

REMARKS ON GENERALIZED (α, β) -DERIVATIONS IN SEMIPRIME RINGS

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ABSTRACT. Let R be an associative ring and $\alpha, \beta : R \rightarrow R$ ring homomorphisms. An additive mapping $d : R \rightarrow R$ is called an (α, β) -derivation of R if $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$ is fulfilled for any $x, y \in R$, and an additive mapping $D : R \rightarrow R$ is called a generalized (α, β) -derivation of R associated with an (α, β) -derivation d if $D(xy) = D(x)\alpha(y) + \beta(x)d(y)$ is fulfilled for all $x, y \in R$. In this note, we intend to generalize a theorem of Vukman [5], and a theorem of Daif and El-Sayiad [2].

1. Introduction

Throughout this paper, R will represent an associative ring with center $Z(R)$ and $\alpha, \beta : R \rightarrow R$ ring homomorphisms. Given an integer $n \geq 2$, a ring R is said to be n -torsion free, if for $x \in R$, $nx = 0$ implies $x = 0$. An additive mapping $d : R \rightarrow R$ is called an (α, β) -derivation of R if $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$ is fulfilled for any $x, y \in R$, and an additive mapping $D : R \rightarrow R$ is called a generalized (α, β) -derivation of R associated with an (α, β) -derivation d if $D(xy) = D(x)\alpha(y) + \beta(x)d(y)$ is fulfilled for all $x, y \in R$, we denote this generalized (α, β) -derivation as (D, d) . Now we call an additive mapping $F : R \rightarrow R$ an (α, β) -G-mapping of R if $F(xy) = F(x)\alpha(y) + \beta(x)D(y)$ is fulfilled for all $x, y \in R$ and for some generalized (α, β) -derivation (D, d) . If $\alpha = \beta$ is an identity map of R , then we call a $(1, 1)$ -derivation d a derivation, we call a generalized $(1, 1)$ -derivation D a generalized derivation associated a derivation d , and we call a $(1, 1)$ -G-mapping F a G -mapping.

Example 1.1. Let S be a semiprime ring, and let $R = \left\{ \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} \mid x, y, z \in S \right\}$. Now, we define maps $F, D, d : R \rightarrow R$ by

$$F \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix}, D \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix}$$

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and

$$d \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, it can be verified that R is a ring which is not semiprime, d is a derivation of R , (D, d) is a generalized derivation and (F, D) is a G-mapping which is not a generalized derivation.

An additive mapping $D : R \rightarrow R$ is called a Jordan (α, β) -derivation if $D(x^2) = D(x)\alpha(x) + \beta(x)D(x)$ is fulfilled for all $x \in R$. An additive mapping $D : R \rightarrow R$ is called a generalized (α, β) -Jordan derivation if $D(x^2) = D(x)\alpha(x) + \beta(x)d(x)$ for all $x \in R$ and for some (α, β) -derivation d . We call a generalized $(1, 1)$ -Jordan derivation a generalized Jordan derivation.

In [5], J. Vukman introduced additive mappings $F : R \rightarrow R$ such that $F(xy) = F(x)y + xF(y)$ for all $x, y \in R$, and $G : R \rightarrow R$ such that $G(xy) = G(x)y + xG(y)$ for all $x, y \in R$. We call this additive mappings F (resp. G) a left (resp. right) V -derivation. In [5], Vukman obtained the following result:

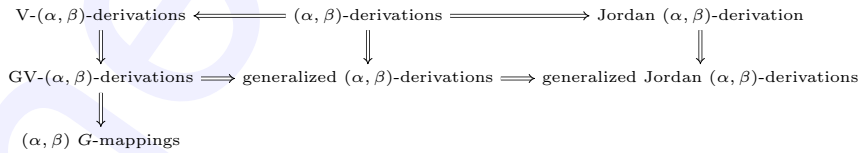
Theorem A. *Let R be a 2-torsion free semiprime ring and let $D : R \rightarrow R$ be an additive mapping. Suppose that either $D(xy) = D(x)y + xD(y)$ or $D(yx) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$. In both cases D is a derivation.*

Further, in [2], M. N. Daif and M. N. Tammam El-Sayiad introduced an additive mapping $G : R \rightarrow R$ such that $G(xy) = G(x)y + xD(y)$ is fulfilled for all $x, y \in R$ and for some derivation D , and we call this additive mapping G a DS -derivation. And, Daif and Tammam El-Sayiad [2] proved the following result.

