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ON THE GAUSS MAP OF HELICOIDAL SURFACES

Dong-Soo Kim, Wonyong Kim, and Young Ho Kim

Abstract. We study the Gauss map G of helicoidal surfaces in the 3-dimensional Euclidean space $\mathbb{E}^3$ with respect to the so called Cheng-Yau operator $\square$ acting on the functions defined on the surfaces. As a result, we completely classify the helicoidal surfaces with Gauss map G satisfying $\square G = AG$ for some 3×3 matrix A.

1. Introduction

We consider a surface M of the Euclidean 3-space $\mathbb{E}^3$. The map $G : M \to S^2$ which sends each point of M to the unit normal vector to M at the point is called the Gauss map of the surface M, where S^2 is the unit sphere in $\mathbb{E}^3$ centered at the origin. The theory of Gauss map of a surface in a Euclidean space and a pseudo-Euclidean space is always one of interesting topics and it has been investigated from the various viewpoints by many differential geometers ([8, 9, 10, 11, 14, 16, 17, 18, 21, 23]).

It is well known that a surface M in the Euclidean 3-space $\mathbb{E}^3$ has constant mean curvature if and only if $\Delta G = ||dG||^2 G$, where Δ is the Laplace operator on M corresponding to the induced metric on M from $\mathbb{E}^3$ ([25]). Surfaces whose Gauss map is an eigenfunction of Laplacian, that is, $\Delta G = \lambda G$ for some constant $\lambda \in R$, are the planes, circular cylinders and spheres ([6]).

Generalizing the equation $\Delta G = \lambda G$, F. Dillen et al. ([12]) and C. Baikoussis et al. ([2]), respectively, studied surfaces of revolution and ruled surfaces in the Euclidean 3-space $\mathbb{E}^3$ such that its Gauss map G satisfies the condition

$$\Delta G = AG, \quad A \in R^{3\times 3}. \tag{1.1}$$

As a result, they proved ([2, 12])

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Proposition 1.1. 1) *Among the surfaces of revolution in* $\mathbb{E}^3$*, the only ones whose Gauss map satisfies* (1.1) *are the planes, the spheres and the circular cylinders.*

2) *Among the ruled surfaces in* $\mathbb{E}^3$*, the only ones whose Gauss map satisfies* (1.1) *are the planes and the circular cylinders.*

A natural generalization of rotation surfaces are the helicoidal surfaces ([13]). For helicoidal surfaces C. Baikoussis et al. proved ([4]):

Proposition 1.2. *Let* M *be a helicoidal surface in* $\mathbb{E}^3$*. Then the Gauss map* G *of* M *satisfies* (1.1) *if and only if it is an open part of either a plane, a sphere or a circular cylinder.*

A natural extension of the Laplace operator Δ is the so-called Cheng-Yau operator $\Box$ (or, L_1) introduced by Cheng and Yau ([7]) for the study of hypersurfaces with constant scalar curvature. For an isometric immersion $X : M \to \mathbb{R}^3$ of a surface M, Alías et al. established the following classification theorem ([1]).

Proposition 1.3. *The only surfaces in* $\mathbb{E}^3$*, which satisfy the condition* $\Box X = AX + b$ *for some constant* 3×3 *matrix* A *and some constant vector* b *are either flat or an open part of a sphere.*

In fact, they classified hypersurfaces in the Euclidean space $\mathbb{E}^n$ satisfying the condition $\Box X = AX + b$ for some constant $n \times n$ matrix A and some constant vector b ([1]), which extends the classification theorem for hypersurfaces in $\mathbb{E}^n$ satisfying $\Delta X = AX + b$ given by Chen and Petrovic ([5]) and Hasanis and Vlachos ([15]).

Hence, following the condition (1.1), it is natural to ask as follows.

Question 1.4. Among helicoidal surfaces in the Euclidean 3-space $\mathbb{E}^3$, which one satisfies the following condition?

$$\Box G = AG, \quad A \in R^{3\times 3}. \tag{1.2}$$

In this paper, we give a complete answer to the above question.

For surfaces of revolution whose Gauss map satisfies (1.2), we refer to [19].

The notion of generalized slant cylindrical surfaces (GSCS's) is also a natural extended one of surfaces of revolution ([20]). Surfaces of revolution, cylindrical surfaces and tubes along a plane curve are special cases of GSCS's. In [22], the first author and B. Song proved that among the GSCS's in $\mathbb{E}^3$, the only ones whose Gauss map satisfies (1.1) are the planes, the spheres and the circular cylinders.

