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GROUP GRADED TYPES OF BÉZOUT MODULES

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Abstract. In this paper, we introduce two group graded types of Bézout modules, namely graded-Bézout modules and weakly graded-Bézout modules, which are two Bézout versions in Graded Module Theory. We investigate the relationship among the three types of Bézout modules, the ordinary Bézout modules and the two graded types of Bézout modules. Also, we study the structure of these new Bézout modules along with different properties; for instance, "A graded-Bézout R-module, with R being a Noetherian ring, is Noetherien iff it is gr-Noetherian".

1. Introduction

Many concepts in Algebra have been extended to graded ring and module theories where the gradation has its impression on these concepts. For instance, gr-simple modules and gr-Noetherian modules are the extensions of simple modules and Noetherian modules, respectively. A similar idea exists in graded ring theory when we talk about gr-maximal ideals, gr-Jacobson radicals,... etc.

Because of its importance in module theory, the concept of Bézout modules is extended in this paper to two types of Bézout modules that involve the group gradation (the graded types of Bézout modules). The first type is graded-Bézout modules (or gr-Bézout modules) in which every finitely generated submodule with homogeneous generators is cyclic with a homogeneous generator. The second type is weakly graded-Bézout modules (or weakly gr-Bézout modules) which satisfy the Bézout property for the graded submodules. These possible extensions of Bézout modules might help extend the results concerning the Bézout property in the ordinary algebra to similar results in graded ring and module theories analogous to the work presented in [1] or [5].

This paper studies the relationship among the three types of Bézout modules: the Bézout graded modules (Bézout modules with group gradation), the gr-Bézout modules, and the weakly gr-Bézout modules. Also, the paper investigates the structure of the graded types of Bézout modules such as the properties of the homogeneous components that build the body of the graded

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types of Bézout modules. Further, we study the relationship between gr-Bézout modules and gr-Noetherian modules, and we show some results explaining how to generate new gr-Bézout modules from old ones.

Section 2 includes the background necessary to this paper.

In Section 3, we study the relationship among the three types of Bézout modules. The relationship is like whether any type of them implies the others. The scope behind this relationship is when gr-Bézout modules are equivalent to either graded type of Bézout modules, it is easier to deal with the new types of Bézout modules since the amount of possible generators (homogeneous elements only) in the new types is less than those in the original type (Bézout modules).

In Section 5, we focus on the structure of gr-Bézout modules following different trends. For example, what components make a module gr-Bézout? If M is gr-Bézout what can one say about the homogeneous components? Or, if M is a gr-Bézout R-module, what can we say about R?

In Section 6, we show that for gr-Bézout modules, gr-Noetherian and Noetherian properties are equivalent. Also, not widely, we list some results that illustrate gr-Bézout modules generating new ones.

2. Preliminaries

This section presents some necessary background of graded rings and graded modules considered in this paper. More details can be found in [1,5]. Throughout this paper, unless otherwise stated, G is a group with identity e, $R = \bigoplus_{g\in G} R_g$ is a G-graded ring with unity 1, and $M = \bigoplus_{g\in G} M_g$ is a G-graded left R-module. The set $supp(R, G) = \{g \in G : \ R_g \neq 0\}$ is called the support of R. The support of M, $supp(M, G)$, is defined similarly. The set $h(R) = \bigcup_{g\in G} R_g$ ($h(M) = \bigcup_{g\in G} M_g$, respectively) is called the set of homogeneous elements of the graded ring R (the graded module M, respectively). All modules considered in this article are left modules. Moreover, all groups, rings, and modules are assumed to be non-trivial.

Definition 2.1 ([1])**.** Let M be a G-graded R-module (an R-module). By a gr-cyclic (a cyclic, respectively) R-submodule of M we mean a cyclic submodule with a homogeneous generator (a generator, not necessarily homogeneous, respectively).

Remark 2.2. Evey *gr*-cyclic R-submodule of M is G-graded. A cyclic submodule with non-homogeneous generator might not be graded as illustrated in the next example.

