

THE PROPERTIES OF JORDAN DERIVATIONS OF SEMIPRIME RINGS AND BANACH ALGEBRAS, I

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ABSTRACT. Let R be a 5!-torsion free semiprime ring, and let $D : R \rightarrow R$ be a Jordan derivation on a semiprime ring R . Then $[D(x), x]D(x)^2 = 0$ if and only if $D(x)^2[D(x), x] = 0$ for every $x \in R$. In particular, let A be a Banach algebra with $\text{rad}(A)$ and if D is a continuous linear Jordan derivation on A , then we show that $[D(x), x]D(x)^2 \in \text{rad}(A)$ if and only if $D(x)^2[D(x), x] \in \text{rad}(A)$ for all $x \in A$ where $\text{rad}(A)$ is the Jacobson radical of A .

1. Introduction

Throughout, R represents an associative ring and A will be a complex Banach algebra. We write $[x, y]$ for the commutator $xy - yx$ for x, y in a ring. A ring R is called *n-torsion free* if $nx = 0$ implies $x = 0$. Recall that R is *prime* if $aRb = (0)$ implies that either $a = 0$ or $b = 0$, and is *semiprime* if $aRa = (0)$ implies $a = 0$. $\text{rad}(A)$ denotes the *Jacobson radical* of a Banach algebra A . We say that A is *semisimple* if $\text{rad}(A) = (0)$ (see Bonsall and Duncan [1]).

An additive mapping D from R to R is called a *derivation* if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$. And an additive mapping D from R to R is called a *Jordan derivation* if $D(x^2) = D(x)x + xD(x)$ holds for all $x \in R$.

Johnson and Sinclair [5] have proved that any linear derivation on a semisimple Banach algebra is continuous. A result of Singer and Wermer [14] states that every continuous linear derivation on a commutative Banach algebra maps the algebra into its radical. From these two results, we can conclude that there are no nonzero linear derivations on a commutative semisimple Banach algebra. Thomas [15] has proved that any linear derivation on a commutative Banach algebra maps the algebra into its radical.

Vukman [17] has proved the following: let R be a 2-torsion free prime ring. If $D : R \rightarrow R$ is a derivation such that $[D(x), x]D(x) = 0$ for all $x \in R$, then $D = 0$.

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Moreover, using the above result, he has proved that the following holds: let A be a noncommutative semisimple Banach algebra. Suppose that

$$[D(x), x]D(x) = 0$$

holds for all $x \in A$. In this case, $D = 0$. See [2], [6], [7], [8], [10], [12], [13] and [16] for further results.

In this paper, we generalize the statement of Theorem 3.1 in [11] except the torsion free condition, and we shall give a generalization of theorem 3.2 and its application to the Banach algebra theory.

2. Preliminaries

The following lemma is due to Chung and Luh [4].

Lemma 2.1. *Let R be a $n!$ -torsion free ring. Suppose there exist elements $y_1, y_2, \dots, y_{n-1}, y_n$ in R such that $\sum_{k=1}^n t^k y_k = 0$ for all $t = 1, 2, \dots, n$. Then we have $y_k = 0$ for every positive integer k with $1 \leq k \leq n$.*

The following theorem is due to Brešar [3].

Theorem 2.2. *Let R be a 2-torsion free semiprime ring and let $D : R \rightarrow R$ be a Jordan derivation. In this case, D is a derivation.*

The following theorems are due to Kim [11].

Theorem 2.3. *Let R be a $3!$ -torsion free semiprime ring. Let $D : R \rightarrow R$ be a Jordan derivation on R . In this case, it follows that*

$$[D(x), x]D(x) = 0 \Leftrightarrow D(x)[D(x), x] = 0$$

for every $x \in R$.

Theorem 2.4. *Let A be a Banach algebra with $\text{rad}(A)$. Let $D : A \rightarrow A$ be a continuous linear Jordan derivation. Then we obtain*

$$[D(x), x]D(x) \in \text{rad}(A) \Leftrightarrow D(x)[D(x), x] \in \text{rad}(A)$$

for every $x \in A$.

3. Main results

We need the following notations. After this, by S_m we denote the set $\{k \in \mathbb{N} \mid 1 \leq k \leq m\}$, where m is a positive integer. When R is a ring, we shall denote the maps $B : R \times R \rightarrow R$, $f, g, h, F : R \rightarrow R$ by $B(x, y) \equiv [D(x), y] + [D(y), x]$, $f(x) \equiv [D(x), x]$, $g(x) \equiv [f(x), x]$, $h(x) \equiv [g(x), x]$, $F(x) \equiv [D^2(x), x]$ for all $x, y \in R$ respectively. And in particular, for a Jordan derivation D on R , we shall denote it by $D^2(x) = (D \circ D)(x) = D(D(x))$, and $D^n(x) = (D \circ D^{n-1})(x) = D(D^{n-1}(x))$ for all integers $n \geq 2$ and all $x \in R$. Moreover, we have the basic properties:

$$\begin{aligned} B(x, y) &= B(y, x), \quad B(x, yz) = B(x, y)z + yB(x, z) + D(y)[z, x] + [y, x]D(z), \\ B(x, yD(x)) &= B(x, y)D(x) + yF(x) + D(y)f(x) + [y, x]D^2(x), \end{aligned}$$

$$B(x, D(x)y) = D(x)B(x, y) + F(x)y + f(x)D(y) + D^2(x)[y, x],$$

$$B(x, x) = 2f(x), B(x, x^2) = 2(f(x)x + xf(x)), x, y, z \in R.$$

Theorem 3.1. *Let R be a 3!-torsion free semiprime ring. Suppose there exists a Jordan derivation $D : R \rightarrow R$ such that*

$$[[D(x), x], x]D(x) = 0$$

for all $x \in R$. Then we have $3f(x)D(x)^2 - D(x)f(x)D(x) = 0$ for all $x \in R$.

Proof. Let R be a commutative ring. Then $f(x) = [D(x), x] = 0$ for all $x \in R$. Hence it is clear that $3f(x)D(x)^2 - D(x)f(x)D(x) = 0$ holds for all $x \in R$. Thus it is sufficient to prove the above statement in the noncommutative case of R . By Theorem 2.2, we can see that D is a derivation on R .

Assume that

$$(1) \quad [[D(x), x], x]D(x) = [f(x), x]D(x) = g(x)D(x) = 0, x \in R.$$

Replacing $x + ty$ for x in (1), we have

$$\begin{aligned} & [[D(x + ty), x + ty], x + ty]D(x + ty) \\ & \equiv [[D(x), x], x]D(x) + t\{[B(x, y), x]D(x) \\ & \quad + [f(x), y]D(x) + g(x)D(y)\} + t^2G_1(x, y) \\ (2) \quad & + t^3G_2(x, y) + t^4g(y)D(y) = 0, x, y \in R, t \in S_3, \end{aligned}$$

where G_1 and G_2 denote the term satisfying the identity (2).

From (1) and (2), we obtain

$$(3) \quad \begin{aligned} & t\{[B(x, y), x]D(x) + [f(x), y]D(x) + g(x)D(y)\} + t^2G_1(x, y) \\ & + t^3G_2(x, y) = 0, x, y \in R, t \in S_3. \end{aligned}$$

Since R is 3!-torsion free, by Lemma 2.1 the relation (3) yields

$$(4) \quad [B(x, y), x]D(x) + [f(x), y]D(x) + g(x)D(y) = 0, x, y \in R.$$

Let $y = x^2$ in (4). Then using (1), we get

$$\begin{aligned} 0 &= 2\{[f(x)x + xf(x), x]\}D(x) + (g(x)x + xg(x))D(x) \\ & \quad + g(x)(D(x)x + xD(x)) \\ &= 2g(x)xD(x) + 2xg(x)D(x) + g(x)xD(x) + xg(x)D(x) \\ & \quad + g(x)D(x)x + g(x)xD(x) \\ &= 4g(x)xD(x) + 3xg(x)D(x) + g(x)D(x)x \\ (5) \quad &= 4g(x)xD(x) = 4h(x)D(x) = -4g(x)f(x) = 0, x \in R. \end{aligned}$$

