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CENTROIDS AND SOME CHARACTERIZATIONS OF CATENARIES

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Abstract. For every interval $[a, b]$, we denote by $(\bar{x}_A, \bar{y}_A)$ and $(\bar{x}_L, \bar{y}_L)$ the geometric centroid of the area under a catenary $y = k\cosh((x-c)/k)$ defined on this interval and the centroid of the curve itself, respectively. Then, it is well-known that $\bar{x}_L = \bar{x}_A$ and $\bar{y}_L = 2\bar{y}_A$.

In this paper, we show that one of $\bar{x}_L = \bar{x}_A$ and $\bar{y}_L = 2\bar{y}_A$ characterizes the family of catenaries among nonconstant C^2 functions. Furthermore, we show that among nonconstant and nonlinear C^2 functions, $\bar{y}_L/\bar{x}_L = 2\bar{y}_A/\bar{x}_A$ is also a characteristic property of catenaries.

1. Introduction

A well-known property of the catenary $y = k\cosh((x-c)/k), k > 0$ is that the ratio of the area under the curve to the arc length of the curve is independent of the interval over which these quantities are concurrently measured. For a positive C^1 function $y(x)$ defined on an interval I and an interval $[a, b] \subset I$, we consider the area $A(a, b)$ over the interval $[a, b]$ and the arc length $L(a, b)$ of the graph of $y(x)$. Then, the catenary $y = k\cosh((x-c)/k), k > 0$ satisfies for every interval $[a, b] \subset I$, $A(a, b) = kL(a, b)$. This property characterizes the family of catenaries $y = k\cosh((x-c)/k)$ among nonconstant C^2 functions ([11]). Thus, we have the following.

Proposition 1.1. *For a nonconstant positive C^2 function $y(x)$ defined on an interval I, the followings are equivalent.*

1. *There exists a positive constant k such that for every interval $[a, b] \subset I$, $A(a, b) = kL(a, b)$.*
2. *The function $y(x)$ satisfies $y(x) = k\sqrt{1 + y'(x)^2}$, where k is a positive constant.*

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(3) *For some* $k > 0$ *and* $c \in \mathbb{R}$,

$$y(x) = k \cosh \left(\frac{x-c}{k} \right).$$

Two higher dimensional generalizations of Proposition 1.1 were established in [1]. For a positive C^1 function $y(x)$ defined on an interval I and an interval $[a, b] \subset I$, we denote by $(\bar{x}_A, \bar{y}_A) = (\bar{x}_A(a, b), \bar{y}_A(a, b))$ and $(\bar{x}_L, \bar{y}_L) = (\bar{x}_L(a, b), \bar{y}_L(a, b))$ the geometric centroid of the area under the graph of $y(x)$ defined on this interval and the centroid of the graph itself, respectively. Then, for a catenary we have the following([11]).

Proposition 1.2. *A catenary* $y = k \cosh((x - c)/k)$ *satisfies the following.*

(1) *For every interval* $[a, b] \subset I$, $\bar{x}_L(a, b) = \bar{x}_A(a, b)$.
(2) *For every interval* $[a, b] \subset I$, $\bar{y}_L(a, b) = 2\bar{y}_A(a, b)$.

In this paper, first of all, in Section 2 we establish the following characterization theorem for catenaries.

Theorem 1.3. *For a nonconstant positive* C^2 *function* $y(x)$ *defined on an interval* I*, the followings are equivalent.*

(1) *For every interval* $[a, b] \subset I$, $\bar{x}_L(a, b) = \bar{x}_A(a, b)$.
(2) *For every interval* $[a, b] \subset I$, $\bar{y}_L(a, b) = 2\bar{y}_A(a, b)$.
(3) *For some* $k > 0$ *and* $c \in \mathbb{R}$,

$$y(x) = k \cosh \left(\frac{x-c}{k} \right).$$

In Section 3, we prove the following characterization theorem for catenaries.