Example 2.3. Let $R = M_2[\mathbb{Z}_2]$ be the ring of 2×2 matrices with entries from $\mathbb{Z}_2$. The ring R is $\mathbb{Z}_3$-graded by $R_0 = [\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}]$, $R_1 = [\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}]$, and $R_2 = [\begin{smallmatrix} 0 & 0 \\ c & 0 \end{smallmatrix}]$, where $a, b, c, d \in \mathbb{Z}_2$.

The principal ideal generated by the non-homogeneous matrix $\left[\begin{smallmatrix}0&1\\0&1\end{smallmatrix}\right]$ is $\mathbb{Z}_3$-graded. However, the principal ideal generated by the non-homogeneous matrix $\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right]$ is not $\mathbb{Z}_3$-graded.

Definition 2.4. A G-graded R-module M is called gr-torsion free (torsion free) if for every $r \in h(R) - \{0\}$ and $m \in h(M) - \{0\}$, $rm \neq 0$ (if for every $r \in R - \{0\}$ and $m \in M - \{0\}$, $rm \neq 0$, respectively).

Definition 2.5 ([1])**.** A G-graded ring R (A ring R) is called a gr-domain (a domain, respectively) if there are no zero divisors in $h(R)$ (in R, respectively) except 0.

Definition 2.6 ([6])**.** A G-graded R-module is said to be flexible if $M = RM_e$.

Proposition 2.7 ([6])**.** *A graded R-module is flexible if and only if $M_g = R_gM_e$, for every $g \in G$*

Proposition 2.8 ([6])**.** *Every graded submodule of a flexible module is flexible.*

Definition 2.9 ([4])**.** Let M be an $R-$module, N an R-submodule of M, and S a nonempty subset of M. We define $(N :_R S) = \{r \in R : \ rS \subseteq N\}$. The annihilator of a nonempty set S of M is defined by $Ann_R(S) = (0 : S)$. If $S = \{x\}$, we write $Ann_R(S) = Ann_R(x)$. If $Ann_R(x) = 0$, we say that x is an $R-$torsion free element.

It is easy to see that the annihilator of a set forms a left ideal of R.

Definition 2.10 ([5])**.** By a gr-Noetherian (a Noetherian) R-module we mean a graded R-module (an R-module, respectively) which satisfies the ascending chain condition on gr-submodules (on R-submodules, respectively), or equivalently in which every graded R-submodule (every R-submodule, respectively) of M is finitely generated.

Definition 2.11 ([5])**.** An R-module homomorphism $f : M \longrightarrow N$ between G-graded R-modules is called a gr-homomorphism if $f(M_g) \subseteq N_g$, for every $g \in G$. If in addition f is an R-module isomorphism, we say f is a gr-isomorphism. In this case we have $f(M_g) = N_g$, for every $g \in G$.

Definition 2.12 ([3])**.** An R-module is Bézout if every finitely generated R-submodule is cyclic (i.e., generated by one element).

3. Graded types of Bézout modules and Bézout modules

In this section, we exhibit the relationship among the three types of Bézout modules. Although the three types are not equivalent, we demonstrate that, under certain circumstances, one type may imply the other types or one of them.

Definition 3.1. A G-graded R-module is said to be a graded-Bézout R-module (or gr-Bézout) if every finitely generated R-submodule with homogeneous generators is gr-cyclic.

Definition 3.2. A G-graded R-module is said to be a weakly graded-Bézout R-module (or weakly gr-Bézout) if every finitely generated graded R-submodule is cyclic.

We shall be careful about the terminology and distinguish between the three types of Bézout modules. The Bézout graded modules are Bézout modules that are given a gradation. That is, the gradation does not affect the Bézout property. The graded-Bézout modules are graded modules satisfying Definition 3.1, while the weakly graded-Bézout modules are graded modules that satisfy the Bézout property for graded submodules.