Since R is 3!-torsion free, it follows from (5) that

$$(6) \quad h(x)D(x) = g(x)f(x) = 0, x \in R.$$

Substituting $yD(x)$ for y in (4), we arrive at

$$0 = [B(x, yD(x)), x]D(x) + [f(x), yD(x)]D(x) + g(x)D(yD(x))$$

$$\begin{aligned}
&= [B(x, y)D(x) + D(y)f(x) + yF(x) + [y, x]D^2(x), x]D(x) \\
&\quad + ([f(x), y]D(x)^2 + y[f(x), D(x)]D(x) + g(x)D(y)D(x) + g(x)yD^2(x)) \\
&= [B(x, y)D(x), x]D(x) + [D(y)f(x), x]D(x) + [yF(x), x]D(x) \\
&\quad + [[y, x]D^2(x), x]D(x) \\
&= [B(x, y), x]D(x)^2 + B(x, y)f(x)D(x) + [D(y), x]f(x)D(x) \\
&\quad + D(y)g(x)D(x) + y[F(x), x]D(x) + [y, x]F(x)D(x) + [y, x]F(x)D(x) \\
&\quad + [[y, x], x]D^2(x)D(x) + [f(x), y]D(x)^2 + y[f(x), D(x)]D(x) \\
(7) \quad &+ g(x)D(y)D(x) + g(x)yD^2(x), \quad x, y \in R.
\end{aligned}$$

Right multiplication of (4) by $D(x)$ leads to

$$(8) \quad [B(x, y), x]D(x)^2 + [f(x), y]D(x)^2 + g(x)D(y)D(x) = 0, \quad x, y \in R.$$

From (7) and (8),

$$\begin{aligned}
&B(x, y)f(x)D(x) + [D(y), x]f(x)D(x) + D(y)g(x)D(x) \\
&\quad + y[F(x), x]D(x) + [y, x]F(x)D(x) + [y, x]F(x)D(x) \\
(9) \quad &+ [[y, x], x]D^2(x)D(x) + y[f(x), D(x)]D(x) + g(x)yD^2(x) = 0, \quad x, y \in R.
\end{aligned}$$

Combining (1) with (9),

$$\begin{aligned}
&B(x, y)f(x)D(x) + [D(y), x]f(x)D(x) + y[F(x), x]D(x) \\
&\quad + [y, x]F(x)D(x) + [y, x]F(x)D(x) + [[y, x], x]D^2(x)D(x) \\
(10) \quad &+ y[f(x), D(x)]D(x) + g(x)yD^2(x) = 0, \quad x, y \in R.
\end{aligned}$$

Writing xy for y in (10), we arrive at

$$\begin{aligned}
0 &= xB(x, y)f(x)D(x) + 2f(x)yf(x)D(x) + D(x)[y, x]f(x)D(x) \\
&\quad + [xD(y) + D(x)y, x]f(x)D(x) + xy[F(x), x]D(x) \\
&\quad + x[y, x]F(x)D(x) + x[y, x]F(x)D(x) + x[[y, x], x]D^2(x)D(x) \\
&\quad + xy[f(x), D(x)]D(x) + g(x)xyD^2(x) \\
&= xB(x, y)f(x)D(x) + 2f(x)yf(x)D(x) + D(x)[y, x]f(x)D(x) \\
&\quad + x[D(y), x]f(x)D(x) + D(x)[y, x]f(x)D(x) \\
&\quad + f(x)yf(x)D(x) + xy[F(x), x]D(x) \\
&\quad + x[y, x]F(x)D(x) + x[y, x]F(x)D(x) + x[[y, x], x]D^2(x)D(x) \\
(11) \quad &+ xy[f(x), D(x)]D(x) + g(x)xyD^2(x), \quad x, y \in R.
\end{aligned}$$

Left multiplication of (10) by x leads to

$$\begin{aligned}
&xB(x, y)f(x)D(x) + x[D(y), x]f(x)D(x) + xy[F(x), x]D(x) \\
&\quad + x[y, x]F(x)D(x) + x[y, x]F(x)D(x) + x[[y, x], x]D^2(x)D(x) \\
(12) \quad &+ xy[f(x), D(x)]D(x) + xg(x)yD^2(x) = 0, \quad x, y \in R.
\end{aligned}$$

From (1), (11) and (12),

$$(13) \quad 3f(x)yf(x)D(x) + 2D(x)[y, x]f(x)D(x) + h(x)yD^2(x) = 0, \quad x, y \in R.$$

Substituting $D(x)y$ for y in (13), we arrive at

$$(14) \quad \begin{aligned} & 3f(x)D(x)yf(x)D(x) + D(x)^2[y, x]f(x)D(x) + D(x)f(x)yf(x)D(x) \\ & + D(x)^2[y, x]f(x)D(x) + D(x)f(x)yf(x)D(x) \\ & + h(x)D(x)yD^2(x) = 0, \quad x, y \in R. \end{aligned}$$

From (6) and (14),

$$(15) \quad \begin{aligned} & \{3f(x)D(x) + 2D(x)f(x)\}yf(x)D(x) + D(x)^2[y, x]f(x)D(x) \\ & + D(x)^2[y, x]f(x)D(x) = 0, \quad x, y \in R. \end{aligned}$$

Left multiplication of (13) by $D(x)$ leads to

$$(16) \quad \begin{aligned} & 3D(x)f(x)yf(x)D(x) + D(x)^2[y, x]f(x)D(x) \\ & + D(x)^2[y, x]f(x)D(x) + D(x)h(x)yD^2(x) = 0, \quad x, y \in R. \end{aligned}$$

From (1), (15) and (16),

$$(17) \quad \{3f(x)D(x) - D(x)f(x)\}yf(x)D(x) - D(x)h(x)yD^2(x) = 0, \quad x, y \in R.$$

Replacing $D(x)y$ for y in (17), we arrive at

$$(18) \quad \begin{aligned} & \{3f(x)D(x)^2 - D(x)f(x)D(x)\}yf(x)D(x) - D(x)h(x)D(x)yD^2(x) \\ & = 0, \quad x, y \in R. \end{aligned}$$

From (6) and (18),

$$(19) \quad \{3f(x)D(x)^2 - D(x)f(x)D(x)\}yf(x)D(x) = 0, \quad x, y \in R.$$

Writing $yD(x)$ for y in (19), we obtain

$$(20) \quad \{3f(x)D(x)^2 - D(x)f(x)D(x)\}yD(x)f(x)D(x) = 0, \quad x, y \in R.$$

Right multiplication of (19) by $3D(x)$ leads to

$$(21) \quad \{3f(x)D(x)^2 - D(x)f(x)D(x)\}y(3f(x)D(x)^2) = 0, \quad x, y \in R.$$

From (20) and (21),

$$(22) \quad \begin{aligned} & \{3f(x)D(x)^2 - D(x)f(x)D(x)\}y(3f(x)D(x)^2 - D(x)f(x)D(x)) \\ & = 0, \quad x, y \in R. \end{aligned}$$

Since R is semiprime, it follows from (22) that

$$(23) \quad 3f(x)D(x)^2 - D(x)f(x)D(x) = 0, \quad x \in R. \quad \square$$

Using the same technique with necessary variations one can prove the following Theorem and statement the Theorem 3.2 without the proof.

Theorem 3.2. *Let R be a $3!$ -torsion free semiprime ring. Suppose there exists a Jordan derivation $D : R \rightarrow R$ such that*

$$D(x)[[D(x), x], x] = 0$$

for all $x \in R$. Then we have $3D(x)^2f(x) - D(x)f(x)D(x) = 0$ for all $x \in R$.

Lemma 3.3. *Let R be a $5!$ -torsion free semiprime ring. Let $D : R \rightarrow R$ be a Jordan derivation on R . And assume that*

$$D(x)g(x)yD^2(x)D(x) = D(x)[f(x), x]yD^2(x)D(x) = 0$$

for all $x, y \in R$. Then we have $D(x)g(x) = D(x)[f(x), x] = 0$ for all $x \in R$.

Proof. When R is commutative, we see that if $f(x) = [D(x), x] = 0$ for all $x \in R$, then $D(x)g(x) = D(x)[f(x), x] = 0$ for all $x \in R$.

Hence it is sufficient to prove the above statement in the noncommutative case of R . By Theorem 2.2, we can see that D is a derivation on R .

Suppose

$$(24) \quad D(x)g(x)yD^2(x)D(x) = [f(x), x]D(x)yD^2(x)D(x) = 0, \quad x \in R.$$

Writing $x + tz$ for x in (24), we get

$$\begin{aligned} & D(x + tz)[[D(x + tz), x + tz], x + tz]yD^2(x + tz)D(x + tz) \\ \equiv & D(x)g(x)yD^2(x)D(x) + t\{(D(z)g(x) + D(x)([B(x, z), x] \\ & + [f(x), z]))yD^2(x)D(x) + D(x)g(x)y(D^2(z)D(x) + D^2(x)D(z))\} \\ & + t^2I_1(x, y, z) + t^3I_2(x, y, z) + t^4I_3(x, y, z) + t^5I_4(x, y, z) \\ (25) \quad & + t^6D(z)g(z)yD^2(z)D(z) = 0, \quad x, y, z \in R, \quad t \in S_5, \end{aligned}$$

where I_1, I_2, I_3 and I_4 denote the term satisfying the identity (25).