Theorem B. *Let R be a 2-torsion free semiprime ring and let $G : R \rightarrow R$ be an additive mapping. If $G(xy) = F(x)y + xD(y)$ for all $x, y \in R$ for some derivation D of R , then G is a generalized Jordan derivation.*

We call an additive mapping $F : R \rightarrow R$ a left (resp. right) Vukman- (α, β) -derivation if $F(xy) = F(x)\alpha(y) + \beta(y)F(x)$ (resp. $F(x)\alpha(y) + \beta(x)F(y)$) for all $x, y \in R$ (abbreviated as V - (α, β) -derivation). And we call an additive mapping F a generalized left (resp. right) Vukman- (α, β) -derivation (abbreviated as GV - (α, β) -derivation) if $F(xy) = F(x)\alpha(y) + \beta(y)D(x)$ (resp. $F(x)\alpha(y) + \beta(x)D(y)$) for all $x, y \in R$ and for some left (resp. right) Vukman- (α, β) -derivation D .

Now, we denote the relationships of above various derivations as follows:



In this note, we intend to generalize above theorem of Vukman [5], and a theorem of Daif and El-Saiyad [2].

2. Results

We will prepare a few lemmas which are essential for developing the proof of our main result.

Lemma 2.1 ([3] Corollary 2.1(1)). *Let R be a 2-torsion free semiprime ring, L be a square-closed Lie ideal of R such that $L \not\subseteq Z(R)$ and let $a \in L$. If $aLa = 0$, then $a = 0$.*

Lemma 2.2 ([4] Theorem 2). *Let R be a 2-torsion-free semiprime ring and D a Jordan (α, β) -derivation of R with α or β an automorphism of R . Then D is an (α, β) -derivation of R .*

Lemma 2.3 ([1] Theorem 3.1). *Let R be a 2-torsion free semiprime ring, α an automorphism of R and β an endomorphism of R . If F is a generalized Jordan (α, β) -derivation with some Jordan (α, β) -derivation D , then F is a generalized (α, β) -derivation associated with D .*

We shall start our investigations with the following proposition concerning (α, β) - G -mappings.

Proposition 2.1. *Let R be a semiprime ring, and β an epimorphism. If F is an (α, β) - G -mapping of R associated with a generalized (α, β) -derivation (D, d) , then $D = d$, and so F is a generalized (α, β) -derivation of R associated with an (α, β) -derivation d .*

Proof. By our hypothesis on F ,

$$F(xy) = F(x)\alpha(y) + \beta(x)D(y) = F(x)\alpha(y) + \beta(x)D(y)\alpha(x) + \beta(xy)d(x)$$

for all $x, y \in R$. While, we have

$$F(xy) = F(xy)\alpha(x) + \beta(xy)D(x) = F(x)\alpha(y) + \beta(x)D(y)\alpha(x) + \beta(xy)D(x)$$

for all $x, y \in R$. Comparing above two equations, we get

$$(2.1) \quad \beta(xy)(D - d)(x) = 0 \text{ for all } x, y \in R.$$

Hence, we obtain that $(D - d)(x)\beta(x)\beta(y)(D - d)(x)\beta(x) = 0$ for all $y \in R$. Since β is an epimorphism and R is semiprime, we have $(D - d)(x)\beta(x) = 0$. While,

$$\beta(x)(D - d)(x)\beta(y)\beta(x)(D - d)(x) = 0 \text{ for all } y \in R$$

by (2.1). So, we have $\beta(x)(D - d)(x) = 0$. By linearizing,

$$\beta(x)(D - d)(z) + \beta(z)(D - d)(x) = 0 \text{ for all } x, z \in R.$$

Multiplying $(D - d)(x)$ from the left, we have

$$\begin{aligned} 0 &= (D - d)(x)\beta(x)(D - d)(z) + (D - d)(x)\beta(z)(D - d)(x) \\ &= (D - d)(x)\beta(z)(D - d)(x) \end{aligned}$$

for all $z \in R$. By semiprimeness of R , we have $(D - d)(x) = 0$ for all $x \in R$. And so, $D = d$, that is, F is a generalized derivation associated with d . \square

Now, we prove our main theorem.