Lemma 3.3. *Let M be a G-graded R-module. A finitely generated R-submodule of M is graded if and only if it is generated by homogeneous elements. In fact, every finitely generated graded submodule is generated by the homogeneous components of its generators.*

Proof. It is easy to see that a finitely generated submodule of a graded module with homogeneous generators is graded. Suppose $N = \sum_{i=1}^{n} Ra_i$ is a graded submodule of M, where $a_1, \dots, a_n \in M$. Let $a_i = a_i^{(1)} + \cdots + a_i^{(k_i)}$, where $a_i^{(1)}, \dots, a_i^{(k_i)} \in h(M)$ for every $i = 1, \dots, n$. We have $N \subseteq \sum_{i=1}^{n} \sum_{j=1}^{k_i} Ra_i^{(j)}$. Since N is graded, $a_i^{(j)} \in N$ for all $j = 1, \dots, k_i$ and $i = 1, \dots, n$. Thus $Ra_i^{(j)} \subseteq N$ for all $j = 1, \dots, k_i$ and $i = 1, \dots, n$ and hence $\sum_{i=1}^{n} \sum_{j=1}^{k_i} Ra_i^{(j)} \subseteq N$. So, $N = \sum_{i=1}^{n} \sum_{j=1}^{k_i} Ra_i^{(j)}$. □

The following corollary states that if a graded cyclic R-submodule has a generator whose homogeneous component owns a zero R-annihilator, then this generator is homogeneous with the same degree of that component and the cyclic R-submodule turns out to be gr-cyclic.

Corollary 3.4. *Let M be a G-graded R-module and Ra be a non-trivial graded cyclic R-submodule. If $Ann_R(a_g) = 0$, where a_g is a nonzero homogeneous component of a, then a is homogeneous of degree g and hence Ra is gr-cyclic.*

Proof. Let $a = \sum\limits_{h \in G} a_h$. Since Ra is graded, $a_g \in Ra$. Let $r = \sum\limits_{\sigma \in G} r_\sigma$ such that $a_g = ra$. Then $a_g = \sum\limits_{\sigma \in G} r_\sigma a_g + \sum\limits_{\sigma \in G, h \in G-g} r_\sigma a_h$. Since the right side of the last equality is homogeneous of degree g, all nonzero terms of $\sum\limits_{\sigma \in G} r_\sigma a_g$ are homogeneous of degree g. So $\sigma = e$ for all $r_\sigma \neq 0$ and hence $r \in R_e$. This implies that a is homogeneous of degree g and that Ra is gr-cyclic. □

Proposition 3.5. *Every graded submodule of a gr-Bézout* (*a weakly gr-Bézout*) *module is gr-Bézout* (*a weakly gr-Bézout, respectively*).

Proof. The proof follows directly from Definitions 3.1, 3.2, and Lemma 3.3. □

Proposition 3.6. *Bézout graded modules and gr-Bézout modules are weakly gr-Bézout modules.*

Proof. The proof follows directly from Definitions 2.12, 3.1, 3.2, and Lemma 3.3. □

In view of Lemma 3.3, we can assume without loss of generality in Definition 3.2 that every finitely generated submodule possesses homogeneous generators. Generally, the three types of Bézout modules are not equivalent. For instance, A gr-Bézout module is not necessarily a Bézout module, for there perhaps exists a non-graded finitely generated submodule which is not cyclic. Also, a weakly gr-Bézout module is not necessarily a gr-Bézout module because there may exist a finitely generated graded submodule which is not gr-cyclic, as shown later in Example 4.2. Our task now is to find out when the three types are equivalent. We list some results about this issue.

Proposition 3.7. *Let M be a graded R-module such that every submodule of M is graded. Then*

(1) *M is weakly gr-Bézout if and only if M is Bézout.*
(2) *If M is gr-Bézout, then M is Bézout.*

Proof. (1) The proof is trivial.

(2) Suppose M is gr-Bézout. Let N be a finitely generated submodule of M. By assumption and Lemma 3.3, N is finitely generated with homogeneous generators. Since M is gr-Bézout, N is gr-cyclic. Thus, M is Bézout. □

The converse of part 2 in Proposition 3.7 is not always true because not every graded cyclic submodule is gr-cyclic. This leads us to the following theorem.

