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SIMPLY CONNECTED COMPLEX SURFACES OF GENERAL TYPE WITH $p_g = 0$ AND $K^2 = 1, 2$

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Abstract. We construct various examples of simply connected minimal complex surfaces of general type with $p_g = 0$ and $K^2 = 1, 2$ using $\mathbb{Q}$-Gorenstein smoothing method.

1. Introduction

In this paper we construct various examples of simply connected minimal complex surfaces of general type with $p_g = 0$ and $K^2 = 1, 2$. We apply the $\mathbb{Q}$-Gorenstein smoothing method used in [3, 4, 5].

The examples of this paper would be useful for studying the Kollár-Shepherd-Barron-Alexeev (KSBA) compactification (developed in Kollár-Shepherd-Barron [2]) of surfaces of general type with $\chi = 1$ and $K^2 = 1, 2$ because of the method of construction. The methods in [3, 4, 5] are to find a rational surface Z which contains several disjoint linear chains of $\mathbb{P}^1$ representing the resolution graphs of quotient surface singularities of class T. We contract these chains of $\mathbb{P}^1$ from the rational surface Z to produce a projective singular surface X with singularities of class T. We then prove that the singular surface X has a $\mathbb{Q}$-Gorenstein smoothing and the general fiber X_t of the $\mathbb{Q}$-Gorenstein smoothing is a simply connected minimal surface of general type with $p_g = 0$ and $K^2 = 1, 2$.

Therefore each singular surface X in this paper determines a codimension one component of the boundary of the KSBA compactifications of moduli space of complex surfaces of general type with $\chi = 1$ and $K_X^2 = 1, 2$; cf. Hacking [1]. For instance Stern and Urzúa [6] identified the minimal models of the general surfaces of the KSBA divisors corresponding to each singular surfaces X in this paper.

It is a very interesting problem to determine whether these examples are diffeomorphic (or deformation equivalent) to each other or to already known surfaces. We leave it for further studies.

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2. $\mathbb{Q}$-Gorenstein smoothing method

We review the method of constructions, so-called $\mathbb{Q}$-*Gorenstein smoothing method*. Since all the proofs are basically the same as the case of the main construction in Lee-Park [3, §3], we briefly sketch the method step by step and we recall some delicate parts of the method.

Procedure

At first we take a pencil of cubic curves in $\mathbb{CP}^2$. We resolve the base points (including infinitely near base points) of the pencil by blowing up 9 times along the base points so that we get a rational elliptic surface Y. We further blow up Y *appropriately* (explained below) to construct a rational surface Z that contains several special linear chains of rational curves. The linear chains can be contracted to special cyclic quotient singular points of type $\frac{1}{n^2}(1, na-1)$ with $1 \leq a < n$ and $(n, a) = 1$, which are called *singularities of class* T, on a singular surface X. Then a general fiber X_t of a $\mathbb{Q}$-Gorenstein smoothing of X will be a complex surface with the desired invariants.

Constraints

In order to guarantee that the singular surface X admits a $\mathbb{Q}$-Gorenstein smoothing and its general fiber X_t has the desired invariants, the rational surface Z should be constructed very carefully from Y. The below explains some constraints of the construction of Z.

Existence of a $\mathbb{Q}$-*Gorenstein smoothing.* Since every singularities of class T on the singular surface X has a local $\mathbb{Q}$-Gorenstein smoothing, it is enough to show that there is no obstruction to globalize the local smoothings. Indeed the obstruction lies in $H^2(X, \mathcal{T}_X)$ where $\mathcal{T}_X$ is the tangent sheaf of X. One can prove the vanishing $H^2(X, \mathcal{T}_X) = 0$ by a similar method in Lee–Park [3] if the rational surface Z is constructed according to the following constraints:

Constraint 1. At most two nodal singular fibers of Y (or their proper transforms on Z) are contained the exceptional divisors of the singularities of class T of X

Constraint 2. The exceptional divisors of the singularities of class T of X should not contain all components of any reducible singular fibers (including their proper transforms on Z) of Y.