From (24) and (25), we obtain

$$\begin{aligned} & t\{(D(z)g(x) + D(x)([B(x, z), x] + [f(x), z]))yD^2(x)D(x) \\ & + D(x)g(x)y(D^2(z)D(x) + D^2(x)D(z))\} + t^2I_1(x, y, z) + t^3I_2(x, y, z) \\ (26) \quad & + t^4I_3(x, y, z) + t^5I_4(x, y, z) = 0, \quad x, y, z \in R, \quad t \in S_5. \end{aligned}$$

Since R is $5!$ -torsion free, by Lemma 2.1 the relation (26) yields

$$\begin{aligned} & \{D(z)g(x) + D(x)([B(x, z), x] + [f(x), z])\}yD^2(x)D(x) \\ (27) \quad & + D(x)g(x)y\{D^2(z)D(x) + D^2(x)D(z)\} = 0, \quad x, y, z \in R. \end{aligned}$$

Writing $wD(x)g(x)y$ for y in (27), we get

$$\begin{aligned} & (D(z)g(x) + D(x)([B(x, z), x] + [f(x), z]))wD(x)g(x)yD^2(x)D(x) \\ (28) \quad & + D(x)g(x)wD(x)g(x)y(D^2(z)D(x) + D^2(x)D(z)) = 0, \quad x, y, z \in R. \end{aligned}$$

From (24) and (28),

$$(29) \quad D(x)g(x)wD(x)g(x)y(D^2(z)D(x) + D^2(x)D(z)) = 0, \quad x, y, z \in R.$$

Replacing $y(D^2(z)D(x) + D^2(x)D(z))w$ for w in (29), we get

$$(30) \quad \begin{aligned} & D(x)g(x)y(D^2(z)D(x) + D^2(x)D(z))wD(x)g(x)y\{D^2(z)D(x) \\ & + D^2(x)D(z)\} = 0, \quad x, y, z \in R. \end{aligned}$$

Since R is semiprime, we get from (30)

$$(31) \quad D(x)g(x)y(D^2(z)D(x) + D^2(x)D(z)) = 0, \quad x, y, z \in R.$$

Writing $x + tw$ for x in (31), we get

$$(32) \quad \begin{aligned} & D(x + tw)[f(x + tw), x + tw]y(D^2(z)D(x + tw) + D^2(x + tw)D(z)) \\ & \equiv D(x)g(x)y(D^2(z)D(x) + D^2(x)D(z)) + t\{D(w)g(x) \\ & + D(x)([B(x, w), x] + [f(x), w])y(D^2(z)D(x) + D^2(x)D(z)) \\ & + D(x)g(x)y(D^2(z)D(w) + D^2(w)D(z))\} + t^2K_1(x, y, z) \\ & + t^3K_2(x, y, z) + t^4K_3(x, y, z) \\ & + t^5D(w)g(w)y(D^2(z)D(w) + D(w)D^2(z)) \\ & = 0, \quad x, y, z \in R, \quad t \in S_5, \end{aligned}$$

where K_1, K_2, K_3 denotes the term satisfying the identity (32).

From (31) and (32), we obtain

$$(33) \quad \begin{aligned} & t\{D(w)g(x) + D(x)([B(x, w), x] + [f(x), w])y(D^2(z)D(x) + D^2(x)D(z)) \\ & + D(x)g(x)y(D^2(z)D(w) + D^2(w)D(z))\} + t^2K_1(x, y, z) + t^3K_2(x, y, z) \\ & + t^4K_3(x, y, z) = 0, \quad w, x, y, z \in R, \quad t \in S_5. \end{aligned}$$

Since R is 5!-torsion free, by Lemma 2.1 the relation (33) yields

$$(34) \quad \begin{aligned} & D(w)g(x) + D(x)([B(x, w), x] + [f(x), w])y(D^2(z)D(x) + D^2(x)D(z)) \\ & + D(x)g(x)y(D^2(z)D(w) + D^2(w)D(z)) = 0, \quad w, x, y, z \in R. \end{aligned}$$

Writing $vD(x)g(x)y$ for y in (34), we get

$$(35) \quad \begin{aligned} & D(w)g(x) + D(x)([B(x, w), x] + [f(x), w])vg(x)D(x)y\{D^2(z)D(x) \\ & + D^2(x)D(z)\} + D(x)g(x)vg(x)D(x)y(D^2(z)D(w) + D^2(w)D(z)) \\ & = 0, \quad w, x, y, z \in R. \end{aligned}$$

From (31) and (35),

$$(36) \quad D(x)g(x)vD(x)g(x)y(D^2(z)D(w) + D^2(w)D(z)) = 0, \quad v, w, x, y, z \in R.$$

Replacing $y(D^2(z)D(w) + D^2(w)D(z))v$ for w in (36), we get

$$(37) \quad \begin{aligned} & D(x)g(x)y(D^2(z)D(w) + D^2(w)D(z))vD(x)g(x)y\{D^2(z)D(w) \\ & + D^2(w)D(z)\} = 0, \quad v, w, x, y, z \in R. \end{aligned}$$

Since R is semiprime, we get from (37)

$$(38) \quad D(x)g(x)y(D^2(z)D(w) + D^2(w)D(z)) = 0, \quad w, x, y, z \in R.$$

Putting $wD(z)$ instead of for w in (38), we have

$$(39) \quad D(x)g(x)y\{D^2(z)D(w)D(z) + D^2(z)wD^2(z) + D^2(w)D(z)^2\} \\ + 2D(w)D^2(z)D(z) + wD^3(z)D(z) = 0, \quad w, x, y, z \in R.$$

From (24), (38) and (39), we obtain

$$(40) \quad D(x)g(x)y\{D^2(z)wD^2(z) + wD^3(z)D(z)\} = 0, \quad w, x, y, z \in R.$$

Writing $D(z)w$ for w in (40), we get

$$(41) \quad D(x)g(x)y\{D^2(z)D(z)wD^2(z) + D(z)wD^3(z)D(z)\} = 0, \quad w, x, y, z \in R.$$

From (24) and (41), we obtain

$$(42) \quad D(x)g(x)yD(z)wD^3(z)D(z) = 0, \quad w, x, y, z \in R.$$

Replacing $yD^3(z)$ for y in (42), we get

$$(43) \quad D(x)g(x)yD^3(z)D(z)wD^3(z)D(z) = 0, \quad w, x, y, z \in R.$$

Substituting $wD(x)g(x)y$ instead of for w in (43), we have

$$(44) \quad D(x)g(x)yD^3(z)D(z)wD(x)g(x)yD^3(z)D(z) = 0, \quad w, x, y, z \in R.$$

Since R is semiprime, it follows from (44) that

$$(45) \quad D(x)g(x)yD^3(z)D(z) = 0, \quad x, y, z \in R.$$

From (40) and (45), we obtain

$$(46) \quad D(x)g(x)yD^2(z)wD^2(z) = 0, \quad w, x, y, z \in R.$$

Substituting $wD(x)g(x)y$ for w in (46), we get

$$(47) \quad D(x)g(x)yD^2(z)wD(x)g(x)yD^2(z) = 0, \quad w, x, y, z \in R.$$

Since R is semiprime, it follows from (47) that

$$(48) \quad D(x)g(x)yD^2(z) = 0, \quad x, y, z \in R.$$

Replacing zw for z in (48), we get

$$(49) \quad D(x)g(x)y(D^2(z)w + 2D(z)D(w) + zD^2(w)) = 0, \quad w, x, y, z \in R.$$

From (48) and (49), we obtain

$$(50) \quad 2D(x)g(x)yD(z)D(w) = 0, \quad w, x, y, z \in R.$$

Since R is 3!-torsionfree, it follows from (50) that

$$(51) \quad D(x)g(x)yD(z)D(w) = 0, \quad x, y, z \in R.$$

Writing wz for w in (51), we get

$$(52) \quad D(x)g(x)y\{D(z)D(w)z + D(z)wD(z)\} = 0, \quad w, x, y, z \in R.$$

From (51) and (52), we obtain

$$(53) \quad D(x)g(x)yD(z)wD(z) = 0, \quad w, x, y, z \in R.$$

Substituting $wD(x)g(x)y$ for w in (53), we get

$$(54) \quad D(x)g(x)yD(z)wD(x)g(x)yD(z) = 0, \quad w, x, y, z \in R.$$

Since R is semiprime, it follows from (54) that

$$(55) \quad D(x)g(x)yD(z) = 0, \quad x, y, z \in R.$$

Right multiplication of (55) by $g(x)$ leads to

$$(56) \quad D(x)g(x)yD(z)g(z) = 0, \quad x, y, z \in R.$$

Since R is semiprime, we obtain from (56)

$$(57) \quad D(x)g(x) = 0, \quad x \in R. \quad \square$$

Using the same technique with necessary variations one can prove the following lemmas and statements Lemmas 3.4 and 3.5 without the proofs.