Theorem 3.8. *Let M be a G-graded R-module. Then*

(1) *The concepts of Bézout and gr-Bézout modules are the same for an R-module M if and only if the concepts of cyclic R-submodules and gr-cyclic R-submodules of M are the same.*
(2) *Every weakly gr-Bézout module M is gr-Bézout if and only if the concepts of graded cyclic submodules and gr-cyclic submodules of M are the same.*
(3) *If M is Bézout and has the property that every graded cyclic R-submodule is gr-cyclic, then M is gr-Bézout.*

Proof. The proofs of (1) and (2) are not difficult. The proof of (3) depends on part (2) and Proposition 3.6. □

Proposition 3.9. *Suppose that R is a local ring and M is a G-graded R-module. If M is Bézout, then M is gr-Bézout.*

Proof. The proof follows from Theorem 3.8 and Proposition 3 of [2]. □

Next, we show in gr-torsion free modules, when graded cyclic and gr-cyclic modules are equivalent, that the three types of Bézout modules are approaching each other.

Theorem 3.10. *Suppose M is a gr-torsion free R-module. Then the two group graded types of Bézout modules are equivalent. In addition, if M is Bézout, then it is gr-Bézout.*

Proof. The proof is immediate from Corollary 3.4 and Theorem 3.8. □

Corollary 3.11. *If R is a Bézout gr-domain, then it is a gr-Bézout gr-domain.*

Proof. The proof follows from the fact that a graded gr-domain is a gr-torsion free module over itself along with Theorem 3.10. □

4. Examples

The aim of this section is to illustrate the three definitions of Bézout modules and exhibit the relationship between them. The main example of this section (Example 4.6) shows that there is an example of a Bézout graded module which is not a gr-Bézout module.

The following examples are applications of Corollary 3.11. Example 4.1 exhibits a Bézout and gr-Bézout module, although not every submodule of which is graded, whereas Example 4.2 exhibits a graded module which is neither gr-Bézout nor Bézout.

Example 4.1. Let $R = K[x]$, where K is a field, and $G = \mathbb{Z}$ the group of integers. Since R is a Euclidean domain, and hence a PID, we have that R is a Bézout domain. R is $\mathbb{Z}$-graded by $R_0 = K$, $R_n = Kx^n$, and $R_{-n} = 0$, for all $n = 1, 2, \ldots$. Let $N = Rx^n + Rx^m$. If $m \geq n \geq 1$, then $N \subseteq Rx^n \subseteq N$ or $N = Rx^n$, i.e., N is gr-cyclic. If $m \geq n = 0$, then $N = R + Rx^m = R$ which is gr-cyclic. Thus, we obtain that R is a gr-Bézout ring (a gr-Bézout module over itself).

Example 4.2. Let $M = K[x]$, where K is a field, and $G = \mathbb{Z}$. Give M the gradation given in Example 4.1 and K the trivial gradation by $\mathbb{Z}$. Consider M as a graded vector space over K. By Lemma 3.3, the only nonzero gr-cyclic submodules are Kx^n, where $n = 0, 1, \ldots$. Thus, M contains non-graded submodules. Since it is impossible for the graded subspace $K + Kx$ to be gr-cyclic, it follows that M is not a gr-Bézout $K-$module. Also, by Lemma 3.3, it is impossible for $K + Kx$ to be cyclic. So, M is also not a Bézout $K-$module.

Practically, we apply Theorem 3.8, Proposition 3.9, or Theorem 3.10 to show that a graded module is Bézout by considering only homogeneous generators instead of arbitrary generators. The following example exposes this idea.

Example 4.3. Consider the Abelian group of Gaussian integers $M = \mathbb{Z} \oplus i\mathbb{Z}$, where $i = \sqrt{-1}$, as a $\mathbb{Z}$-module. The domain $\mathbb{Z}$ is $\mathbb{Z}_2$-graded by the trivial gradation, and M is a gr-torsion free $\mathbb{Z}_2$-module graded by $M_0 = \mathbb{Z}$ and $M_1 = i\mathbb{Z}$. Consider the graded submodule $N = 2\mathbb{Z} \oplus i\mathbb{Z}$. It is easy to see that $N \neq m\mathbb{Z}$ and $N \neq im\mathbb{Z}$ for every $m \in \mathbb{Z}$. Which means that it is impossible for N to be generated by a homogeneous element of M. So M is not gr-Bézout. Now, Theorem 3.10 guarantees that M is not Bézout.