The desired invariants. At first, the geometric genus $p_g(X_t)$ is zero because X is constructed from a rational surface Z. It is not difficult to show that X_t is simply connected by van Kampen theorem. Indeed if Z_0 is an open 4-manifold obtained by deleting a small open neighborhood of the singular points of X, then it is enough to show that Z_0 is simply connected in order to show that X_t is simply connected. One can show by van Kampen theorem that $\pi_1(Z_0)$ is generated by (roughly speaking) normal circles around the exceptional divisors of the singularities of class T. But the normal circles lie on (-1)-spheres connecting the exceptional divisors. Hence there are relations on the generators of $\pi_1(Z_0)$ and one can show that they should be zero by solving the relations. The self-intersection number K^2 can be computed by the formula

$$\begin{aligned} K^2 = 9 - {} & \text{the number of blowing-ups needed to construct } Z \text{ from } Y \\ + {} & \text{the number of irreducible components} \\ & \text{of the exceptional divisors of the singularities of class } T \text{ of } X \end{aligned}$$

Finally, one of the main constraints arises because X_t should be of general type. For this it is enough to show that K_X is nef. One can easily show that its pull-back f^*K_X on Z is effective. Therefore it is needed to show that the intersection number of f^*K_X with the (-1)-curves on Z are nonnegative. Since every (-1)-curve on Z intersects the exceptional divisor of the singularities of class T, the nefness of K_X follows from the following final constraints.

Constraint 3. Every (-1)-curve on Z should intersect at least two components of the exceptional divisors of the singularities of class T and the sum of the discrepancies of the components of the exceptional divisors intersecting a given (-1)-curve should be not less that one.

Here a *discrepancy* is defined as follows. Let $(X, 0)$ be a normal surface singularity with the minimal good resolution $f\colon (V, E) \to (X, 0)$. Let $E = \sum_{i=1}^{n} E_i$ be the decomposition of the exceptional divisor E with irreducible components E_i. Then

$$K_V = f^*K_X + \sum_{i=1}^{n} a_i E_i$$

for some $a_i \in \mathbb{Z}$. The coefficients a_i is called the *discrepancy* of E_i.

3. Various examples

In the following we list pencils of cubics in $\mathbb{CP}^2$, elliptic fibrations Y obtained from the pencils, and the rational surfaces Z obtained by blowing-up Y several times appropriately. In the rational surfaces Z, we indicate the configurations of linear chains of $\mathbb{P}^1$ which will be contracted so that we obtain a singular surface X which has a $\mathbb{Q}$-Gorenstein smoothing.

Type of singular fibers. The index, for example $I_9 + 3I_1$, denotes the type of singular fibers of elliptic fibrations.

Pencils of cubics. The pencils of cubic curves are presented by two plane cubic curves Γ and Γ' which give rise to the special singular fibers indicated in the index. We describe how they intersect as follows: In the figure of the pencil, the pair $(k, 1)$ denotes the intersection numbers of a curve with the two branches of another curve at a node. We denote by Γ and Γ' the solid curve and the dotted curve, respectively.

Sections. Blowing up several times at each intersection points of two cubic curves, we get a rational elliptic surface Y admitting an elliptic fibration $Y \to \mathbb{CP}^1$. We describe the way how sections S_i of $Y \to \mathbb{CP}^1$ intersect with special singular fibers of the elliptic fibrations. We indicate on Z which sections are used to construct the rational surface Z. We abbreviate S_i to i.

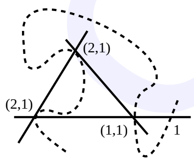
Rational surfaces Z. The number n in $Z = E(1)\#n\overline{\mathbb{CP}}^2$ indicates how many times we blow up to get Z from Y. The numbers in the figures of Z indicate the self-intersection numbers of each curves and all rational curves without self-intersection numbers are -2-curves.

The exceptional divisors. On the dual graphs of the exceptional divisors of the singularities of class T in Z, we denote the discrepancies.

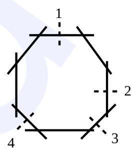
3.1. Examples with $K^2 = 1$

Example 3.1. • *Types of singular fibers*: $I_8 + 4I_1$

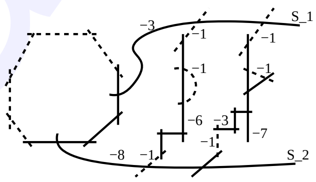
- *Pencils of cubics*



- *Sections*



- *Rational surfaces* $Z = \mathbb{CP}^2\sharp 13\overline{\mathbb{CP}}^2$



- *The exceptional divisors*

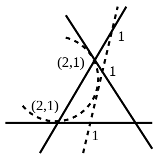
$$C_{4,1} : \underset{-6}{\overset{3/4}{\circ}} - \underset{-2}{\overset{2/4}{\circ}} - \underset{-2}{\overset{1/4}{\circ}}$$

$$C_{9,5}: \underset{-2}{\overset{4/9}{\circ}} - \underset{-7}{\overset{8/9}{\circ}} - \underset{-2}{\overset{7/9}{\circ}} - \underset{-2}{\overset{6/9}{\circ}} - \underset{-3}{\overset{5/9}{\circ}}$$

