ϕ -prime spectrum of R , denoted by $\text{Spec}_\phi(R)$ or simply X_ϕ , is the set of all ϕ -prime ideals of R . For an ideal I of R , let $V_\phi(I)$ denotes the set of all ϕ -prime ideals P of R such that $P \supseteq I$, i.e., $V_\phi(I) = \{P \in X_\phi : P \supseteq I\}$. When $\phi = \phi_\emptyset$, we use X and $V(I)$ instead of X_{ϕ_\emptyset} and $V_{\phi_\emptyset}(I)$, respectively. The following lemma collects some elementary facts about the sets $V_\phi(I)$.

Lemma 3.1. *With the above notations we have,*

- (1) $V_\phi(\emptyset) = X_\phi$ and $V_\phi(R) = \emptyset$.
- (2) $\bigcap_{\gamma \in \Gamma} V_\phi(I_\gamma) = V_\phi(\sum_{\gamma \in \Gamma} I_\gamma)$ for every family $I_\gamma (\gamma \in \Gamma)$ of ideals of R .
- (3) $V_\phi(I) \cup V_\phi(J) \subseteq V_\phi(I \cap J) \subseteq V_\phi(IJ)$ for any ideals I, J of R .
- (4) $V_\phi(I) = V_\phi(\sqrt[\phi]{I})$ for any ideal I of R .

Proof. Clear. \square

From the above lemma, we can easily see that there exists a topology, τ_ϕ say, on X_ϕ having the set $\zeta_\phi(R) = \{V_\phi(I_\gamma) \mid I_\gamma \text{ is an ideal of } R\}$ as the collection of all closed sets if and only if $\zeta_\phi(R)$ is closed under finite union. When this is the case, we call the ring R a ϕ -top ring. It is well-known that any ring R is a ϕ_\emptyset -top ring.

Example 3.2. (1) Let R be a ring which is an injective R -module, and $0 \neq \mathcal{M}$ be a maximal ideal of R with $\mathcal{M}^2 = 0$. By [8, Lemma 2.25], $0 \subsetneq \mathcal{M} \subsetneq R$ is the only ideal of R . Hence we have $X = V(\mathcal{M}) = V(0) = \{\mathcal{M}\}$. Moreover, for any $\phi \geq \phi_0$, we have $X_\phi = \{0, \mathcal{M}\} = \mathcal{I}^*(R)$, $V_\phi(0) = \mathcal{I}^*(R)$ and $V_\phi(\mathcal{M}) = \{\mathcal{M}\}$. Now, it is easily seen that R is a ϕ -top ring for all $\phi \geq \phi_0$ and hence R is a ϕ -top ring for each $\phi \in \mathcal{A}$.

(2) Let R be a valuation ring and $\phi : \mathcal{I}(R) \rightarrow \mathcal{I}(R) \cup \{\emptyset\}$ a function. It is well-known that the ideals of R are totally ordered by inclusion. Hence, for any ideals I and J of R , $V_\phi(I) \subseteq V_\phi(J)$ or $V_\phi(J) \subseteq V_\phi(I)$. Therefore, every valuation ring is a ϕ -top ring.

(3) Let $R = K[X, Y]/(X, Y)^2$, where K is a field. Then R is not a ϕ -top ring, for each $\phi \geq \phi_0$. To see this, assume the contrary. Let $\mathcal{M} = (X, Y)/(X, Y)^2$, $I = (\bar{X})$ and $J = (\bar{Y})$. Since $\mathcal{M}^2 = 0$, every proper ideal of R is ϕ_0 -prime and hence ϕ -prime. Since R is assumed to be ϕ -top, there exists $K \in \mathcal{S}^*(R)$ such that $V_\phi(I) \cup V_\phi(J) = V_\phi(K)$. This implies that $I, J \in V_\phi(K)$ and hence $I \supseteq K$ and $J \supseteq K$. On the other hand $K \in V_\phi(I)$ or $K \in V_\phi(J)$. Let $K \in V_\phi(I)$. Then $I \subseteq K$ which shows that $I \subseteq K \subseteq J$, a contradiction.