Another usage of the above results is to prove that a graded module is gr-Bézout using the fact that Bézout modules are already studied in the literature. That is, the possibility of replacing a generator of a cyclic submodule with a homogeneous generator. The following example illustrates the last statement.

Example 4.4. Recall Example 2.3. Since the entries of the matrix ring are taken from a field, R is a Bézout ring (see Section C.36 of [4]). Direct computations show that a cyclic module is graded if and only if it has a homogeneous generator. Thus, Theorem 3.8 implies R is a gr-Bézout ring.

Now we present an example of a gr-Bézout domain which is not a PID.

Example 4.5. Let $R = \mathbb{Z}+x\mathbb{Q}[x]$. Then R is a Bézout domain but not a PID. Set a $\mathbb{Z}_2$-gradation for R as follows:

$$R_0=\{a_0+a_1x^2+a_2x^4+\cdots+a_nx^{2n} : n = 0, 1, 2, \dots ; a_0\in\mathbb{Z};\text{ and } a_1, \dots , a_n\in\mathbb{Q}\}$$

and

$$R_1 = \{b_1x + b_2x^3 + \cdots + b_mx^{2m-1} : m = 1, 2, \dots \text{ and } b_1, \dots , b_m \in \mathbb{Q}\}.$$

By Corollary 3.11, R is a gr-Bézout domain but not a PID.

The next example shows that there is an example of a Bézout graded module that is not a gr-Bézout module. Which demonstrates the non-equivalence between Bézout and gr-Bézout properties in the general situation.

Example 4.6. Consider the ring $M = \mathbb{Z}_2 \oplus \mathbb{Z}_4$ and its subring $R = \mathbb{Z}_2 \oplus \langle 2\rangle$, where $\langle 2\rangle$ is the cyclic subgroup of $\mathbb{Z}_4$ generated by 2. We have M is a $\mathbb{Z}_2$-graded R-module, where R has the trivial gradation and the gradation of M is defined by $M_0 = \mathbb{Z}_2 \oplus 0$ and $M_1 = 0 \oplus \mathbb{Z}_4$. Direct calculations show that the sum of any two cyclic R-submodules is again cyclic. This means that M is a Bézout R-module. However, M is not a gr-Bézout R-module because $R(1, 0) + R(0, 1) = R(1, 1) = R(1, 3)$ can't be generated by any homogeneous element, and $(1, 1)$ and $(1, 3)$ are not homogeneous.

5. Structure of gr-Bézout modules

This section is devoted to study the structure of gr-Bézout modules such as whether the components of gr-Bézout modules carry the Bézout property or not.

Proposition 5.1. *Suppose that R_g is a cyclic R_e-module for all $g \in G$, and M is a gr-Bézout R-module. Then M_e is a Bézout R_e-module.*

Proof. Let $N = R_ea + R_eb$, where $a, b \in M_e$. Then $L = Ra + Rb$ is a finitely generated R-submodule of M. Because M is gr-Bézout, $L = Rc$, where $c \in M_g$, for some $g \in G$. Hence, $N = L_e = R_{g^{-1}}c = (R_et)c$, where $t \in R_{g^{-1}}$. So, $N = R_e(tc)$, and $tc \in M_e$. Consequently, N is a cyclic R_e-submodule of M_e and hence M_e is a Bézout R_e-module. □

The converse of Proposition 5.1 is not generally true, for instance consider the graded module given in Example 4.3. A partial converse of Proposition 5.1 is given in the following proposition.

Proposition 5.2. *Suppose that M is a flexible R-module and R_g is a cyclic R_e-module for all $g \in supp(R, G)$. If M_e is a Bézout R_e-module, then M is a gr-Bézout R-module.*