Lemma 3.4. *Let R be a $5!$ -torsion free semiprime ring. Let $D : R \rightarrow R$ be a Jordan derivation on R . Assume that*

$$g(x)D(x)yD^2(x)D(x) = [f(x), x]D(x)yD^2(x)D(x) = 0$$

for all $x, y \in R$. Then we have $g(x)D(x) = [f(x), x]D(x) = 0$ for all $x \in R$.

Lemma 3.5. *Let R be a $3!$ -torsion free semiprime ring. Let $D : R \rightarrow R$ be a Jordan derivation on R . Assume that*

$$D(x)D^2(x)yD(x)g(x) = D(x)D^2(x)yD(x)[f(x), x] = 0$$

for all $x, y \in R$. Then we have $D(x)g(x) = D(x)[f(x), x] = 0$ for all $x \in R$.

Lemma 3.6. *Let R be a $3!$ -torsion free semiprime ring. Let $D : R \rightarrow R$ be a Jordan derivation on R . Assume that*

$$D(x)D^2(x)yg(x)D(x) = D(x)D^2(x)y[f(x), x]D(x) = 0$$

for all $x, y \in R$. Then we have $g(x)D(x) = [f(x), x]D(x) = 0$ for all $x \in R$.

Proof. In the commutative case of R , we see that if $f(x) = [D(x), x] = 0$ for all $x \in R$. Hence it is clear that $D(x)g(x) = D(x)[f(x), x] = 0$ for all $x \in R$. It is sufficient to prove the above statement in the noncommutative case of R .

By Theorem 2.2, we can see that D is a derivation on R .

In any semiprime ring, we see that $ayb = 0 \iff bya = 0$ for all $y \in R$. Thus it follows that

$$D(x)D^2(x)yD(x) = 0 \iff g(x)D(x)yD(x)D^2(x) = 0, \quad x \in R.$$

Hence we may assume that

$$(58) \quad g(x)D(x)yD(x)D^2(x) = [f(x), x]D(x)yD(x)D^2(x) = 0, \quad x \in R.$$

Writing $x + tz$ for x in (58), we get

$$\begin{aligned} & [[D(x + tz), x + tz], x + tz]D(x + tz)yD(x + tz)D^2(x + tz) \\ & \equiv g(x)D(x)yD(x)D^2(x) + t\{([B(x, z), x] + [f(x), z])D(x) \} \end{aligned}$$

$$\begin{aligned}
& + g(x)D(z)yD(x)D^2(x) + g(x)D(x)y(D(z)D^2(x) + D(x)D^2(z))\} \\
& + t^2N_1(x, y, z) + t^3N_2(x, y, z) + t^4N_3(x, y, z) + t^5N_4(x, y, z) \\
(59) \quad & + t^6g(z)D(z)yD(z)D^2(z) = 0, \quad x, y, z \in R, \quad t \in S_5,
\end{aligned}$$

where N_1, N_2, N_3 and N_4 denote the term satisfying the identity (59).

From (58) and (59), we obtain

$$\begin{aligned}
& t\{([B(x, z), x] + [f(x), z])D(x) \\
& + g(x)D(z)yD(x)D^2(x) + g(x)D(x)y(D(z)D^2(x) + D(x)D^2(z))\} \\
& + t^2N_1(x, y, z) + t^3N_2(x, y, z) + t^4N_3(x, y, z) + t^5N_4(x, y, z) \\
(60) \quad & = 0, \quad x, y, z \in R, \quad t \in S_5.
\end{aligned}$$

Since R is 5!-torsion free, by Lemma 2.1 the relation (60) yields

$$\begin{aligned}
& ([B(x, z), x] + [f(x), z])D(x) + g(x)D(z)yD(x)D^2(x) \\
(61) \quad & + g(x)D(x)y(D(z)D^2(x) + D(x)D^2(z)) = 0, \quad x, y, z \in R.
\end{aligned}$$

Writing $wg(x)D(x)y$ for y in (61), we get

$$\begin{aligned}
& ([B(x, z), x] + [f(x), z])D(x) + g(x)D(z)wg(x)D(x)yD(x)D^2(x) \\
(62) \quad & + g(x)D(x)wg(x)D(x)y(D(z)D^2(x) + D(x)D^2(z)) = 0, \quad x, y, z \in R.
\end{aligned}$$

From (58) and (62),

$$(63) \quad g(x)D(x)wg(x)D(x)y(D(z)D^2(x) + D(x)D^2(z)) = 0, \quad x, y, z \in R.$$

Replacing $y(D(z)D^2(x) + D(x)D^2(z))w$ for w in (63), we get

$$\begin{aligned}
& g(x)D(x)y(D(z)D^2(x) + D(x)D^2(z))wg(x)D(x)y\{D(z)D^2(x) \\
(64) \quad & + D(x)D^2(z)\} = 0, \quad x, y, z \in R.
\end{aligned}$$

Since R is semiprime, we get from (64)

$$(65) \quad g(x)D(x)y(D(z)D^2(x) + D(x)D^2(z)) = 0, \quad x, y, z \in R.$$

Writing $x + tw$ for x in (31), we get

$$\begin{aligned}
& [f(x + tw), x + tw]D(x + tw)y(D(z)D^2(x + tw) + D(x + tw)D^2(z)) \\
& \equiv g(x)D(x)y(D(z)D^2(x) + D(x)D^2(z)) + t\{D(w)g(x) + ([B(x, w), x] \\
& + [f(x), w])D(x) + g(x)D(w)y(D(z)D^2(x) + D(x)D^2(z)) \\
& + g(x)D(x)y(D(z)D^2(w) + D(w)D^2(z))\} + t^2P_1(x, y, z) + t^3P_2(x, y, z) \\
& + t^4P_3(x, y, z) + t^5g(w)D(w)y(D(z)D^2(w) + D(w)D^2(z)) \\
(66) \quad & = 0, \quad x, y, z \in R, \quad t \in S_5,
\end{aligned}$$

where P_1, P_2 and P_3 denote the term satisfying the identity (66).

From (65) and (66), we obtain

$$t\{([B(x, w), x] + [f(x), w])D(x)$$

$$\begin{aligned}
& + g(x)D(w)y(D(z)D^2(x) + D(x)D^2(z)) + g(x)D(x)y(D(z)D^2(w) \\
& + D(w)D^2(z))\} + t^2P_1(x, y, z) + t^3P_2(x, y, z) \\
(67) \quad & + t^4P_3(x, y, z) = 0, \quad w, x, y, z \in R, \quad t \in S_5.
\end{aligned}$$

Since R is 5!-torsion free, by Lemma 2.1 the relation (67) yields

$$\begin{aligned}
& ([B(x, w), x] + [f(x), w])D(x) \\
& + g(x)D(w)y(D(z)D^2(x) + D(x)D^2(z)) + g(x)D(x)y(D(z)D^2(w) \\
(68) \quad & + D(w)D^2(z)) = 0, \quad w, x, y, z \in R.
\end{aligned}$$

Writing $vg(x)D(x)y$ for y in (68), we get

$$\begin{aligned}
& ([B(x, w), x] + [f(x), w])D(x) \\
& + g(x)D(w)vg(x)D(x)y(D(z)D^2(x) + D(x)D^2(z)) \\
(69) \quad & + g(x)D(x)vg(x)D(x)y(D(z)D^2(w) + D(w)D^2(z)) = 0, \quad w, x, y, z \in R.
\end{aligned}$$

From (65) and (69),

$$(70) \quad g(x)D(x)vg(x)D(x)y(D(z)D^2(w) + D(w)D^2(z)) = 0, \quad v, w, x, y, z \in R.$$