Throughout this paper, we assume that all objects are smooth and connected, unless otherwise mentioned.

2. Preliminaries

A natural generalization of rotation surfaces are the helicoidal surfaces that can be defined as follows. Let $\mathbb{R}^3$ have coordinates (x, y, z). Consider the one-parameter subgroup $g_t : \mathbb{R}^3 \to \mathbb{R}^3$ of the group of rigid motions of $\mathbb{R}^3$ given by

$g_t : \mathbb{R}^3 \to \mathbb{R}^3$, $g_t(x, y, z) = (x \cos t - y \sin t, x \sin t + y \cos t, z + ht)$. The motion g_t is called a helicoidal motion with axis Oz and pitch h. A helicoidal surface with axis Oz and pitch h is a surface that is invariant by g_t, for all t. When $h = 0$, they reduce to rotation surfaces.

For a surface M in $\mathbb{E}^3$ with Gauss map G, we denote by S the shape operator of M with respect to the Gauss map G. For each $k = 0, 1$, we put $P_0 = I, P_1 = tr(S)I - S$, where I is the identity operator acting on the tangent bundle of M. Let us define an operator $L_k : C^\infty(M) \to C^\infty(M)$ by $L_k(f) = -tr(P_k \circ \nabla^2 f)$, where $\nabla^2 f : \chi(M) \to \chi(M)$ denotes the self-adjoint linear operator metrically equivalent to the hessian of f. Then, up to signature, L_k is the linearized operator of the first variation of the $(k + 1)$-th mean curvature arising from normal variations of the surface. Note that the operator L_0 is nothing but the Laplace operator acting on M, i.e., $L_0 = \Delta$ and $L_1 = \Box$ is called the Cheng-Yau operator introduced in [7].

Now, we state a useful lemma as follows ([1]).

Lemma 2.1. *Let M be an oriented surface in $\mathbb{E}^3$ with Gaussian curvature K and mean curvature H. Then, the Gauss map G of M satisfies*

$$\Box G = \nabla K + 2HKG, \tag{2.1}$$

where ∇K denotes the gradient of K.

Next, we need the well-known classification theorem for isoparametric surfaces in $\mathbb{E}^3$, which are surfaces with constant principal curvatures.

Proposition 2.2 ([24])**.** *Let M be an isoparametric surface in $\mathbb{E}^3$. Then M is an open part of either a plane, a sphere or a circular cylinder.*

Finally in this section, using Lemma 2.1 we give some examples of surfaces with Gauss map satisfying (1.2).

Example 2.3. (1) Flat surfaces. In this case, we have $\Box G = 0$, and hence flat surfaces satisfy $\Box G = AG$ for some 3×3 matrix A. Note that the matrix A must be singular.

(2) Spheres: $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$. In this case, we have $G = \frac{1}{r}(x - a, y - b, z - c)$ so the sphere satisfies $\Box G = AG$ with $A = -\frac{2}{r^3} I$, where I denotes the identity matrix.

(3) Circular cylinders: $x^2 + y^2 = r^2$. In this case, we have $G = \frac{1}{r}(x, y, 0)$. Hence, the surface M satisfies $\Box G = AG$ for some nonzero matrix A of the following form:

$$A = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{pmatrix}.$$

3. Gauss map of helicoidal surfaces

We consider a regular plane curve $\alpha(s) = (x(s), 0, z(s))$ with $x(s) > 0$ in the xz plane which is defined on an interval I. A surface M in the Euclidean space

$\mathbb{E}^3$ defined by

$$X(s,t) = (x(s)\cos t, x(s)\sin t, z(s) + ht), \tag{3.1}$$

where h is a constant, is said to be the helicoidal surface with axis Oz, pitch h and profile curve α. If $h = 0$, then a helicoidal surface is just a surface of revolution. If $h \neq 0$, we call M a genuine helicoidal surface. For the helicoidal rigid motion $g_t : \mathbb{R}^3 \to \mathbb{R}^3$ given by $g_t(x, y, z) = (x\cos t - y\sin t, x\sin t + y\cos t, z + ht)$, $t \in R$, the helicoidal surface (3.1) is invariant under g_t for all t.