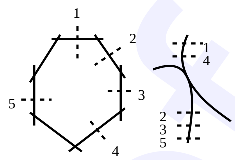
$$C_{11,6}: \underset{-2}{\overset{5/11}{\circ}} - \underset{-8}{\overset{10/11}{\circ}} - \underset{-2}{\overset{9/11}{\circ}} - \underset{-2}{\overset{8/11}{\circ}} - \underset{-2}{\overset{7/11}{\circ}} - \underset{-3}{\overset{6/11}{\circ}}$$

Example 3.2. • *Types of singular fibers*: $I_7 + III + 2I_1$

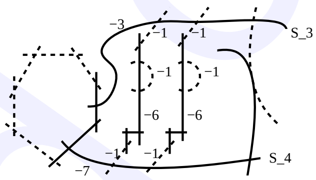
- *Pencils of cubics*



- *Sections*



- *Rational surfaces* $Z = \mathbb{CP}^2 \sharp 10 \overline{\mathbb{CP}}^2$



- *The exceptional divisors*

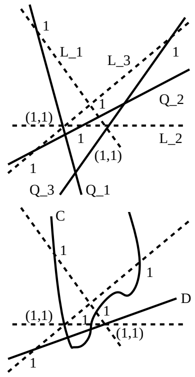
$$C_{4,1}: \underset{-6}{\overset{3/4}{\circ}} - \underset{-2}{\overset{2/4}{\circ}} - \underset{-2}{\overset{1/4}{\circ}}$$

$$C_{4,1}: \underset{-6}{\overset{3/4}{\circ}} - \underset{-2}{\overset{2/4}{\circ}} - \underset{-2}{\overset{1/4}{\circ}}$$

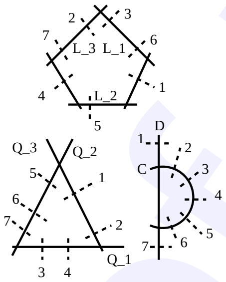
$$C_{9,5}: \underset{-2}{\overset{4/9}{\circ}} - \underset{-7}{\overset{8/9}{\circ}} - \underset{-2}{\overset{7/9}{\circ}} - \underset{-2}{\overset{6/9}{\circ}} - \underset{-3}{\overset{5/9}{\circ}}$$

Example 3.3. • *Types of singular fibers*: $I_5 + I_3 + I_2 + 2I_1$

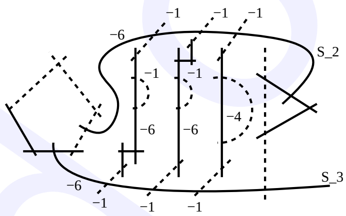
- *Pencils of cubics*



- *Sections*



- *Rational surfaces* $Z = \mathbb{CP}^2 \sharp 12\overline{\mathbb{CP}}^2$



- *The exceptional divisors*

$$C_{2,1}: \underset{-4}{\overset{1/2}{\circ}}$$

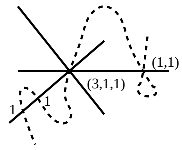
$$C_{4,1}: \underset{-6}{\overset{3/4}{\circ}} - \underset{-2}{\overset{2/4}{\circ}} - \underset{-2}{\overset{1/4}{\circ}}$$

$$C_{4,1}: \underset{-6}{\overset{3/4}{\circ}} - \underset{-2}{\overset{2/4}{\circ}} - \underset{-2}{\overset{1/4}{\circ}}$$

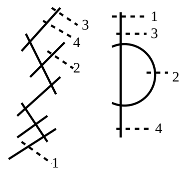
$$C_{4,1}: \underset{-6}{\overset{3/4}{\circ}} - \underset{-2}{\overset{2/4}{\circ}} - \underset{-2}{\overset{1/4}{\circ}}$$

$$C_{4,1}: \underset{-6}{\overset{3/4}{\circ}} - \underset{-2}{\overset{2/4}{\circ}} - \underset{-2}{\overset{1/4}{\circ}}$$

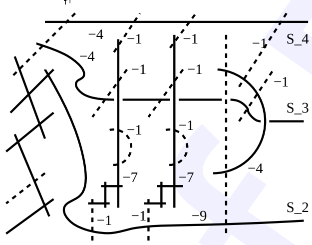
Example 3.4.
- *Types of singular fibers*: $I_2^* + I_2 + 2I_1$
- *Pencils of cubics*