Replacing $y(D(z)D^2(w) + D(w)D^2(z))v$ for v in (70), we get

$$\begin{aligned}
& g(x)D(x)y(D(z)D^2(w) + D(w)D^2(z))vg(x)D(x)y\{D(z)D^2(w) \\
(71) \quad & + D(w)D^2(z)\} = 0, \quad v, w, x, y, z \in R.
\end{aligned}$$

Since R is semiprime, we get from (71)

$$(72) \quad g(x)D(x)y(D(z)D^2(w) + D(w)D^2(z)) = 0, \quad w, x, y, z \in R.$$

Putting $D(z)w$ instead of for w in (72), we have

$$\begin{aligned}
& g(x)D(x)y\{D(z)^2D^2(w) + 2D(z)D^2(z)w + D(z)D^3(z)w \\
(73) \quad & + D(z)D(w)D^2(z) + D^2(z)wD^2(z)\} = 0, \quad w, x, y, z \in R.
\end{aligned}$$

From (58), (72) and (73), we obtain

$$(74) \quad g(x)D(x)y\{D(z)D^3(z)w + D^2(z)wD^2(z)\} = 0, \quad w, x, y, z \in R.$$

Writing $wD(z)$ for w in (74), we get

$$(75) \quad g(x)D(x)y\{D(z)D^3(z)wD(z) + D^2(z)wD(z)D^2(z)\} = 0, \quad w, x, y, z \in R.$$

From (58) and (75), we obtain

$$(76) \quad g(x)D(x)yD(z)D^3(z)wD(z) = 0, \quad w, x, y, z \in R.$$

Right multiplication of (76) by $g(x)$ leads to

$$(77) \quad g(x)D(x)yD(z)D^3(z)wD(z)D^3(z) = 0, \quad w, x, y, z \in R.$$

Substituting $wg(x)D(x)y$ for w in (77), we have

$$(78) \quad g(x)D(x)yD(z)D^3(z)wg(x)D(x)yD(z)D^3(z) = 0, \quad w, x, y, z \in R.$$

Since R is semiprime, it follows from (78) that

$$(79) \quad g(x)D(x)yD(z)D^3(z) = 0, \quad x, y, z \in R.$$

From (74) and (79), we obtain

$$(80) \quad g(x)D(x)yD^2(z)wD^2(z) = 0, \quad w, x, y, z \in R.$$

Substituting $wg(x)D(x)y$ for w in (80), we get

$$(81) \quad g(x)D(x)yD^2(z)wg(x)D(x)yD^2(z) = 0, \quad w, x, y, z \in R.$$

Since R is semiprime, it follows from (81) that

$$(82) \quad g(x)D(x)yD^2(z) = 0, \quad x, y, z \in R.$$

Replacing xz for z in (82), we get

$$(83) \quad D(x)g(x)y(xD^2(z)w + 2D(x)D(z) + D^2(x)z) = 0, \quad x, y, z \in R.$$

From (82) and (83), we obtain

$$(84) \quad 2g(x)D(x)yD(x)D(z) = 0, \quad x, y, z \in R.$$

Since R is 3!-torsion free, it follows from (84) that

$$(85) \quad g(x)D(x)yD(x)D(z) = 0, \quad x, y, z \in R.$$

Writing zx for z in (85), we get

$$(86) \quad g(x)D(x)y\{D(x)D(z)x + D(x)zD(x)\} = 0, \quad x, y, z \in R.$$

From (85) and (86), we obtain

$$(87) \quad g(x)D(x)yD(x)zD(x) = 0, \quad x, y, z \in R.$$

Substituting $zg(x)D(x)y$ for z in (87), we get

$$(88) \quad g(x)D(x)yD(x)zg(x)D(x)yD(x) = 0, \quad x, y, z \in R.$$

Since R is semiprime, it follows from (88) that

$$(89) \quad g(x)D(x)yD(x) = 0, \quad x, y, z \in R.$$

Putting $yg(x)$ for y in (89), we get

$$(90) \quad g(x)D(x)yg(x)D(x) = 0, \quad x, y \in R.$$

Since R is semiprime, we obtain from (90)

$$(91) \quad g(x)D(x) = 0, \quad x \in R. \quad \square$$

Theorem 3.7. *Let R be a 5!-torsion free semiprime ring. Let $D : R \rightarrow R$ be a Jordan derivation on R . Then*

$$[D(x), x]D(x)^2 = 0 \Leftrightarrow D(x)^2[D(x), x] = 0$$

for every $x \in R$.

Proof. The proof of the commutative case is trivial. Thus it suffices to prove the case that R is noncommutative.

Necessity: Assume that

$$(92) \quad [D(x), x]D(x)^2 = f(x)D(x)^2 = 0, \quad x \in R.$$

Replacing $x + ty$ for x in (92), we have

$$(93) \quad \begin{aligned} & [D(x + ty), x + ty]D(x + ty)^2 \\ & \equiv f(x)D(x)^2 + t\{B(x, y)D(x)^2 + f(x)D(y)D(x) + f(x)D(x)D(y)\} \\ & + t^2H_1(x, y) + t^3H_2(x, y) + t^4f(y)D(y)^2 = 0, \quad x, y \in R, \quad t \in S_3, \end{aligned}$$

where H_1, H_2 denote the term satisfying the identity (93).

From (92) and (93), we obtain

$$(94) \quad \begin{aligned} & t\{B(x, y)D(x)^2 + f(x)D(y)D(x) + f(x)D(x)D(y)\} \\ & + t^2H_1(x, y) + t^3H_2(x, y) = 0, \quad x, y \in R, \quad t \in S_5. \end{aligned}$$

Since R is 5!-torsion free, by Lemma 2.1 the relation (94) yields

$$(95) \quad B(x, y)D(x)^2 + f(x)D(y)D(x) + f(x)D(x)D(y) = 0, \quad x, y \in R.$$

Writing xy for y in (95), we have

$$(96) \quad \begin{aligned} & xB(x, y)D(x)^2 + 2f(x)yD(x)^2 + D(x)[y, x]D(x)^2 \\ & + f(x)xD(y)D(x) + f(x)D(x)yD(x) + f(x)D(x)xD(y) \\ & + f(x)D(x)^2y = 0, \quad x, y \in R. \end{aligned}$$

Left multiplication of (95) by x leads to

$$(97) \quad xB(x, y)D(x)^2 + xf(x)D(y)D(x) + xf(x)D(x)D(y) = 0, \quad x, y \in R.$$

From (96) and (97), we obtain

$$(98) \quad \begin{aligned} & 2f(x)yD(x)^2 + D(x)[y, x]D(x)^2 + g(x)D(y)D(x) \\ & + f(x)D(x)yD(x) + \{g(x)D(x) + f(x)^2\}D(y) \\ & + f(x)D(x)^2y = 0, \quad x, y \in R. \end{aligned}$$

From (92) and (98), we obtain

$$(99) \quad \begin{aligned} & 2f(x)yD(x)^2 + D(x)[y, x]D(x)^2 + g(x)D(y)D(x) \\ & + f(x)D(x)yD(x) + \{g(x)D(x) + f(x)^2\}D(y) = 0, \quad x, y \in R. \end{aligned}$$

Writing $yD(x)$ for y in (99), we have

$$(100) \quad \begin{aligned} & 2f(x)yD(x)^3 + D(x)[y, x]D(x)^3 + D(x)yf(x)D(x)^2 + g(x)D(y)D(x)^2 \\ & + g(x)yD^2(x)D(x) + f(x)D(x)yD(x)^2 + \{g(x)D(x) + f(x)^2\}D(y)D(x) \\ & + \{g(x)D(x) + f(x)^2\}yD^2(x) = 0, \quad x, y \in R. \end{aligned}$$

Right multiplication of (99) by $D(x)$ leads to

$$(101) \quad \begin{aligned} & 2f(x)yD(x)^3 + D(x)[y, x]D(x)^3 + g(x)D(y)D(x)^2 \\ & + f(x)D(x)yD(x)^2 + \{g(x)D(x) + f(x)^2\}D(y)D(x) = 0, \quad x, y \in R. \end{aligned}$$

From (100) and (101), we obtain

$$(102) \quad \begin{aligned} & D(x)yf(x)D(x)^2 + g(x)yD^2(x)D(x) \\ & + \{g(x)D(x) + f(x)^2\}yD^2(x) = 0, \quad x, y \in R. \end{aligned}$$

From (92) and (102), we obtain

$$(103) \quad g(x)yD^2(x)D(x) + \{g(x)D(x) + f(x)^2\}yD^2(x) = 0, \quad x, y \in R.$$