We assume that the profile curve $\alpha(s) = (x(s), 0, z(s))$ is of unit speed with $x(s) > 0$. The adapted frame field $\{e_1, e_2, G\}$ on the helicoidal surface M are given by

$$\begin{aligned} e_1 &= X_s = (x'\cos t, x'\sin t, z'),\\ e_2 &= \frac{1}{Q}\{-hz'X_s + X_t\},\\ G &= e_1 \times e_2 = \frac{1}{Q}(-xz'\cos t + hx'\sin t, -hx'\cos t - xz'\sin t, xx'), \end{aligned} \tag{3.2}$$

where X_s and X_t denote the derivative of X with respect to s and t, respectively and $'$ means the derivative with respect to s. We put

$$Q = Q(s) = \{x^2 + h^2(x')^2\}^{1/2}. \tag{3.3}$$

The metric (g_{ij}) and the second fundamental form (h_{ij}) on the helicoidal surface M are, respectively, given by

$$g_{11} = 1, \quad g_{12} = g_{21} = hz', \quad g_{22} = x^2 + h^2 \tag{3.4}$$

and

$$h_{11} = \frac{x\kappa}{Q}, \quad h_{12} = h_{21} = -\frac{(x')^2h}{Q}, \quad h_{22} = \frac{x^2z'}{Q}, \tag{3.5}$$

where $\kappa = \kappa(s)$ denotes the curvature function of the curve α.

It follows from (3.4) and (3.5) that the Gaussian curvature K and the mean curvature H of M are given by

$$\begin{aligned} K &= \frac{\det(h_{ij})}{\det(g_{ij})} = \frac{x^3z'\kappa - h^2(x')^4}{Q^4},\\ 2H &= \frac{g_{11}h_{22} + g_{22}h_{11} - 2g_{12}h_{12}}{\det(g_{ij})}\\ &= \frac{\kappa x(x^2 + h^2) + x^2z' + 2h^2(x')^2z'}{Q^3}. \end{aligned} \tag{3.6}$$

Note that both of the curvature functions K and H depend only on the parameter s. Using (3.2), ∇K can be computed as follows:

$$\nabla K = e_1(K)e_1 + e_2(K)e_2 = PX_s + RX_t, \tag{3.7}$$

where we put

$$P = P(s) = \frac{x^2 + h^2}{Q^2} K'(s), \quad R = R(s) = -\frac{hz'}{Q^2} K'(s). \tag{3.8}$$

Now, we suppose that the Gauss map G on the helicoidal surface M satisfies $\Box G = AG$ for some matrix $A = (a_{ij})$. Then, it follows from Lemma 2.1 and (3.7) that

$$PX_s + RX_t + 2HKG = AG. \tag{3.9}$$

Putting X_s, $X_t = (-x\sin t, x\cos t, h)$ and G in (3.2) into the equation (3.9), we get the following:

$$\begin{aligned} &QPx'\cos t - QRx\sin t + 2HK(hx'\sin t - xz'\cos t) \\ &= a_{11}(hx'\sin t - xz'\cos t) + a_{12}(-hx'\cos t - xz'\sin t) + a_{13}xx', \end{aligned} \tag{3.10}$$

$$\begin{aligned} &QPx'\sin t + QRx\cos t + 2HK(-hx'\cos t - xz'\sin t) \\ &= a_{21}(hx'\sin t - xz'\cos t) + a_{22}(-hx'\cos t - xz'\sin t) + a_{23}xx' \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} &QPz' + QRh + 2HKxx' \\ &= a_{31}(hx'\sin t - xz'\cos t) + a_{32}(-hx'\cos t - xz'\sin t) + a_{33}xx'. \end{aligned} \tag{3.12}$$

Since $\cos t, \sin t$ and 1 are linearly independent and their coefficients in (3.10) depend only on the parameter s, we get from (3.10)

$$QPx' - 2HKxz' = -a_{11}xz' - a_{12}hx', \tag{3.13}$$

$$-QRx + 2HKhx' = a_{11}hx' - a_{12}xz' \tag{3.14}$$

and

$$a_{13}xx' = 0. \tag{3.15}$$

Similarly, we obtain from (3.11)

$$QRx - 2HKhx' = -a_{21}xz' - a_{22}hx', \tag{3.16}$$

$$QPx' - 2HKxz' = a_{21}hx' - a_{22}xz' \tag{3.17}$$

and

$$a_{23}xx' = 0. \tag{3.18}$$

From (3.12), we also get the following:

$$a_{31}hx' - a_{32}xz' = 0, \tag{3.19}$$

$$-a_{31}xz' - a_{32}hx' = 0 \tag{3.20}$$

and

$$QPz' + QRh + 2HKxx' = a_{33}xx'. \tag{3.21}$$

It follows from (3.13) and (3.17) that

$$(a_{12}+a_{21})hx' = (a_{22}-a_{11})xz'. \tag{3.22}$$

From (3.14) and (3.16) we also get

$$(a_{12}+a_{21})xz' = -(a_{22}-a_{11})hx'. \tag{3.23}$$

Combining (3.22) and (3.23), we have

$$\{(a_{12}+a_{21})^2 + (a_{22}-a_{11})^2\}hxx'z' = 0. \tag{3.24}$$

Let us put $J = \{s \in I | x'(s)z'(s) \neq 0\}$. We divide by two cases as follows.
Case 1. J is empty. In this case, $x'(s)z'(s)$ identically vanishes on the domain I of s. If $x(s)$ is a constant r, then $X(s,t)$ is nothing but a parametrization of the circular cylinder M given by $x^2+y^2=r^2$, which is flat.

If $z(s)$ is a constant, then we may assume $x(s) = s + c$, and hence M is a helicoid. In this case, from (3.3), (3.6) and (3.8) we have

$$Q = \{(s+c)^2 + h^2\}^{1/2}, K = -\frac{h^2}{Q^4}, H = 0, P = K'(s), R = 0. \tag{3.25}$$

Hence, it follows from (3.17) that

$$QP = a_{21}h, \tag{3.26}$$

which contradicts to the equations in (3.25). Therefore, this case cannot occur.
Case 2. Suppose that the subinterval J is nonempty. Then, from (3.15), (3.18) and (3.24) we see that the matrix A is of the following form:

$$A = \begin{pmatrix} \lambda & \mu & 0 \\ -\mu & \lambda & 0 \\ 0 & 0 & \nu \end{pmatrix}.$$

Hence, (3.14) and (3.21), respectively, reduce to

$$-QRx + 2HKhx' = \lambda hx' - \mu xz' \tag{3.27}$$

and

$$QPz' + QRh + 2HKxx' = \nu xx'. \tag{3.28}$$

On the other hands, (3.8) shows that

$$QPz' + QRh = \frac{x^2z'}{Q}K'(s) = -\frac{x^2}{h}QR. \tag{3.29}$$

Hence, (3.28) becomes

$$-QRx + 2HKhx' = \nu hx'. \tag{3.30}$$

Together with (3.27), this equation implies

$$(\lambda - \nu)hx' = \mu xz'. \tag{3.31}$$

We divide by two subcases as follows.
Subcase 2-1. Suppose that $\lambda = \nu$. In this case, $\mu = 0$. Hence we have $A = \lambda I$, where I is the identity matrix. Thus, Lemma 2.1 yields that K is constant and

$2HK = \lambda$. If the Gaussian curvature K is nonzero, then the mean curvature H is also constant. Hence, Proposition 2.2 implies that M is an open part of a sphere, which leads to a contradiction. Thus, the Gaussian curvature K vanishes identically, that is, M is flat.

Subcase 2-2. Suppose that $\lambda \neq \nu$. In this case, (3.31) shows that $\mu \neq 0$ because J is not empty. It follows from (3.31) that

$$z = a \ln x + b, \tag{3.32}$$

where $a = (\lambda - \nu)h/\mu$ and b is a constant. Without loss of generality, we may assume that $b = 0$.

Next, we claim the following.

Claim. Let M denote a helicoidal surface given by

$$X(s,t) = (s\cos t, s\sin t, a\ln s + ht).$$

Then, the Gauss map G of the surface M does not satisfy $\Box G = AG$ for any matrix A.

Proof. Note that the generating curve is not of unit speed. For simplicity, we assume that $a = 1$. Then, just as in the above argument we proceed as follows. The adapted frame field $\{e_1, e_2, G\}$ on the helicoidal surface M are given by

$$\begin{aligned} e_1 &= \frac{s}{\sqrt{s^2+1}} X_s, \\ e_2 &= \frac{1}{Q}\{\frac{-h}{\sqrt{s^2+1}} X_s + \frac{\sqrt{s^2+1}}{s} X_t\}, \\ G = e_1 \times e_2 &= \frac{1}{Q}(-\cos t + h\sin t, -h\cos t - \sin t, s), \end{aligned} \tag{3.33}$$

where we put

$$Q = Q(s) = \{s^2 + h^2 + 1\}^{1/2}. \tag{3.34}$$

The metric (g_{ij}) and the second fundamental form (h_{ij}) on the helicoidal surface M are, respectively, given by

$$g_{11} = 1 + \frac{1}{s^2}, \quad g_{12} = g_{21} = \frac{h}{s}, \quad g_{22} = s^2 + h^2 \tag{3.35}$$

and

$$h_{11} = \frac{-1}{sQ}, \quad h_{12} = h_{21} = \frac{-h}{Q}, \quad h_{22} = \frac{s}{Q}. \tag{3.36}$$