(4) Let R be an integral domain. It is clear that R is a ϕ_0 -top ring. Now, let I and J be two ideals of R such that $I \not\subseteq J$, $J \not\subseteq I$ and $I \cap J$ be an almost prime ideal of R . Then $I \cap J \in V_{\phi_2}(I \cap J)$ but $I \cap J \notin V_{\phi_2}(I)$ and $I \cap J \notin V_{\phi_2}(J)$ and hence $I \cap J \notin V_{\phi_2}(I) \cup V_{\phi_2}(J)$, that is, R is not a ϕ_2 -top ring.

Proposition 3.3. *Let R be a ring, $I, J \in \mathcal{S}(R)$ and $\phi : \mathcal{S}(R) \rightarrow \mathcal{S}(R) \cup \{\emptyset\}$ a function. If for every ϕ -prime ideal P of R containing IJ , $IJ \not\subseteq \phi(P)$, then R is a ϕ -top ring.*

Proof. Let $P \in V_\phi(IJ)$. Then, by [2, Theorem 13], $P \in V_\phi(I)$ or $P \in V_\phi(J)$. Thus, by the Lemma 3.1(3), $V_\phi(I) \cup V_\phi(J) = V_\phi(I \cap J) = V_\phi(IJ)$, that is, R is a ϕ -top ring. \square

Proposition 3.4. *Let R be a ring, $\phi : \mathcal{S}(R) \rightarrow \mathcal{S}(R) \cup \{\emptyset\}$ be a function and $P \in X_\phi - X$. Then $V(\phi(P)) = V(P)$.*

Proof. Since $\phi(P) \subseteq P$, we have $V(P) \subseteq V(\phi(P))$. On the other hand, since P is not a prime ideal of R , by [2, Theorem 5], $P^2 \subseteq \phi(P)$. Now, if $Q \in V(\phi(P))$, then $P^2 \subseteq Q$; hence $Q \supseteq P$. So $V(\phi(P)) \subseteq V(P)$. \square

Corollary 3.5. *Let R be a ring and $P \in X_{\phi_0} - X$. Then $V(P) = V(0)$.*

Proposition 3.6. *Let R be a ring and $\phi, \psi : \mathcal{S}(R) \rightarrow \mathcal{S}(R) \cup \{\emptyset\}$ be two functions such that $\psi \leq \phi$. If R is a ϕ -top ring, then R is a ψ -top ring.*

Proof. For any ideal I of R we have $V_\psi(I) = V_\phi(I) \cap X_\psi$. Now let I_1 and I_2 be two ideals of R . Then $V_\psi(I_1) \cup V_\psi(I_2) = (V_\phi(I_1) \cap X_\psi) \cup (V_\phi(I_2) \cap X_\psi) = (V_\phi(I_1) \cup V_\phi(I_2)) \cap X_\psi$. Since R is a ϕ -top ring, there exists an ideal K of R such that $V_\phi(I_1) \cup V_\phi(I_2) = V_\phi(K)$. Thus $V_\psi(I_1) \cup V_\psi(I_2) = V_\phi(K) \cap X_\psi = V_\psi(K)$. Therefore R is a ψ -top ring. \square

Theorem 3.7. *Let R be a ring and $\phi : \mathcal{S}(R) \rightarrow \mathcal{S}(R) \cup \{\emptyset\}$ a function. Then R is a ϕ -top ring if and only if $V_\phi(I) \cup V_\phi(J) = V_\phi(I \cap J)$ for any ϕ -radical ideals I and J of R .*