On the other hand, we get from (92)

$$(104) \quad \begin{aligned} 0 &= [f(x)D(x)^2, x] \\ &= g(x)D(x)^2 + f(x)^2D(x) + f(x)D(x)f(x), \quad x \in R. \end{aligned}$$

On the one hand, let $y = x^2$ in (95). Then we obtain

$$(105) \quad \begin{aligned} 0 &= B(x, x^2)D(x)^2 + f(x)D(x^2)D(x) + f(x)D(x)D(x^2) \\ &= 2(f(x)x + xf(x))D(x)^2 + f(x)(D(x)x + xD(x))D(x) \\ &\quad + f(x)D(x)(D(x)x + xD(x)) \\ &= 3f(x)xD(x)^2 + 2xf(x)D(x)^2 + 2f(x)D(x)xD(x) + f(x)D(x)^2x, \quad x \in R. \end{aligned}$$

From (92) and (105), we obtain

$$(106) \quad \begin{aligned} & 3f(x)xD(x)^2 + 2xf(x)D(x)^2 + 2f(x)D(x)xD(x) + f(x)D(x)^2x \\ &= 0, \quad x \in R. \end{aligned}$$

Since $3xf(x)D(x)^2 = 0$, $2xf(x)D(x)^2 = 0$ holds for all $x \in R$ from (92), we get from (106)

$$(107) \quad \begin{aligned} 0 &= 3g(x)D(x)^2 + 2(g(x)D(x) + f(x)^2)D(x) \\ &= 5g(x)D(x)^2 + 2f(x)^2D(x), \quad x \in R. \end{aligned}$$

Substituting $D(x)y$ for y in (107), we have

$$(108) \quad g(x)D(x)yD^2(x)D(x) + \{g(x)D(x)^2 + f(x)^2D(x)\}yD^2(x) = 0, \quad x, y \in R.$$

From (104) and (108), we obtain

$$(109) \quad g(x)D(x)yD^2(x)D(x) - f(x)D(x)f(x)yD^2(x) = 0, \quad x, y \in R.$$

Replacing $D(x)^2y$ for y in (109), we get

$$(110) \quad g(x)D(x)^3yD^2(x)D(x) - f(x)D(x)f(x)D(x)^2yD^2(x) = 0, \quad x, y \in R.$$

From (92) and (110), it follows that

$$(111) \quad g(x)D(x)^3yD^2(x)D(x) = 0, \quad x, y \in R.$$

Putting $D(x)y$ instead of y in (108), we arrive at

$$(112) \quad g(x)D(x)^2yD^2(x)D(x) + \{g(x)D(x)^3 + f(x)^2D(x)^2\}yD^2(x) = 0, x, y \in R.$$

From (92), (111) and (112), one obtains

$$(113) \quad g(x)D(x)^2yD^2(x)D(x) = 0, x, y \in R.$$

Substituting $2yD^2(x)D(x)z$ for y in (108), we have

$$(114) \quad \begin{aligned} & 2g(x)D(x)yD^2(x)D(x)zD^2(x)D(x) \\ & + \{2g(x)D(x)^2 + 2f(x)^2D(x)\}yD^2(x)D(x)zD^2(x) = 0, x, y, z \in R. \end{aligned}$$

From (113) and (114), we get

$$(115) \quad \begin{aligned} & 2g(x)D(x)yD^2(x)D(x)zD^2(x)D(x) \\ & + 2f(x)^2D(x)yD^2(x)D(x)zD^2(x) = 0, x, y, z \in R. \end{aligned}$$

From (107) and (115),

$$(116) \quad \begin{aligned} & 2g(x)D(x)yD^2(x)D(x)zD^2(x)D(x) \\ & - 5g(x)D(x)^2yD^2(x)D(x)zD^2(x) = 0, x, y, z \in R. \end{aligned}$$

From (113) and (116), we arrive at

$$(117) \quad 2g(x)D(x)yD^2(x)D(x)zD^2(x)D(x) = 0, x, y, z \in R.$$

Since R is 3!-torsion free, it follows from (117) that

$$(118) \quad g(x)D(x)yD^2(x)D(x)zD^2(x)D(x) = 0, x, y, z \in R.$$

Replacing $zg(x)D(x)y$ for z in (118), we obtain

$$(119) \quad g(x)D(x)yD^2(x)D(x)zg(x)D(x)yD^2(x)D(x) = 0, x, y, z \in R.$$

By the semiprimeness of R , we get from (119)

$$(120) \quad g(x)D(x)yD^2(x)D(x) = 0, x, y \in R.$$

Substituting $D(x)y$ for y in (99), we have

$$(121) \quad \begin{aligned} & 2f(x)D(x)yD(x)^2 + D(x)^2[y, x]D(x)^2 + D(x)f(x)yD(x)^2 \\ & + g(x)D(x)D(y)D(x) + g(x)D^2(x)yD(x) + f(x)D(x)^2yD(x) \\ & + \{g(x)D(x)^2 + f(x)^2D(x)\}D(y) + \{g(x)D(x)D^2(x) + f(x)^2D^2(x)\}y \\ & = 0, x, y \in R. \end{aligned}$$

Left multiplication of (99) by $D(x)$ leads to

$$(122) \quad \begin{aligned} & 2D(x)f(x)yD(x)^2 + D(x)^2[y, x]D(x)^2 + D(x)g(x)D(y)D(x) \\ & + D(x)f(x)D(x)yD(x) + \{D(x)g(x)D(x) + D(x)f(x)^2\}D(y) \\ & = 0, x, y \in R. \end{aligned}$$

From (121) and (122), we obtain

$$\{2f(x)D(x) - D(x)f(x)\}yD(x)^2$$

$$\begin{aligned}
& + \{g(x)D(x) - D(x)g(x)\}D(y)D(x) + g(x)D^2(x)yD(x) \\
& + \{f(x)D(x)^2 - D(x)f(x)D(x)\}yD(x) \\
& + \{g(x)D(x)^2 + f(x)^2D(x) - D(x)g(x)D(x) - D(x)f(x)^2\}D(y) \\
(123) \quad & + \{g(x)D(x)D^2(x) + f(x)^2D^2(x)\}y = 0, \quad x, y \in R.
\end{aligned}$$

From (92) and (123), we obtain

$$\begin{aligned}
& \{2f(x)D(x) - D(x)f(x)\}yD(x)^2 \\
& + \{g(x)D(x) - D(x)g(x)\}D(y)D(x) + g(x)D^2(x)yD(x) \\
& - D(x)f(x)D(x)yD(x) \\
& + \{g(x)D(x)^2 + f(x)^2D(x) - D(x)g(x)D(x) - D(x)f(x)^2\}D(y) \\
(124) \quad & + \{g(x)D(x)D^2(x) + f(x)^2D^2(x)\}y = 0, \quad x, y \in R.
\end{aligned}$$

Writing $yD(x)$ for y in (124), we have

$$\begin{aligned}
& \{2f(x)D(x) - D(x)f(x)\}yD(x)^3 \\
& + \{g(x)D(x) - D(x)g(x)\}D(y)D(x)^2 \\
& + \{g(x)D(x) - D(x)g(x)\}yD^2(x)D(x) \\
& + g(x)D^2(x)yD(x)^2 - D(x)f(x)D(x)yD(x)^2 \\
& + \{g(x)D(x)^2 + f(x)^2D(x) - D(x)g(x)D(x) - D(x)f(x)^2\}D(y)D(x) \\
(125) \quad & + \{g(x)D(x)D^2(x) + f(x)^2D^2(x)\}yD(x) = 0, \quad x, y \in R.
\end{aligned}$$

Right multiplication of (124) by $D(x)$ leads to

$$\begin{aligned}
& \{2f(x)D(x) - D(x)f(x)\}yD(x)^3 \\
& + \{g(x)D(x) - D(x)g(x)\}D(y)D(x)^2 + g(x)D^2(x)yD(x)^2 \\
& - D(x)f(x)D(x)yD(x)^2 \\
& + \{g(x)D(x)^2 + f(x)^2D(x) - D(x)g(x)D(x) - D(x)f(x)^2\}D(y)D(x) \\
(126) \quad & + \{g(x)D(x)D^2(x) + f(x)^2D^2(x)\}yD(x) = 0, \quad x, y \in R.
\end{aligned}$$