Theorem 2.1. *Let R be a 2-torsion free semiprime ring and $L \not\subseteq Z(R)$ be a square-closed Lie ideal of R . Let $F, D : R \rightarrow R$ be additive mappings such that $F(L) \subseteq L$ and $D(L) \subseteq L$, and let α, β be ring homomorphisms of R such that $\alpha(L) \subseteq L$ and $\beta(L) \subseteq L$.*

- (i) *If $F(xy) = F(xy)\alpha(x) + \beta(xy)D(x)$ holds for all $x, y \in L$ and $\beta(L) = L$, then D is a Jordan (α, β) -derivation on L .*
- (ii) *If $F(xy) = F(x)\alpha(y) + \beta(x)D(y)$ and $D(xy) = D(x)\alpha(y) + \beta(x)D(y)$ hold for all $x, y \in L$ and $\alpha(L) = L$, then F is a generalized Jordan (α, β) -derivation associated with a Jordan (α, β) -derivation D on L .*
- (iii) *If $F(xy) = \alpha(x)F(y) + D(x)\beta(y)$ holds for all $x, y \in L$ and $\beta(L) = L$, then D is a Jordan (β, α) -derivation on L .*
- (iv) *If $F(xy) = \alpha(xy)F(x) + D(xy)\beta(x)$ and $D(xy) = \alpha(xy)D(x) + D(xy)\beta(x)$ hold for all $x, y \in L$ and $\alpha(L) = L$, then F is a generalized Jordan (β, α) -derivation associated with a Jordan (β, α) -derivation D on L .*

Proof. (i) We have

$$(2.2) \quad F(xy) = F(xy)\alpha(x) + \beta(xy)D(x)$$

for all $x, y \in L$. Linearizing above relation, we have

$$(2.3) \quad F(xyz + zyx) = F(xy)\alpha(z) + F(zx)\alpha(y) + \beta(xy)D(z) + \beta(zx)D(y)$$

for all $x, y \in L$. Replacing z by x^2 in (2.3), we get

$$(2.4) \quad F(xy x^2 + x^2 yx) = F(xy)\alpha(x^2) + F(x^2 y)\alpha(x) + \beta(xy)D(x^2) + \beta(x^2 y)D(x).$$

On the other hand, in (2.2), substituting $xy + yx$ for y , we obtain that

$$(2.5) \quad \begin{aligned} F(x^2 yx + xy x^2) &= F(x^2 y + xyx)\alpha(x) + \beta(x^2 y + xyx)D(x) \\ &= F(x^2 y)\alpha(x) + F(xy)\alpha(x^2) + \beta(xy)D(x)\alpha(x) \\ &\quad + \beta(x^2 y + xyx)D(x). \end{aligned}$$

Comparing (2.4) with (2.5), we have

$$\beta(x)\beta(y)\{D(x^2) - D(x)\alpha(x) - \beta(x)D(x)\} = 0$$

for all $x, y \in L$. Since β is a ring homomorphism and $\beta(L) = L$, we find that

$$\beta(x)z\{D(x^2) - D(x)\alpha(x) - \beta(x)D(x)\} = 0$$

for all $x, z \in L$. Now, we set $A(x) = D(x^2) - D(x)\alpha(x) - \beta(x)D(x)$. Since $D(L) \subseteq L$, $\alpha(L) \subseteq L$ and $\beta(L) \subseteq L$, we find that

$$\beta(x)A(x)z\beta(x)A(x) = 0$$

and

$$A(x)\beta(x)zA(x)\beta(x) = 0.$$

Since R is semiprime, we have

$$(2.6) \quad A(x)\beta(x) = 0$$

and

$$(2.7) \quad \beta(x)A(x) = 0$$

by Lemma 2.1. In (2.6), substituting $x + z$ for x , we have

$$(2.8) \quad A(x)\beta(z) + A(z)\beta(x) + B(x, z)\beta(x) + B(x, z)\beta(z) = 0,$$

where

$$B(x, z) = D(xz + zx) - D(x)\alpha(z) - D(z)\alpha(x) - \beta(x)D(z) - \beta(z)D(x).$$