Proof. (\Rightarrow) Let P be any ϕ -prime ideal of R and let I and J be ϕ -radical ideals of R such that $I \cap J \subseteq P$. Since R is a ϕ -top ring, there exists an ideal K of R such that $V_\phi(I) \cup V_\phi(J) = V_\phi(K)$. Now, since I is assumed to be ϕ -radical, we have $I = \bigcap_{\gamma \in \Gamma} P_\gamma$ for some ϕ -prime ideals P_γ ($\gamma \in \Gamma$) of R . So $P_\gamma \in V_\phi(I) \subseteq V_\phi(K)$ (for each $\gamma \in \Gamma$) and hence $K \subseteq P_\gamma$ (for each $\gamma \in \Gamma$) which shows that $K \subseteq \bigcap_{\gamma \in \Gamma} P_\gamma = I$. Similarly, $K \subseteq J$. Thus $K \subseteq I \cap J$ and

therefore $V_\phi(I) \cup V_\phi(J) \subseteq V_\phi(I \cap J) \subseteq V_\phi(K) = V_\phi(I) \cup V_\phi(J)$ which implies that $V_\phi(I) \cup V_\phi(J) = V_\phi(I \cap J)$. (\Leftarrow) Let I and J be ideals of R . Then, by Lemma 3.1 and hypothesis, we have $V_\phi(I) \cup V_\phi(J) = V_\phi(\sqrt[\phi]{I}) \cup V_\phi(\sqrt[\phi]{J}) = V_\phi(\sqrt[\phi]{I} \cap \sqrt[\phi]{J})$. Thus, R is a ϕ -top ring. \square

Corollary 3.8. *Let R be a ring, and M an R -module. Let $\phi : \mathcal{I}(R) \rightarrow \mathcal{I}(R) \cup \{\emptyset\}$, and $\psi : \mathcal{I}(R_{(+)})M \rightarrow \mathcal{I}(R_{(+)})M \cup \{\emptyset\}$ be two functions such that $\psi(I_{(+)})M = \phi(I_{(+)})M$. If $R_{(+)})M$ is a ψ -top ring, then R is a ϕ -top ring.*

Proof. Let I_1 and I_2 be two ϕ -radical ideals of R and $P \in V_\phi(I_1 \cap I_2)$. By Proposition 2.7, $J_1 = I_{1(+)})M$ and $J_2 = I_{2(+)})M$ are ψ -radical ideals of $R_{(+)})M$, and $P_{(+)})M$ is a ψ -prime ideal of $R_{(+)})M$. So $P_{(+)})M \in V_\psi(J_1 \cap J_2)$. Thus, by Theorem 3.7, $P_{(+)})M \in V_\psi(J_1) \cup V_\psi(J_2)$. It follows that $P \supseteq I_1$ or $P \supseteq I_2$. Hence $V_\phi(I_1) \cup V_\phi(I_2) = V_\phi(I_1 \cap I_2)$, i.e., R is ϕ -top. \square

Theorem 3.9. *Let R_1 and R_2 be rings, $\psi_i : \mathcal{I}(R_i) \rightarrow \mathcal{I}(R_i) \cup \{\emptyset\}$ (for $i = 1, 2$) be functions, and let $\phi = \psi_1 \times \psi_2$. If R_i is a ψ_i -top ring such that for any non-trivial ideal I_i of R_i , $\psi_i(I_i) \neq I_i$ (for $i = 1, 2$), then $R_1 \times R_2$ is a ϕ -top ring.*

Proof. Let $I_1 \times I_2$ and $J_1 \times J_2$ be ideals of $R_1 \times R_2$. Since R_i is ψ_i -top, $V_{\psi_i}(I_i) \cup V_{\psi_i}(J_i) = V_{\psi_i}(K_i)$ for some ideal K_i of R_i ($i = 1, 2$). Now, since by [2, Theorem 16], every ϕ -prime ideal of $R_1 \times R_2$ has the form $P_1 \times R_2$ or $R_1 \times P_2$, where P_i is a ψ_i -prime ideal of R_i ($i = 1, 2$), we have $V_\phi(I_1 \times I_2) \cup V_\phi(J_1 \times J_2) = V_\phi(K_1 \times K_2)$. \square