From (125) and (126), we obtain

$$(127) \quad \{g(x)D(x) - D(x)g(x)\}yD^2(x)D(x) = 0, \quad x, y \in R.$$

From (120) and (127), we obtain

$$(128) \quad D(x)g(x)yD^2(x)D(x) = 0, \quad x, y \in R.$$

By Lemma 3.4, we get from (128)

$$(129) \quad D(x)g(x) = 0, \quad x \in R.$$

Hence by Lemma 3.2, we obtain from (129), we get

$$(130) \quad 3D(x)^2f(x) - D(x)f(x)D(x) = 0, \quad x \in R.$$

Right multiplication of (107) by $D(x)$ leads to

$$(131) \quad 5g(x)D(x)^3 + 2f(x)^2D(x)^2 = 0, \quad x \in R.$$

From (92) and (131), we obtain

$$(132) \quad 5g(x)D(x)^3 = 0, \quad x \in R.$$

Since R is 5!-torsion free, we get from (132)

$$(133) \quad g(x)D(x)^3 = 0, \quad x, y \in R.$$

Replacing $D(x)^2y$ for y in (103), we have

$$(134) \quad g(x)D(x)^2yD^2(x)D(x) + \{g(x)D(x)^3 + f(x)^2D(x)^2\}yD^2(x) = 0, \quad x, y \in R.$$

From (92), (133) and (134), we obtain

$$(135) \quad g(x)D(x)^2yD^2(x)D(x) = 0, \quad x, y \in R.$$

Writing $2D(x)y$ for y in (103), we have

$$(136) \quad 2g(x)D(x)yD^2(x)D(x) + \{2g(x)D(x)^2 + 2f(x)^2D(x)\}yD^2(x) = 0, \quad x, y \in R.$$

From (107) and (136), we obtain

$$(137) \quad 2g(x)D(x)yD^2(x)D(x) - 3g(x)D(x)^2yD^2(x) = 0, \quad x, y \in R.$$

Replacing $yD^2(x)D(x)z$ for y in (137), we have

$$(138) \quad \begin{aligned} & 2g(x)D(x)yD^2(x)D(x)zD^2(x)D(x) \\ & - 3g(x)D(x)^2yD^2(x)D(x)zD^2(x) = 0, \quad x, y, z \in R. \end{aligned}$$

From (135) and (138), we obtain

$$(139) \quad 2g(x)D(x)yD^2(x)D(x)zD^2(x)D(x) = 0, \quad x, y, z \in R.$$

Since R is 5!-torsion free, we get from (139)

$$(140) \quad g(x)D(x)yD^2(x)D(x)zD^2(x)D(x) = 0, \quad x, y, z \in R.$$

Replacing $zg(x)D(x)y$ for z in (140), we have

$$(141) \quad g(x)D(x)yD^2(x)D(x)zg(x)D(x)yD^2(x)D(x) = 0, \quad x, y, z \in R.$$

Since R is semiprime, it follows from (141) that

$$(142) \quad g(x)D(x)yD^2(x)D(x) = 0, \quad x, y \in R.$$

By Lemma 3.3, we get from (142)

$$(143) \quad g(x)D(x) = 0, \quad x \in R.$$

Hence by Lemma 3.1, we obtain from (143), we get

$$(144) \quad 3f(x)D(x)^2 - D(x)f(x)D(x) = 0, \quad x \in R.$$

Thus combining (130) with (144), we have

$$(145) \quad 3(f(x)D(x)^2 - D(x)^2f(x)^2) = 0, \quad x \in R.$$

Since R is 3!-torsion free, it follows from (145) that

$$(146) \quad f(x)D(x)^2 - D(x)^2f(x) = 0, \quad x \in R.$$

Thus from (92) and (146), we get

$$D(x)^2f(x) = 0, \quad x \in R.$$

Sufficiency: Assume that

$$(147) \quad D(x)^2[D(x), x] = D(x)^2f(x) = 0, \quad x \in R.$$

Replacing $x + ty$ for x in (147), we have

$$(148) \quad \begin{aligned} & D(x + ty)^2[D(x + ty), x + ty] \\ & \equiv D(x)^2f(x) + t\{D(y)D(x)f(x) + D(x)D(y)f(x) + D(x)^2B(x, y)\} \\ & + t^2Q_1(x, y) + t^3Q_2(x, y) + t^4D(y)^2f(y) = 0, \quad x, y \in R, \quad t \in S_3, \end{aligned}$$

where Q_1 and Q_2 denote the term satisfying the identity (148).

From (147) and (148), we obtain

$$(149) \quad \begin{aligned} & t\{D(y)D(x)f(x) + D(x)D(y)f(x) + D(x)^2B(x, y)\} \\ & + t^2P_1(x, y) + t^3P_2(x, y) = 0, \quad x, y \in R, \quad t \in S_3. \end{aligned}$$

Since R is 3!-torsion free, by Lemma 2.1 the relation (149) yields

$$(150) \quad D(y)D(x)f(x) + D(x)D(y)f(x) + D(x)^2B(x, y) = 0, \quad x, y \in R.$$

Right multiplication of (150) by x leads to

$$(151) \quad D(y)D(x)f(x)x + D(x)D(y)f(x)x + D(x)^2B(x, y)x = 0, \quad x, y \in R.$$

Substituting yx for y in (150), we have

$$(152) \quad \begin{aligned} & D(y)xD(x)f(x) + yD(x)^2f(x) + D(x)D(y)xf(x) + D(x)yD(x)f(x) \\ & + D(x)^2B(x, y)x + 2D(x)^2yf(x) + D(x)^2[y, x]D(x) \\ & = 0, \quad x, y \in R. \end{aligned}$$

From (151) and (152), we obtain

$$(153) \quad \begin{aligned} & D(y)\{f(x)^2 + D(x)g(x)\} - yD(x)^2f(x) - D(x)yD(x)f(x) \\ & + D(x)D(y)g(x) - 2D(x)^2yf(x) - D(x)^2[y, x]D(x) = 0, \quad x, y \in R. \end{aligned}$$

From (147) and (153), we obtain

$$(154) \quad \begin{aligned} & D(y)\{f(x)^2 + D(x)g(x)\} - D(x)yD(x)f(x) + D(x)D(y)g(x) \\ & - 2D(x)^2yf(x) - D(x)^2[y, x]D(x) = 0, \quad x, y \in R. \end{aligned}$$

On the one hand, let $y = x^2$ in (150). Then we arrive at

$$\begin{aligned} 0 & = D(x^2)D(x)f(x) + D(x)D(x^2)f(x) + D(x)^2B(x, x^2) \\ & = \{D(x)x + xD(x)\}D(x)f(x) + D(x)\{D(x)x + xD(x)\}f(x) \\ & \quad + 2D(x)^2\{f(x)x + xf(x)\} \end{aligned}$$

$$\begin{aligned}
&= D(x)xD(x)f(x) + xD(x)^2f(x) + D(x)^2xf(x) + D(x)xD(x)f(x) \\
&\quad + 2D(x)^2\{f(x)x + xf(x)\} \\
&= D(x)xD(x)f(x) + xD(x)^2f(x) + D(x)^2xf(x) + D(x)xD(x)f(x) \\
&\quad + 2D(x)^2f(x)x + 2D(x)^2xf(x) \\
(155) \quad &= 2D(x)xD(x)f(x) + xD(x)^2f(x) + 3D(x)^2xf(x) + 2D(x)^2f(x)x, \quad x \in R.
\end{aligned}$$

From (147) and (155), we obtain

$$(156) \quad 2D(x)xD(x)f(x) + 3D(x)^2xf(x) = 0, \quad x \in R.$$

From (147) and (156), we have

$$\begin{aligned}
0 &= -\{2D(x)[x, D(x)f(x)] + 3D(x)^2[x, f(x)]\} \\
&= 2D(x)f(x)^2 + 2D(x)^2g(x) + 3D(x)^2g(x) \\
(157) \quad &= 2D(x)f(x)^2 + 5D(x)^2g(x), \quad x \in R.
\end{aligned}$$

Writing $D(x)y$ for y in (154), we have

$$\begin{aligned}
&D(x)D(y)\{f(x)^2 + D(x)g(x)\} + D^2(x)y\{f(x)^2 + D(x)g(x)\} \\
&\quad - D(x)^2yD(x)f(x) + D(x)^2D(y)g(x) + D(x)D^2(x)yg(x) \\
(158) \quad &- 2D(x)^3yf(x) - D(x)^3[y, x]D(x) - D(x)^2f(x)yD(x) = 0, \quad x, y \in R.
\end{aligned}$$

Left multiplication of (154) by $D(x)$ leads to

$$\begin{aligned}
&D(x)D(y)\{f(x)^2 + D(x)g(x)\} - D(x)^2yD(x)f(x) + D(x)^2D(y)g(x) \\
(159) \quad &- 2D(x)^3yf(x) - D(x)^3[y, x]D(x) = 0, \quad x, y \in R.
\end{aligned}$$