Proof. Assume $R_g = R_e r_g$, where $r_g \in R_g$ and $g \in G$. Let $N = Ra_g + Ra_h$, where $a_g, a_h \in h(M)$ and $g, h \in G$. If either $a_g = 0$ or $a_h = 0$, then N is gr-cyclic. Assume $a_g \neq 0$ and $a_h \neq 0$. We have $N_e = R_{g^{-1}}a_g + R_{h^{-1}}a_h = R_e r_{g^{-1}}a_g + R_e r_{h^{-1}}a_h$ is a finitely generated R_e-submodule of M_e. Since M_e is a Bézout R_e-module, we obtain that $N_e = R_e x_e$, where $x_e \in M_e$. Since M is a flexible R-module, by Proposition 2.8, N is a flexible R-module and hence $N = RN_e = RR_e x_e = Rx_e$. So, N is gr-cyclic. Consequently, M is gr-Bézout. □

The following lemma and proposition are true for graded and ungraded cases. We present both results for ungraded case. The lemma has a simple proof and well-known in the literature, whereas the proof of the proposition might exist in the literature. Proposition 5.4 states that a cyclic regular module over a Bézout ring is a Bézout module.

Lemma 5.3. *If $M = Ra$ and N is an R-submodule of M, then $N = (N :_R a)a$.*

Proposition 5.4. *If R is a Bézout ring and X is a free R-module of dimension 1 (i.e., a cyclic R-module with a generator x such that $Ann_R(x) = 0$), then X is a Bézout R-module.*

Proof. Let $N = Rz + Ry$, where $z, y \in X$. By Lemma 5.3, $N = (N :_R x)x$, $Rz = (Rz :_R x)x$, and $Ry = (Ry :_R x)x$. Since $Ann_R(x) = 0$, we get

$$(N :_R x) = (Rz :_R x) + (Ry :_R x).$$

Moreover, from $Rz = (Rz :_R x)x$, we can write $z = r_1 x$, for some $r_1 \in (Rz :_R x)$. Thus, $(Rz :_R x)x = R(r_1 x)$. Again, the fact that $Ann_R(x) = 0$ yields $(Rz :_R x) = Rr_1$. That is, $(Rz :_R x)$ is a principal ideal of R. Similarly, $(Ry :_R x)$ is a principal ideal of R. It follows that, $(N :_R x)$ is a principal ideal because R is a Bézout ring. So, $(N :_R x) = Rw$, for some $w \in R$. Consequently, $N = R(wx)$ which is a cyclic $R-$submodule of X. Therefore, we obtain that X is a Bézout R-module. □

Examples 4.1 and 4.2 show that there is an example of a gr-Bézout module that is not a Bézout R_e-module. However a graded R-module which is an R_e-Bézout module is a gr-Bézout module. This is demonstrated in Proposition 5.5, which provides us with a useful method to build gr-Bézout modules.

Proposition 5.5. *If M is a graded R-module such that M is a Bézout R_e-module, then M is gr-Bézout.*

Proof. Suppose M is a graded R-module which is a Bézout R_e-module. Let $L = Rx_g + Ry_h$, where $x_g, y_h \in h(M)$. Since M is a Bézout R_e-module, we have $R_e x_g + R_e y_h = R_e z_k$, where $z_k \in h(M)$. There exist $\alpha_e, \beta_e \in R_e$ such that $x_g = \alpha_e z_k$ and $y_h = \beta_e z_k$. Thus, we get that $k = g = h$. Hence

$$\begin{aligned} Rz_g &= R(R_e z_g) = R(R_e x_g + R_e y_g) \\ &\subseteq RR_e x_g + RR_e y_g \\ &\subseteq Rx_g + Ry_g = R\alpha_e z_g + R\beta_e z_g \\ &\subseteq Rz_g + Rz_g = Rz_g. \end{aligned}$$

Finally, we obtain that $L = Rx_g + Ry_h = Rz_g$. So, M is a gr-Bézout. □

Given a graded R-module M, if R is a gr-Bézout ring, then M is not necessarily a gr-Bézout R-module; Example 4.2 demostrates the last statement. Now, we try to answer the questions: "If R is a gr-Bézout ring, what conditions should be applied to M to transfer it to a gr-Bézout R-module?" and "If M is a gr-Bézout R-module, is R necessarily a Bézout ring?". An answer to the first question is given in Proposition 5.6, whereas an answer for the second question is given in Proposition 5.7.