A topological space T is Noetherian provided that the open (respectively, closed) subsets of T satisfy the ascending (respectively, descending) chain condition, or the maximal (respectively, minimal) condition [6, §4.2]. Recall that a ring R has Noetherian spectrum (i.e., $\text{Spec}(R)$ is a Noetherian space with the Zariski topology) if and only if the ascending chain condition (ACC) for radical ideals holds [9, p. 631]. We next generalize this result.

Theorem 3.10. *Let R be a ϕ -top ring, where $\phi : \mathcal{I}(R) \rightarrow \mathcal{I}(R) \cup \{\emptyset\}$ is a function. Then (X_ϕ, τ_ϕ) is a Noetherian space if and only if ACC holds for ϕ -radical ideals of R . In particular, if R is a Noetherian ϕ -top ring, then (X_ϕ, τ_ϕ) is a Noetherian space.*

Proof. (\Rightarrow) Let $I_1 \subseteq I_2 \subseteq \dots$ be an ascending chain of ϕ -radical ideals of R . Then we have the descending chain $V_\phi(I_1) \supseteq V_\phi(I_2) \supseteq \dots$ of closed subsets of (X_ϕ, τ_ϕ) . Now, by hypothesis, there is a positive integer n such that $V_\phi(I_n) = V_\phi(I_{n+1}) = \dots$. It follows that $\sqrt[\phi]{I_n} = \sqrt[\phi]{I_{n+1}} = \dots$; hence $I_n = I_{n+1} = \dots$.

(\Leftarrow) Let $V_\phi(I_1) \supseteq V_\phi(I_2) \supseteq \dots$ be a descending chain of closed subsets of X_ϕ . Then $\sqrt[\phi]{I_1} \subseteq \sqrt[\phi]{I_2} \subseteq \dots$. Since ACC holds for ϕ -radical ideals, there is a positive integer n such that $\sqrt[\phi]{I_n} = \sqrt[\phi]{I_{n+1}} = \dots$, and hence $V_\phi(\sqrt[\phi]{I_n}) = V_\phi(\sqrt[\phi]{I_{n+1}}) = \dots$. Now, by Lemma 3.1(4), we have $V_\phi(I_n) = V_\phi(I_{n+1}) = \dots$.

The ‘‘In particular’’ statement follows immediately from the first part. \square

Corollary 3.11. *Let R be a ring, and let $\phi, \psi : \mathcal{S}(R) \rightarrow \mathcal{S}(R) \cup \{\emptyset\}$ be two functions such that $\phi \leq \psi$. If (X_ψ, τ_ψ) is a Noetherian space, then (X_ϕ, τ_ϕ) is a Noetherian space, and in particular, R has Noetherian spectrum.*

Proof. Apply Theorem 2.5(4) and Theorem 3.10. \square

Corollary 3.12. *Let R be a Noetherian domain. Then (X_ϕ, τ_ϕ) is a Noetherian space for all $\phi \in \mathcal{A} - \{\phi_1, \phi_2\}$. In particular, if R is a PID, then (X_ϕ, τ_ϕ) is a Noetherian space for all $\phi \in \mathcal{A} - \{\phi_1\}$.*

Proof. If R is a Noetherian domain, by [2, Corollary 10], $V_\phi(I) = V(I)$ for all $\phi \in \mathcal{A} - \{\phi_1, \phi_2\}$, and if R is a PID, by [2, Theorem 12], $V_\phi(I) = V(I)$ for all $\phi \in \mathcal{A} - \{\phi_1\}$. Thus R is a ϕ -top ring for these functions ϕ , and hence (X_ϕ, τ_ϕ) is a Noetherian space by Theorem 3.10. \square

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