Theorem 1.4. *For a nonconstant and nonlinear positive* C^2 *function* $y(x)$ *defined on an interval* I*, the followings are equivalent.*

(1) *For every interval* $[a, b] \subset I$,

$$\frac{\bar{y}_L}{\bar{x}_L} = 2\frac{\bar{y}_A}{\bar{x}_A}.$$

(2) *For some* $k > 0$ *and* $c \in \mathbb{R}$,

$$y(x) = k \cosh \left(\frac{x-c}{k} \right).$$

In order to find the centroid of polygons, see [3]. For the perimeter centroid of a polygon, we refer to [2]. In [9], mathematical definitions of centroid of planar bounded domains were given. For various centroids of higher dimensional simplexes, see [10]. The relationships between various centroids of a quadrangle were given in [5, 8]

Archimedes proved the area properties of parabolic sections and then formulated the centroid of parabolic sections ([12]). Some characterizations of parabolas using these properties were given in [4, 6, 7].

2. Proof of Theorem 1.3

In this section, we prove Theorem 1.3 stated in Section 1.

Suppose that a nonconstant positive C^2 function $y(x)$ defined on an interval I satisfies $\bar{x}_L(a,b) = \bar{x}_A(a,b)$. Then for all $a, b \in I$ with $a < b$, we have

$$\int_a^b y(x)dx \int_a^b xw(x)dx = \int_a^b w(x)dx \int_a^b xy(x)dx, \tag{2.1}$$

where $w(x)$ is a function defined by $w(x) = \sqrt{1 + y'(x)^2}$. Note that (2.1) is valid for all $a, b \in I$.

By differentiating (2.1) with respect to the variable b, the fundamental theorem of calculus gives

$$y(b)\int_a^b xw(x)dx + bw(b)\int_a^b y(x)dx = w(b)\int_a^b xy(x)dx + by(b)\int_a^b w(x)dx. \tag{2.2}$$

With respect to a, we differentiate (2.2) to have

$$y(b)aw(a) + bw(b)y(a) = w(b)ay(a) + by(b)w(a), \tag{2.3}$$

which shows that for all $a, b \in I$

$$(b - a)\{y(b)w(a) - y(a)w(b)\} = 0. \tag{2.4}$$

It follows from (2.4) that on the interval I, $y(x)/w(x)$ is a constant k. That is, the function $y(x)$ satisfies $y(x) = k\sqrt{1 + y'(x)^2}$. Hence, Proposition 1.1 implies that (1) $\Rightarrow$ (3).

Now, suppose that a nonconstant positive C^2 function $y(x)$ defined on an interval I satisfies $\bar{y}_L(a,b) = \bar{y}_A(a,b)$. Then for all $a, b \in I$ with $a < b$, we have

$$\int_a^b y(x)dx \int_a^b y(x)w(x)dx = \int_a^b w(x)dx \int_a^b y(x)^2dx, \tag{2.5}$$

where $w(x) = \sqrt{1 + y'(x)^2}$. Note that (2.5) is valid for all $a, b \in I$.

Differentiating (2.5) with respect to b gives

$$\begin{aligned} y(b)\int_a^b y(x)w(x)dx + y(b)w(b)\int_a^b y(x)dx \\ = w(b)\int_a^b y(x)^2dx + y(b)^2\int_a^b w(x)dx. \end{aligned} \tag{2.6}$$

We differentiate (2.6) with respect to a. Then we have

$$y(b)y(a)w(a) + y(b)w(b)y(a) = w(b)y(a)^2 + y(b)^2w(a), \tag{2.7}$$

from which for all $a, b \in I$ we get

$$\{y(b) - y(a)\}\{y(b)w(a) - y(a)w(b)\} = 0. \tag{2.8}$$

We fix a point $a_0 \in I$. Then we have from (2.8)

$$\{y(b) - y(a_0)\}\{y(b)w(a_0) - y(a_0)w(b)\} = 0. \tag{2.9}$$

Let us denote by J the open set given by

$$J = \{b \in I \mid y(b)w(a_0) - y(a_0)w(b) \neq 0\}.$$

We divide by two cases as follows.