- *Sections*



- *Rational surfaces* $Z = \mathbb{CP}^2 \sharp 16 \overline{\mathbb{CP}}^2$



- *The exceptional divisors*

$$C_{2,1} : \overset{1/2}{\underset{-4}{\circ}}$$

$$C_{2,1} : \overset{1/2}{\underset{-4}{\circ}}$$

$$C_{2,1} : \overset{1/2}{\underset{-4}{\circ}}$$

$$C_{5,1} : \overset{4/5}{\underset{-7}{\circ}} - \overset{3/5}{\underset{-2}{\circ}} - \overset{2/5}{\underset{-2}{\circ}} - \overset{1/5}{\underset{-2}{\circ}}$$

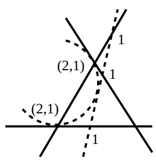
$$C_{5,1} : \overset{4/5}{\underset{-7}{\circ}} - \overset{3/5}{\underset{-2}{\circ}} - \overset{2/5}{\underset{-2}{\circ}} - \overset{1/5}{\underset{-2}{\circ}}$$

$$C_{7,1} : \overset{6/7}{\underset{-9}{\circ}} - \overset{5/7}{\underset{-2}{\circ}} - \overset{4/7}{\underset{-2}{\circ}} - \overset{3/7}{\underset{-2}{\circ}} - \overset{2/7}{\underset{-2}{\circ}} - \overset{1/7}{\underset{-2}{\circ}}$$

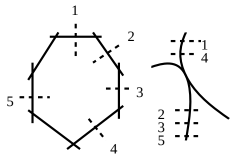
3.2. Examples with $K^2 = 2$

Example 3.5. • *Types of singular fibers*: $I_7 + III + 2I_1$

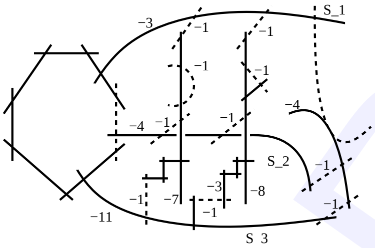
- *Pencils of cubics*



- *Sections*



- *Rational surfaces* $Z = \mathbb{CP}^2 \sharp 19 \overline{\mathbb{CP}}^2$



- *The exceptional divisors*

$C_{2,1}$: $\overset{1/2}{\underset{-4}{\circ}}$

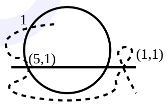
$C_{2,1}$: $\overset{1/2}{\underset{-4}{\circ}}$

$C_{5,1}$: $\overset{4/5}{\underset{-7}{\circ}} - \overset{3/5}{\underset{-2}{\circ}} - \overset{2/5}{\underset{-2}{\circ}} - \overset{1/5}{\underset{-2}{\circ}}$

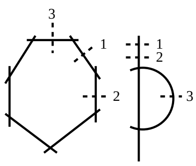
$C_{11,6}$: $\overset{5/11}{\underset{-2}{\circ}} - \overset{10/11}{\underset{-8}{\circ}} - \overset{9/11}{\underset{-2}{\circ}} - \overset{8/11}{\underset{-2}{\circ}} - \overset{7/11}{\underset{-2}{\circ}} - \overset{6/11}{\underset{-3}{\circ}}$

$C_{17,9}$: $\overset{8/17}{\underset{-2}{\circ}} - \overset{16/17}{\underset{-11}{\circ}} - \overset{15/17}{\underset{-2}{\circ}} - \overset{14/17}{\underset{-2}{\circ}} - \overset{13/17}{\underset{-2}{\circ}} - \overset{12/17}{\underset{-2}{\circ}} - \overset{11/17}{\underset{-2}{\circ}} - \overset{10/17}{\underset{-2}{\circ}} - \overset{9/17}{\underset{-3}{\circ}}$

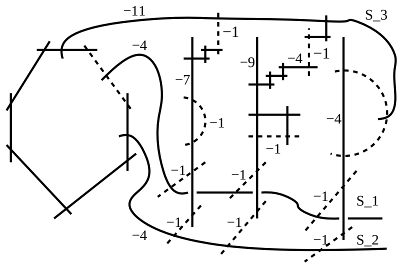
Example 3.6.
- *Types of singular fibers*: $I_7 + I_2 + 3I_1$
- *Pencils of cubics*