In (2.8), substituting $-x$ for x , we get

$$(2.9) \quad A(x)\beta(z) - A(z)\beta(x) + B(x, z)\beta(x) - B(x, z)\beta(z) = 0.$$

By comparing (2.8) and (2.9), we get

$$2\{A(x)\beta(z) + B(x, z)\beta(x)\} = 0.$$

Since R is 2-torsion free, we have

$$A(x)\beta(z) + B(x, z)\beta(x) = 0.$$

And so we have

$$0 = A(x)\beta(z)A(x) + B(x, z)\beta(x)A(x) = A(x)\beta(z)A(x)$$

by (2.7). Since $\beta(L) = L$, we get

$$A(x)yA(x) = 0 \text{ for all } x, y \in L.$$

By semiprimeness of R , we obtain that $A(x) = 0$ for all $x \in L$ by Lemma 2.1 and hence, D is a Jordan (α, β) -derivation on L .

(ii) Now, assume that

$$(2.10) \quad F(xy) = F(x)\alpha(y) + \beta(x)D(y)$$

and

$$(2.11) \quad D(xy) = D(x)\alpha(y) + \beta(x)D(y)$$

for all $x, y \in L$. In (2.10), by linearizing, we have

$$F(xyz + zyx) = F(x)\alpha(yz) + F(z)\alpha(yx) + \beta(x)D(yz) + \beta(z)D(yx).$$

Now, substituting x^2 for x , we have

$$(2.12) \quad F(xy^2 + x^2yx) = F(x)\alpha(yx^2) + F(x^2)\alpha(yx) + \beta(x)D(yx^2) + \beta(x^2)D(yx).$$

In (2.10), substituting $xy + yx$ for y , we have

$$\begin{aligned}
 (2.13) \quad F(x^2yx + xyx^2) &= F(x)\alpha(xyx + yx^2) + \beta(x)D(xyx + yx^2) \\
 &= F(x)\alpha(xyx) + F(x)\alpha(yx^2) \\
 &\quad + \beta(x)\{D(x)\alpha(yx) + \beta(x)D(yx) + D(yx^2)\}.
 \end{aligned}$$

By comparing (2.12) with (2.13), we get

$$\{F(x^2) - F(x)\alpha(x) - \beta(x)D(x)\}\alpha(y)\alpha(x) = 0$$

for all $x \in L$. Now, we set

$$E(x) = F(x^2) - F(x)\alpha(x) - \beta(x)D(x).$$

Since $\alpha(L) = L$, we have $E(x)z\alpha(x) = 0$ for all $x, z \in L$. As a similar way to the proof of (i), we obtain $E(x) = 0$, that is,

$$F(x^2) = F(x)\alpha(x) + \beta(x)D(x) \text{ for all } x \in L.$$

In the case of $D(xyx) = D(x)\alpha(yx) + \beta(x)D(yx)$, D is a Jordan (α, β) -derivation on L by the similar arguments to the above arguments, and so F is a generalized Jordan (α, β) -derivation on L associated with a Jordan (α, β) -derivation D on L .

(iii) The proof is similar to that of (i).

(iv) The proof is similar to that of (ii). \square

In the following there are some immediate consequences of the above theorem.

Corollary 2.1. *Let R be a 2-torsion free semiprime ring, α, β endomorphisms of R , and let $F, D : R \rightarrow R$ be additive mappings.*