Proposition 5.6. *Let R be a gr-Bézout ring and M a flexible R-module such that M_e is a cyclic R_e-module. Then M is a gr-Bézout R-module.*

Proof. Suppose $N = Rx_g + Ry_h$, where $x_g, y_h \in h(M)$, and $g, h \in G$. By assumption, $M_k = R_k M_e = R_k R_e m_e = R_k m_e$, where $m_e \in M_e$, for every $k \in G$. Hence, there exist $r_g \in R_g$ and $r_h \in R_h$ such that $x_g = r_g m_e$ and $y_h = r_h m_e$. Thus, $N = (Rr_g + Rr_h)m_e$. Since R is gr-Bézout, $Rr_g + Rr_h = Rt$, for some $t \in h(R)$. Hence, $N = R(tm)$ with $tm \in h(M)$. Therefore, M is a gr-Bézout R-module. □

The next proposition is a partial converse of Proposition 5.6 for the graded case when the underlying module is cyclic.

Proposition 5.7. *Let M be a gr-Bézout R-module that contains an R-torsion free homogeneous element. Then R is a gr-Bézout ring.*

Proof. Let $x_k \in h(M)$ be an R-torsion free element and $I = Rr_g + Rr_h$, where $r_g, r_h \in h(R)$, and $k, g, h \in G$. Then $Ix_k = Rr_g x_k + Rr_h x_k$. Since M is gr-Bézout, there exists $y_f \in h(M)$, with $f \in G$ such that $Ix_k = Ry_f$. So, $y_f = ix_k$ for some $i \in I$. Thus, $i \in R_{fk^{-1}}$. Now, we have $Ix_k = Rix_k$. Since x_k is an R-torsion free element, $I = Ri$ is a gr-principle ideal. Consequently, R is a gr-Bézout ring. □

6. Gr-Bézout modules and other types of graded modules

This section studies the relationship between gr-Bézout modules and other types of graded modules such as gr-Noetherian modules, gr-simple modules, . . .

etc. In addition, this section studies the possibility of generating new gr-Bézout modules from old ones.

One of the issues that matters in graded ring theory is whether a gr-property (a property that involves the group gradation such as gr-Noetherian, gr-semisimple, ... etc) implies the same property without gradation (such as Noetherian, semisimple, ... etc). The equivalence of the properties of being Noetherian and of being gr-Noetherian was demonstrated under different settings and in many articles. In the following theorem, which is the main theorem of this section, we give new settings that guarantee this equivalence.

Theorem 6.1. *Let M be a gr-Bézout R-module with R being a Noetherian ring. Then, M is a Noetherian R-module if and only if M is a gr-Noetherian R-module.*

Proof. It is obvious that if M is Noetherian, then it is gr-Noetherian. Suppose M is gr-Noetherian and N is an R-submodule of M. Fix $x = x_{g_1} + \cdots + x_{g_n} \in N$, where $x_{g_1}, \ldots, x_{g_n} \in h(M)$. We have $x \in Rx_{g_1} + \cdots + Rx_{g_n}$. Because M is gr-Bézout, there exists $a_x \in h(M)$ such that $Rx_{g_1} + \cdots + Rx_{g_n} = Ra_x$. Hence, $N \subseteq \sum_{x\in N} Ra_x$. Since $\sum_{x\in N} Ra_x$ is a graded R-submodule of M and M is gr-Noetherian, $\sum_{x\in N} Ra_x$ is finitely generated with homogeneous generators. Again, since M is gr-Bézout, $\sum_{x\in N} Ra_x$ is gr-cyclic. Thus $N \subseteq Ra$, for some $a \in h(M)$. Now, $N = (N :_R a)a$. Since R is Noetherian, $(N :_R a) = Rb_1 + \cdots + Rb_m$, where $b_1, \ldots, b_m \in R$. So, we get $N = R(b_1a) + \cdots + R(b_ma)$ a finitely generated submodule of M. Consequently, M is Noetherian. □

Next, we prove some results concerning generating new Bézout modules from old ones.