- *Sections*



- *Rational surfaces* $Z = \mathbb{CP}^2 \sharp 22 \overline{\mathbb{CP}}^2$



- *The exceptional divisors*

$C_{2,1}: \overset{1/2}{\underset{-4}{\circ}}$

$C_{2,1}: \overset{1/2}{\underset{-4}{\circ}}$

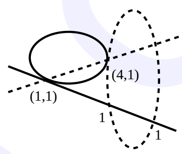
$C_{5,1}: \overset{4/5}{\underset{-7}{\circ}} - \overset{3/5}{\underset{-2}{\circ}} - \overset{2/5}{\underset{-2}{\circ}} - \overset{1/5}{\underset{-2}{\circ}}$

$C_{19,4}: \overset{6/19}{\underset{-2}{\circ}} - \overset{12/19}{\underset{-2}{\circ}} - \overset{18/19}{\underset{-9}{\circ}} - \overset{17/19}{\underset{-2}{\circ}} - \overset{16/19}{\underset{-2}{\circ}} - \overset{15/19}{\underset{-2}{\circ}} - \overset{14/19}{\underset{-2}{\circ}} - \overset{13/19}{\underset{-4}{\circ}}$

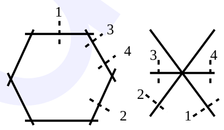
$C_{25,8}: \overset{8/25}{\underset{-2}{\circ}} - \overset{16/25}{\underset{-2}{\circ}} - \overset{24/25}{\underset{-11}{\circ}} - \overset{23/25}{\underset{-2}{\circ}} - \overset{22/25}{\underset{-2}{\circ}} - \overset{21/25}{\underset{-2}{\circ}} - \overset{20/25}{\underset{-2}{\circ}} - \overset{19/25}{\underset{-2}{\circ}} - \overset{18/25}{\underset{-2}{\circ}} - \overset{17/25}{\underset{-4}{\circ}}$

Example 3.7. • *Types of singular fibers*: $I_6 + IV + 2I_1$

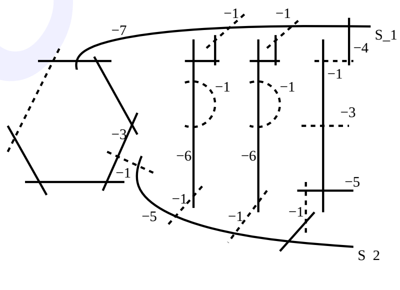
- *Pencils of cubics*



- *Sections*



- *Rational surfaces* $Z = \mathbb{CP}^2 \sharp 15 \overline{\mathbb{CP}}^2$



- *The exceptional divisors*

$$C_{3,1}: \overset{2/3}{\underset{-5}{\circ}} - \overset{1/3}{\underset{-2}{\circ}}$$

$$C_{3,1}: \overset{2/3}{\underset{-5}{\circ}} - \overset{1/3}{\underset{-2}{\circ}}$$

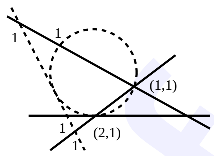
$$C_{4,1}: \overset{3/4}{\underset{-6}{\circ}} - \overset{2/4}{\underset{-2}{\circ}} - \overset{1/4}{\underset{-2}{\circ}}$$

$$C_{4,1}: \overset{3/4}{\underset{-6}{\circ}} - \overset{2/4}{\underset{-2}{\circ}} - \overset{1/4}{\underset{-2}{\circ}}$$

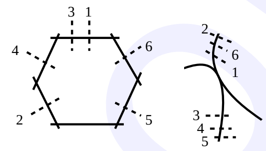
$$C_{19,5}: \overset{14/19}{\underset{-4}{\circ}} - \overset{18/19}{\underset{-7}{\circ}} - \overset{17/19}{\underset{-2}{\circ}} - \overset{16/19}{\underset{-2}{\circ}} - \overset{15/19}{\underset{-3}{\circ}} - \overset{10/19}{\underset{-2}{\circ}} - \overset{5/19}{\underset{-2}{\circ}}$$

Example 3.8.