It follows from (3.35) and (3.36) that the Gaussian curvature K and the mean curvature H of M are, respectively, given by

$$K = \frac{-(h^2+1)}{Q^4}, \quad 2H = \frac{(h^2+1)}{sQ^3}. \tag{3.37}$$

The gradient ∇K is also given by

$$\begin{aligned} \nabla K &= e_1(K)e_1 + e_2(K)e_2 \\ &= PX_s + RX_t, \end{aligned} \tag{3.38}$$

where we put

$$P = \frac{s^2+h^2}{Q^2}K'(s), \quad R = \frac{-h}{sQ^2}K'(s). \tag{3.39}$$

Suppose that the Gauss map G of the surface M satisfies $\Box G = AG$ for a matrix A. Then the matrix A must be of the form in Case 2 and we get

$$QP - 2HK = -\lambda - \mu h \tag{3.40}$$

and

$$-QRs + 2HKh = \lambda h - \mu. \tag{3.41}$$

Hence, combining (3.40) and (3.41) we obtain

$$Q(Ph - Rs) = -\mu(h^2+1). \tag{3.42}$$

Since $Ph - Rs = hK'(s)$, (3.42) shows that

$$K'(s) = \frac{-\mu(h^2+1)}{Qh}, \tag{3.43}$$

which contradicts (3.37). This completes the proof of Claim. □

Finally, combining Cases 1 and 2 we get the following classification theorem.

Theorem 3.1. *Let M be a genuine helicoidal surface in the Euclidean 3-space $\mathbb{E}^3$. Then the Gauss map G of M satisfies $\Box G = AG$ for some 3×3 matrix A if and only if it is flat.*

For flat helicoidal surfaces, we refer to [3].

For a genuine helicoidal surface M with Gauss map G satisfying $\Box G = AG$ for some nonzero 3×3 matrix A, as a corollary we get the following.

Corollary 3.2. *Let M be a genuine helicoidal surface in the Euclidean 3-space $\mathbb{E}^3$. Then the Gauss map G of M satisfies $\Box G = AG$ for some nonzero 3×3 matrix A if and only if it is an open part of a circular cylinder.*

Proof. Suppose that $\Box G = AG$ for some nonzero 3×3 matrix A. Then, it follows from Theorem 3.1 that the surface M is flat, hence we have $AG = \Box G = 0$. If we denote by V the kernel space of the matrix A, the image of the Gauss map G lies in the space V. Since A is nonzero, there exists a unit vector $a = (a_1, a_2, a_3)$ which is orthogonal to V. It follows from (3.2) that

$$a_1(-xz'\cos t + hx'\sin t) + a_2(-hx'\cos t - xz'\sin t) + a_3xx' = 0, \tag{3.44}$$

which shows that

$$a_1xz' + a_2hx' = a_1hx' - a_2xz' = a_3xx' = 0. \tag{3.45}$$

The equations in (3.45) reduce to

$$(a_1^2 + a_2^2)xz' = a_3xx' = 0. \tag{3.46}$$

If $J = \{s \in I | x'(s)z'(s) \neq 0\}$ is nonempty, then (3.46) implies that $a = 0$. This contradiction shows that J is empty. Hence, it follows from Case 1 in the proof of Theorem 3.1 that the surface M is an open part of a circular cylinder.

The converse follows from Example 2.3. □

With the help of the classification theorem for surfaces of revolution ([19]), we obtain

Theorem 3.3. *Let M be a helicoidal surface in the Euclidean 3-space $\mathbb{E}^3$. Then we have the following.*

1) *The Gauss map G of M satisfies $\Box G = AG$ for some 3×3 matrix A if and only if it is either a flat surface or an open part of a sphere.*

2) *The Gauss map G of M satisfies $\Box G = AG$ for some nonzero 3×3 matrix A if and only if it is isoparametric, that is, an open part of either a plane, a sphere or a circular cylinder.*

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Dong-Soo Kim
Department of Mathematics
Chonnam National University
Kwangju 500-757, Korea
E-mail address: dosokim@chonnam.ac.kr

Wonyong Kim
Department of Mathematics
Chonnam National University
Kwangju 500-757, Korea
E-mail address: yong4625@naver.com

Young Ho Kim
Department of Mathematics
Kyungpook National University
Taegu 702-701, Korea
E-mail address: yhkim@knu.ac.kr