- (i) *If $F(xyx) = F(xy)\alpha(x) + \beta(xy)D(x)$ holds for all $x, y \in R$, and β is an automorphism of R , then D is an (α, β) -derivation.*
- (ii) *If $F(xyx) = F(x)\alpha(yx) + \beta(x)D(yx)$ and $D(xyx) = D(x)\alpha(yx) + \beta(x)D(yx)$ hold for all $x, y \in R$ and α is an automorphism of R , then F is a generalized (α, β) -derivation associated with an (α, β) -derivation D .*
- (iii) *If $F(xyx) = \alpha(x)F(yx) + D(x)\beta(yx)$ holds for all $x, y \in R$, and β is an automorphism of R , then D is a (β, α) -derivation.*
- (iv) *If $F(xyx) = \alpha(xy)F(x) + D(xy)\beta(x)$ and $D(xyx) = \alpha(xy)D(x) + D(xy)\beta(x)$ hold for all $x, y \in R$, and α is an automorphism of R , then F is a generalized (β, α) -derivation associated with a (β, α) -derivation D .*

Corollary 2.2. *Let R be a 2-torsion free semiprime ring, $D : R \rightarrow R$ an additive mapping. Then the followings are equivalent:*

- (1) $D(xyx) = D(xy)x + xy(D(x))$ for all $x, y \in R$.
- (2) $D(xyx) = D(x)yx + xD(yx)$ for all $x, y \in R$.

- (3) $D(xy) = D(xy)x + xyD(x)$ or $D(xy) = D(x)yx + xyD(x)$ for all $x, y \in R$.
 (4) D is a derivation.

Proof. (1) \rightarrow (4). In Corollary 1, by putting $F = D$, D is a derivation.

Similarly, (2) \rightarrow (4) is proved.

(3) \rightarrow (4). We put $R_x = \{y \in R \mid D(xy) = D(xy)x + xD(x)\}$ for all $x \in R$ and $R_x^* = \{y \in R \mid D(xy) = D(x)yx + xD(y)\}$ for all $x \in R$. Then we have $R = R_x \cup R_x^*$. Since R_x and R_x^* are additive groups, $R = R_x$ or $R = R_x^*$ by Brauer's Trick. By the same method, we have $R = \{x \in R \mid R = R_x\}$ or $R = \{x \in R \mid R = R_x^*\}$. Therefore, by (1) and (2), D is a derivation.

(4) \rightarrow (1), (4) \rightarrow (2) and (4) \rightarrow (3) are clear. \square

Corollary 2.3. *Let R be a 2-torsion free semiprime ring, and let $F, D : R \rightarrow R$ be additive mappings.*

(i) *If one of the following conditions is fulfilled, then F is a generalized derivation associated with a derivation D .*

(1) $F(xy) = F(x)yx + xD(y)$ and $D(xy) = D(x)yx + xD(y)$ for all $x, y \in R$.

(2) $F(xy) = xyF(x) + D(xy)x$ and $D(xy) = xyD(x) + D(xy)x$ for all $x, y \in R$.

(3) $F(xy) = F(x)yx + xD(y)$ and $D(xy) = D(x)yx + xD(y)$, or $F(xy) = xyF(x) + D(xy)x$ and $D(xy) = xyD(x) + D(xy)x$ for all $x, y \in R$.

(ii) *If one of the following conditions is fulfilled, then D is a derivation.*

(4) $F(xy) = F(xy)x + xyD(x)$ for all $x, y \in R$.

(5) $F(xy) = xF(y) + D(x)yx$ for all $x, y \in R$.

(6) $F(xy) = F(xy)x + xyD(x)$ or $F(xy) = xF(y) + D(x)yx$ for all $x, y \in R$.

Proof. By the similar method of Corollary 2.2, this corollary is proved. \square

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References

- [1] S. Ali and C. Haetinger, *Jordan α -centralizer in rings and some applications*, Bol. Soc. Paran. Mat. **26** (2008), no. 1-2, 71–80.
- [2] M. N. Daif and M. S. Tammam El-Sayiad, *An Identity related to generalized derivations*, Int. J. Algebra **1** (2007), no. 9-12, 547–550.
- [3] M. Hongan, N. Rehman, and R. Al-Omary, *Lie ideals and Jordan Triple derivations in rings*, Rend. Sem. Mat. Univ. Padova **125** (2011), 147–156.
- [4] C. Lanski, *Generalized derivations and n -th power maps in rings*, Comm. Algebra **35** (2007), no. 11, 3660–3672.
- [5] J. Vukman, *Some remarks on derivations in semiprime rings and standard operator algebras*, Glas. Mat. Ser. III **46** (2011), no. 1, 43–48.

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