From (158) and (159), we obtain

$$\begin{aligned}
&D^2(x)y\{f(x)^2 + D(x)g(x)\} + D(x)D^2(x)yg(x) \\
(160) \quad &- D(x)^2f(x)yD(x) = 0, \quad x, y \in R.
\end{aligned}$$

From (147) and (160), we obtain

$$(161) \quad D^2(x)y\{f(x)^2 + D(x)g(x)\} + D(x)D^2(x)yg(x) = 0, \quad x, y \in R.$$

Putting $2D(x)y$ instead of y in (161), we have

$$(162) \quad D^2(x)y\{2D(x)f(x)^2 + 2D(x)^2g(x)\} + 2D(x)D^2(x)yD(x)g(x) = 0, \quad x, y \in R.$$

From (157) and (162), we obtain

$$(163) \quad -3D^2(x)yD(x)^2g(x) + 2D(x)D^2(x)yD(x)g(x) = 0, \quad x, y \in R.$$

Left multiplication of (157) by $D(x)$ yields

$$(164) \quad 2D(x)^2f(x)^2 + 5D(x)^3g(x) = 0, \quad x \in R.$$

From (147) and (164), we have

$$(165) \quad 5D(x)^3g(x) = 0, \quad x \in R.$$

Since R is $5!$ torsion free, it follows from (165) that

$$(166) \quad D(x)^3 g(x) = 0, \quad x \in R.$$

Replacing $yD(x)$ for y in (163), we have

$$(167) \quad -3D^2(x)yD(x)^3 g(x) + 2D(x)D^2(x)yD(x)^2 g(x) = 0, \quad x, y \in R.$$

From (166) and (167), we have

$$(168) \quad 2D(x)D^2(x)yD(x)^2 g(x) = 0, \quad x \in R.$$

Since R is $5!$ torsion free, it follows from (168) that

$$(169) \quad D(x)D^2(x)yD(x)^2 g(x) = 0, \quad x \in R.$$

Replacing $zD(x)D^2(x)y$ for y in (163), we obtain

$$(170) \quad \begin{aligned} & -3D^2(x)zD(x)D^2(x)yD(x)^2 g(x) \\ & + 2D(x)D^2(x)zD(x)D^2(x)yD(x)g(x) = 0, \quad x, y \in R. \end{aligned}$$

From (169) and (170), we have

$$(171) \quad 2D(x)D^2(x)zD(x)D^2(x)yD(x)g(x) = 0, \quad x, y \in R.$$

Since R is $5!$ torsion free, we get from (171)

$$(172) \quad D(x)D^2(x)zD(x)D^2(x)yD(x)g(x) = 0, \quad x, y \in R.$$

Substituting $yD(x)g(x)z$ for z in (172), we obtain

$$(173) \quad D(x)D^2(x)yD(x)g(x)zD(x)D^2(x)yD(x)g(x) = 0, \quad x, y \in R.$$

By the semiprimeness of R , it follows from (173) that

$$(174) \quad D(x)D^2(x)yD(x)g(x) = 0, \quad x, y \in R.$$

Thus by Lemma 3.4, we get from (174)

$$(175) \quad D(x)g(x) = 0, \quad x \in R.$$

Hence by Lemma 3.2, we have from (175)

$$(176) \quad 3D(x)^2 f(x) - D(x)f(x)D(x) = 0, \quad x \in R.$$

From (147) and (176), we get

$$(177) \quad D(x)f(x)D(x) = 0, \quad x \in R.$$

From (177), we get

$$(178) \quad \begin{aligned} 0 &= [D(x)f(x)D(x), x] \\ &= f(x)^2 D(x) + D(x)g(x)D(x) + D(x)f(x)^2, \quad x \in R. \end{aligned}$$

From (175) and (178), we get

$$(179) \quad f(x)^2 D(x) + D(x)f(x)^2 = 0, \quad x \in R.$$

From (161) and (175), we obtain

$$(180) \quad D^2(x)yf(x)^2 + D(x)D^2(x)yg(x) = 0, \quad x, y \in R.$$

From (157) and (175), we have

$$(181) \quad 2D(x)f(x)^2 = 0, \quad x \in R.$$

Since R is $5!$ torsion free, we get from (181)

$$(182) \quad D(x)f(x)^2 = 0, \quad x \in R.$$

From (179) and (182), we have

$$(183) \quad f(x)^2D(x) = 0, \quad x \in R.$$

Right multiplication of (180) by $D(x)$ yields

$$(184) \quad D^2(x)yf(x)^2D(x) + D(x)D^2(x)yg(x)D(x) = 0, \quad x, y \in R.$$

From (183) and (184), we have

$$(185) \quad D(x)D^2(x)yg(x)D(x) = 0, \quad x \in R.$$

Thus Lemma 3.3, (185) yields

$$(186) \quad g(x)D(x) = 0, \quad x \in R.$$

By Theorem 3.1, we obtain from (186)

$$(187) \quad 3f(x)D(x)^2 - D(x)f(x)D(x) = 0, \quad x \in R.$$

From (176) and (187), we obtain

$$(188) \quad 3(f(x)D(x)^2 - D(x)^2f(x)) = 0, \quad x \in R.$$

Since R is $5!$ torsion free, we get from (188)

$$(189) \quad f(x)D(x)^2 - D(x)^2f(x) = 0, \quad x \in R.$$

From (147) and (189), we get

$$f(x)D(x)^2 = 0, \quad x \in R. \quad \square$$

Remark 3.8. Let R be a $3!$ -torsion free semiprime ring. Let $D : R \rightarrow R$ be a Jordan derivation on R . In this case, by some calculations, it is checked that if $[D(x), x]D(x)^2 = 0$ for every $x \in R$, then $f(x) = [D(x), x] = 0$ for all $x \in R$.

The following theorem is nearly proved by the same arguments as in the proof of J. Vukman's theorem [17].

Theorem 3.9. *Let A be a Banach algebra with $\text{rad}(A)$. Let $D : A \rightarrow A$ be a continuous linear Jordan derivation. In this case, we show that*

$$[D(x), x]D(x)^2 \in \text{rad}(A) \iff D(x)^2[D(x), x] \in \text{rad}(A)$$

for every $x \in A$.

Proof. It suffices to prove the case that A is noncommutative. By the result of B. E. Johnson and A. M. Sinclair [5] any linear derivation on a semisimple Banach algebra is continuous. Sinclair [9] has proved that every continuous linear Jordan derivation on a Banach algebra leaves the primitive ideals of A invariant. Hence for any primitive ideals $P \subset A$ one can introduce a derivation $D_P : A/P \rightarrow A/P$, where A/P is a prime and factor Banach algebra, by $D_P(\hat{x}) = D(x) + P$, $\hat{x} = x + P$. We see that if $[D(x), x]D(x)^2 \in \text{rad}(A)$, we obtain $[D(x), x]D(x)^2 \in \text{rad}(A) \subset P$ for all primitive ideals of A , then $[D_P(\hat{x}), \hat{x}](D_P(\hat{x}))^2 = \hat{0}$. Then since A/P is a prime factor Banach algebra for all primitive ideals of A , by Theorem 3.7, we get $[D_P(\hat{x}), \hat{x}](D_P(\hat{x}))^2 = \hat{0} \iff (D_P(\hat{x}))^2[D_P(\hat{x}), \hat{x}] = \hat{0}$, $\hat{x} \in A/P$ for all primitive ideals of A . Hence we conclude that $D(x)^2[D(x), x] \in P$ for all $x \in A$ and for all primitive ideals P of A . Therefore since $\text{rad}(A) = \bigcap \{P : P \text{ is any primitive ideals of } A\}$, it follows that

$$[D(x), x]D(x)^2 \in \text{rad}(A) \iff D(x)^2[D(x), x] \in \text{rad}(A)$$

for every $x \in A$. □

As a special case of Theorem 3.9 we get the following result which characterizes commutative semisimple Banach algebras.

Corollary 3.10. *Let A be a semisimple Banach algebra. Suppose*

$$[[y, x], x][y, x]^2 = 0 \iff [y, x]^2[[y, x], x] = 0$$

for every $x, y \in A$.

Proof. Let $\delta_y(x) = [y, x]$, $[[y, x], x] = [\delta_y(x), x]$, $D = \delta_y$ for all $x, y \in R$. Hence we see that δ_y is a continuous (Jordan) derivation on A . Since A is semisimple, $\text{rad}(A) = (0)$. Thus all the conditions of Theorem 3.10 are fulfilled. □

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