Proposition 6.2. *Let M be a gr-Bézout R-module and N a G-graded R-submodule of M. Then, both N and $\frac{M}{N}$ are gr-Bézout R-modules.*

Proof. Recall that $\frac{M}{N}$ is a G-graded R-module with gradation $\left(\frac{M}{N}\right)_g = \frac{M_g+N}{N}$, for $g \in G$. It is easy to see that N is a gr-Bézout R-module. Let $L = R(a + N) + R(b + N)$, where $a, b \in h(M)$. Then $L = \frac{Ra+Rb+N}{N}$. Since M is gr-Bézout, $Ra + Rb = Rc$, for some $c \in h(M)$. Thus $L = \frac{Rc+N}{N} = R(c + N)$, i.e., L is a gr-cyclic R-submodule of $\frac{M}{N}$. Therefore, $\frac{M}{N}$ is gr-Bézout. □

The converse of Proposition 6.2 is not necessarily true. That is, if M is a graded R-module which possesses a graded $R-$submodule N such that both N and $\frac{M}{N}$ are Bézout R-modules, then M needs not be a gr-Bézout R-module. To see this, consider Example 4.3. If we set $N = \mathbb{Z}$, we have N and $\frac{M}{N}$, which is isomorphic to $\mathbb{Z}$, are both gr-Bézout $\mathbb{Z}$-modules. However M itself is not a gr-Bézout R-module. The following proposition gives a partial converse of Proposition 6.2.

Proposition 6.3. *Let M be a G-graded R-module. Suppose there is a graded R-submodule N such that $\frac{M}{N}$ and $N + Rc$ are gr-Bézout R-modules, for all $c \in h(M)$. Then M is a gr-Bézout R-module.*

Proof. Suppose N and $\frac{M}{N}$ are gr-Bézout R-modules. Let $L = Ra + Rb$, where $a, b \in h(M)$. Then $\frac{L+N}{N} = R(a+N)+R(b+N) \subseteq \frac{M}{N}$. Because $\frac{M}{N}$ is gr-Bézout, there is $c \in h(M)$ such that $\frac{L+N}{N} = R(c+N) = \frac{Rc+N}{N}$. The last equality yields $L \subseteq Rc + N$. Since $Rc + N$ is gr-Bézout, there exists $d \in h(N) \subseteq h(M)$ such that $L = Rd$. This proves that M is a gr-Bézout R-module. □

Propositions 6.2 and 6.3 produce the following theorem which provides a criterion for a graded module to be gr-Bézout from a submodule and the related quotient module .

Theorem 6.4. *Let M be a G-graded R-module. Then M is a gr-Bézout R-module if and only if there exists a G-graded R-submodule N of M such that $\frac{M}{N}$ is a gr-Bézout R-module and $N + Rc$ is a gr-Bézout R-module for every homogeneous element c of M.*

The proof of the following proposition is straightforward.

Proposition 6.5. *Let $f : M \longrightarrow N$ be a gr-isomorphism between G-graded R-modules. Then M is gr-Bézout if and only if N is gr-Bézout*

Proposition 6.6. *Let $f : M \longrightarrow N$ be a gr-homomorphism between G-graded R-modules. If M is a gr-Bézout R-module, then $Im(f)$ is a gr-Bézout R-module. Moreover, if f is an epimorphism, then N is a gr-Bézout R-module.*

Proof. We have $\frac{M}{Ker(f)} \cong Im(f)$. Since M is gr-Bézout, by Proposition 6.2 $\frac{M}{Ker(f)}$ is a gr-Bézout R-module. By Proposition 6.5, $Im(f)$ is a gr-Bézout R-module. The rest is obvious. □

Recall that if M_i is a G_i-graded R_i-module, for $i = 1, 2$, then $M_1 \times M_2$ is a $(G_1 \times G_2)$-graded $(R_1 \times R_2)$-module with gradation $(M_1 \times M_2)_{(g_1,g_2)} = M_{g_1} \times M_{g_2}$, for each $g_1 \in G_1$ and $g_2 \in G_2$.

Proposition 6.7. *Let M_i be a G_i-graded R_i-module, for $i = 1, 2$. Then $M_1 \times M_2$ is a gr-Bézout $(G_1 \times G_2)$-graded $(R_1 \times R_2)$-module if and only if M_i is a gr-Bézout R_i-module, for every $i = 1, 2$.*

Proof. The proof is straightforward from Proposition 6.6. □

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