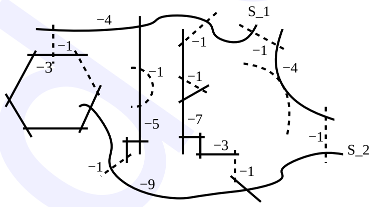
- *Types of singular fibers*: $I_6 + III + 3I_1$
- *Pencils of cubics*



- *Sections*



- *Rational surfaces* $Z = \mathbb{CP}^2 \sharp 15 \overline{\mathbb{CP}}^2$



- *The exceptional divisors*

$$C_{2,1}: \overset{1/2}{\underset{-4}{\circ}}$$

$$C_{7,2}: \overset{5/7}{\underset{-4}{\circ}} - \overset{6/7}{\underset{-5}{\circ}} - \overset{4/7}{\underset{-2}{\circ}} - \overset{2/7}{\underset{-2}{\circ}}$$

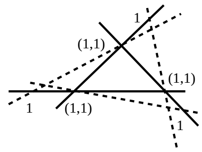
$$C_{9,5}: \overset{4/9}{\underset{-2}{\circ}} - \overset{8/9}{\underset{-7}{\circ}} - \overset{7/9}{\underset{-2}{\circ}} - \overset{6/9}{\underset{-2}{\circ}} - \overset{5/9}{\underset{-3}{\circ}}$$

$$C_{13,7}: \overset{6/13}{\underset{-2}{\circ}} - \overset{12/13}{\underset{-9}{\circ}} - \overset{11/13}{\underset{-2}{\circ}} - \overset{10/13}{\underset{-2}{\circ}} - \overset{9/13}{\underset{-2}{\circ}} - \overset{8/13}{\underset{-2}{\circ}} - \overset{7/13}{\underset{-3}{\circ}}$$

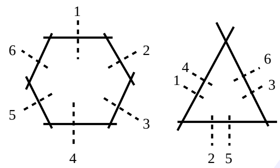
Example 3.9.

- *Types of singular fibers*: $I_6 + I_3 + 3I_1$

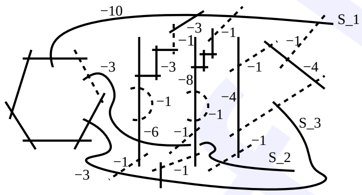
- *Pencils of cubics*



- *Sections*



- *Rational surfaces* $Z = \mathbb{CP}^2 \sharp 19 \overline{\mathbb{CP}}^2$



- *The exceptional divisors*

$C_{2,1} : \underset{-4}{\overset{1/2}{\circ}}$

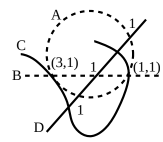
$C_{2,1} : \underset{-4}{\overset{1/2}{\circ}}$

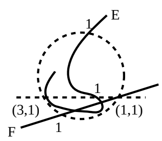
$C_{6,1} : \underset{-8}{\overset{5/6}{\circ}} - \underset{-2}{\overset{4/6}{\circ}} - \underset{-2}{\overset{3/6}{\circ}} - \underset{-2}{\overset{2/6}{\circ}} - \underset{-2}{\overset{1/6}{\circ}}$

$C_{11,4} : \underset{-3}{\overset{7/11}{\circ}} - \underset{-6}{\overset{10/11}{\circ}} - \underset{-2}{\overset{9/11}{\circ}} - \underset{-3}{\overset{8/11}{\circ}} - \underset{-2}{\overset{4/11}{\circ}}$

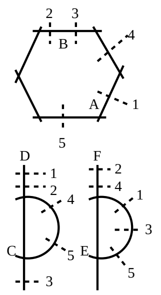
$C_{23,8} : \underset{-3}{\overset{13/23}{\circ}} - \underset{-10}{\overset{22/23}{\circ}} - \underset{-2}{\overset{21/23}{\circ}} - \underset{-2}{\overset{20/23}{\circ}} - \underset{-2}{\overset{19/23}{\circ}} - \underset{-2}{\overset{18/23}{\circ}} - \underset{-2}{\overset{17/23}{\circ}} - \underset{-3}{\overset{16/23}{\circ}} - \underset{-2}{\overset{8/23}{\circ}}$

Example 3.10.
- *Types of singular fibers*: $I_6 + 2I_2 + 2I_1$
- *Pencils of cubics*

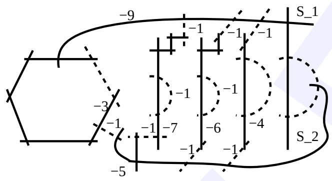




- *Sections*



- *Rational surfaces* $Z = \mathbb{CP}^2 \sharp 15 \overline{\mathbb{CP}}^2$



- *The exceptional divisors*

$$C_{2,1} : \overset{1/2}{\underset{-4}{\circ}}$$

$$C_{3,1} : \overset{2/3}{\underset{-5}{\circ}} - \overset{1/3}{\underset{-2}{\circ}}$$

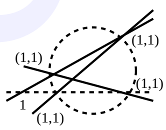
$$C_{4,1} : \overset{3/4}{\underset{-6}{\circ}} - \overset{2/4}{\underset{-2}{\circ}} - \overset{1/4}{\underset{-2}{\circ}}$$

$$C_{5,1} : \overset{4/5}{\underset{-7}{\circ}} - \overset{3/5}{\underset{-2}{\circ}} - \overset{2/5}{\underset{-2}{\circ}} - \overset{1/5}{\underset{-2}{\circ}}$$

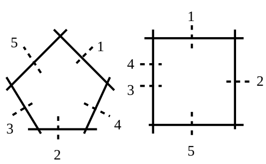
$$C_{13,7} : \overset{6/13}{\underset{-2}{\circ}} - \overset{12/13}{\underset{-9}{\circ}} - \overset{11/13}{\underset{-2}{\circ}} - \overset{10/13}{\underset{-2}{\circ}} - \overset{9/13}{\underset{-2}{\circ}} - \overset{8/13}{\underset{-2}{\circ}} - \overset{7/13}{\underset{-3}{\circ}}$$

Example 3.11. • *Types of singular fibers*: $I_5 + I_4 + 3I_1$

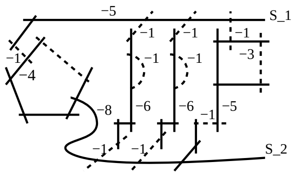
- *Pencils of cubics*



- *Sections*



- *Rational surfaces* $Z = \mathbb{CP}^2 \sharp 16 \overline{\mathbb{CP}}^2$



- *The exceptional divisors*

$$C_{3,1}: \overset{2/3}{\underset{-5}{\circ}} - \overset{1/3}{\underset{-2}{\circ}}$$

$$C_{4,1}: \overset{3/4}{\underset{-6}{\circ}} - \overset{2/4}{\underset{-2}{\circ}} - \overset{1/4}{\underset{-2}{\circ}}$$

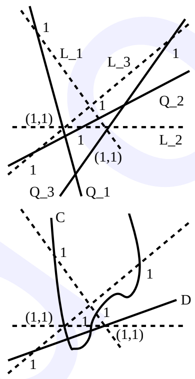
$$C_{4,1}: \overset{3/4}{\underset{-6}{\circ}} - \overset{2/4}{\underset{-2}{\circ}} - \overset{1/4}{\underset{-2}{\circ}}$$

$$C_{5,3}: \overset{2/5}{\underset{-2}{\circ}} - \overset{4/5}{\underset{-5}{\circ}} - \overset{3/5}{\underset{-3}{\circ}}$$

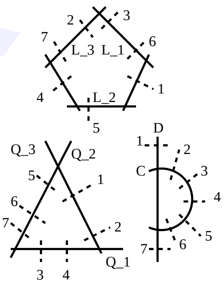
$$C_{16,5}: \overset{5/16}{\underset{-2}{\circ}} - \overset{10/16}{\underset{-2}{\circ}} - \overset{15/16}{\underset{-8}{\circ}} - \overset{14/16}{\underset{-2}{\circ}} - \overset{13/16}{\underset{-2}{\circ}} - \overset{12/16}{\underset{-2}{\circ}} - \overset{11/16}{\underset{-4}{\circ}}$$

Example 3.12. • *Types of singular fibers*: $I_5 + I_3 + I_2 + 2I_1$

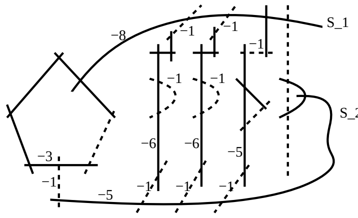
- *Pencils of cubics*



- *Sections*



- *Rational surfaces* $Z = \mathbb{CP}^2 \sharp 14 \overline{\mathbb{CP}}^2$



- *The exceptional divisors*

$$C_{3,1} : \overset{2/3}{\underset{-5}{\circ}} - \overset{1/3}{\underset{-2}{\circ}}$$

$$C_{3,1} : \overset{2/3}{\underset{-5}{\circ}} - \overset{1/3}{\underset{-2}{\circ}}$$

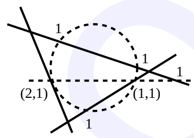
$$C_{4,1} : \overset{3/4}{\underset{-6}{\circ}} - \overset{2/4}{\underset{-2}{\circ}} - \overset{1/4}{\underset{-2}{\circ}}$$

$$C_{4,1} : \overset{3/4}{\underset{-6}{\circ}} - \overset{2/4}{\underset{-2}{\circ}} - \overset{1/4}{\underset{-2}{\circ}}$$

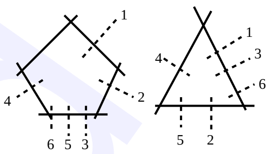
$$C_{11,6} : \overset{5/11}{\underset{-2}{\circ}} - \overset{10/11}{\underset{-8}{\circ}} - \overset{9/11}{\underset{-2}{\circ}} - \overset{8/11}{\underset{-2}{\circ}} - \overset{7/11}{\underset{-2}{\circ}} - \overset{6/11}{\underset{-3}{\circ}}$$

Example 3.13.

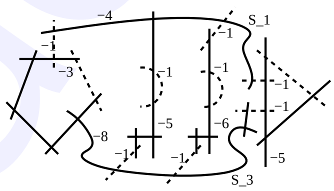
- *Types of singular fibers*: $I_5 + I_3 + 4I_1$
- *Pencils of cubics*



- *Sections*



- *Rational surfaces* $Z = \mathbb{CP}^2 \sharp 13 \overline{\mathbb{CP}}^2$



- *The exceptional divisors*

$$C_{3,1} : \overset{2/3}{\underset{-5}{\circ}} - \overset{1/3}{\underset{-2}{\circ}}$$

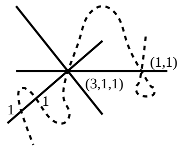
$$C_{4,1} : \overset{3/4}{\underset{-6}{\circ}} - \overset{2/4}{\underset{-2}{\circ}} - \overset{1/4}{\underset{-2}{\circ}}$$

$$C_{7,2} : \underset{-4}{\overset{5/7}{\circ}} - \underset{-5}{\overset{6/7}{\circ}} - \underset{-2}{\overset{4/7}{\circ}} - \underset{-2}{\overset{2/7}{\circ}}$$

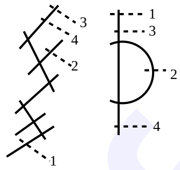
$$C_{11,6} : \underset{-2}{\overset{5/11}{\circ}} - \underset{-8}{\overset{10/11}{\circ}} - \underset{-2}{\overset{9/11}{\circ}} - \underset{-2}{\overset{8/11}{\circ}} - \underset{-2}{\overset{7/11}{\circ}} - \underset{-3}{\overset{6/11}{\circ}}$$

Example 3.14.

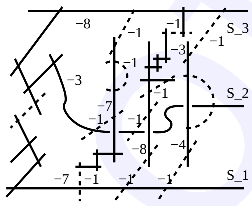
- *Types of singular fibers*: $I_2^* + I_2 + 2I_1$
- *Pencils of cubics*



- *Sections*



- *Rational surfaces* $Z = \mathbb{CP}^2 \sharp 19 \overline{\mathbb{CP}}^2$



- *The exceptional divisors*

$$C_{2,1} : \underset{-4}{\overset{1/2}{\circ}}$$

$$C_{5,1} : \underset{-7}{\overset{4/5}{\circ}} - \underset{-2}{\overset{3/5}{\circ}} - \underset{-2}{\overset{2/5}{\circ}} - \underset{-2}{\overset{1/5}{\circ}}$$

$$C_{5,1} : \underset{-7}{\overset{4/5}{\circ}} - \underset{-2}{\overset{3/5}{\circ}} - \underset{-2}{\overset{2/5}{\circ}} - \underset{-2}{\overset{1/5}{\circ}}$$

$$C_{11,6} : \underset{-2}{\overset{5/11}{\circ}} - \underset{-8}{\overset{10/11}{\circ}} - \underset{-2}{\overset{9/11}{\circ}} - \underset{-2}{\overset{8/11}{\circ}} - \underset{-2}{\overset{7/11}{\circ}} - \underset{-3}{\overset{6/11}{\circ}}$$

$$C_{11,6} : \underset{-2}{\overset{5/11}{\circ}} - \underset{-8}{\overset{10/11}{\circ}} - \underset{-2}{\overset{9/11}{\circ}} - \underset{-2}{\overset{8/11}{\circ}} - \underset{-2}{\overset{7/11}{\circ}} - \underset{-3}{\overset{6/11}{\circ}}$$

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