Case 1. $J = \phi$**.** In this case, from the definition of the open set J we get

$$\frac{y(b)}{w(b)} = \frac{y(a_0)}{w(a_0)}(= k). \tag{2.10}$$

Hence, the function $y(x)$ satisfies $y(x) = k\sqrt{1 + y'(x)^2}$. Thus, Proposition 1.1 implies that the function $y(x)$ is a catenary.

Case 2. $J \neq \phi$**.** In this case, it follows from (2.9) that for all $x \in J$, $y(x) = y(a_0)$. We let $k = y(a_0)$ and fix a point $x_0 \in J$. We denote by $K = (x_1, x_2)$ the maximal open interval containing x_0 such that $y(x) = k$. If the maximal interval K satisfies $K = I$, then the function $y(x)$ is a constant function. This contradiction shows that $K \neq I$, and hence one of x_1 and x_2 belongs to I. Thus we may assume that $x_2 \in I$. For a sufficiently small $\epsilon > 0$, the interval $(x_2, x_2 + \epsilon)$ does not intersect J. Hence, it follows from (2.8) with $a = x_2$ that for all $b \in (x_2, x_2 + \epsilon)$

$$\frac{y(b)}{w(b)} = \frac{y(x_2)}{w(x_2)}(= k), \tag{2.11}$$

where we use $y(x_2) = k, y'(x_2) = 0$ and $w(x_2) = 1$. Hence, on the interval $(x_2, x_2 + \epsilon)$ the function $y(x)$ satisfies $y(x) = k\sqrt{1 + y'(x)^2}$. Together with (2.11) and Proposition 1.1, the maximality of K shows that

$$y(x) = \begin{cases} k, & \text{if } x \in (x_1, x_2], \\ k\cosh\left(\frac{x - x_2}{k}\right), & \text{if } x \in (x_2, x_2 + \epsilon). \end{cases} \tag{2.12}$$

This yields that the function $y(x)$ cannot be C^2, a contradiction.

Summarizing the above cases, we see that (2) $\Rightarrow$ (3).

Conversely, it follows from Proposition 1.2 that (3) $\Rightarrow$ (1) and (2). This completes the proof of Theorem 1.3.

3. Proof of Theorem 1.4

In this section, we prove Theorem 1.4 stated in Section 1.

Suppose that a nonconstant positive C^2 function $y(x)$ defined on an interval I satisfies $\bar{y}_L/\bar{x}_L = 2\bar{y}_A/\bar{x}_A$. Then for all $a, b \in I$ with $a < b$, we have

$$\int_a^b xy(x)dx \int_a^b y(x)w(x)dx = \int_a^b xw(x)dx \int_a^b y(x)^2 dx, \tag{3.1}$$

where $w(x) = \sqrt{1 + y'(x)^2}$. Note that (2.1) is valid for all $a, b \in I$.

By differentiating (3.1) with respect to b, we obtain

$$\begin{aligned} & by(b)\int_a^b y(x)w(x)dx + y(b)w(b)\int_a^b xy(x)dx \\ & = bw(b)\int_a^b y(x)^2dx + y(b)^2\int_a^b xw(x)dx. \end{aligned} \tag{3.2}$$

We differentiate (3.2) with respect to a. Then we have

$$by(b)y(a)w(a) + y(b)w(b)ay(a) = bw(b)y(a)^2 + y(b)^2aw(a), \tag{3.3}$$

which shows that for all $a, b \in I$

$$\{by(a) - ay(b)\}\{y(b)w(a) - y(a)w(b)\} = 0. \tag{3.4}$$

We fix a nonzero $a_0 \in I$. Then we get from (3.4)

$$\{by(a_0) - a_0y(b)\}\{y(b)w(a_0) - y(a_0)w(b)\} = 0. \tag{3.5}$$

Putting $J = \{b \in I \,|\, y(b)w(a_0) - y(a_0)w(b) \neq 0\}$, we divide by three cases as follows.

Case 1. $J = \phi$. In this case, from the definition of the open set J we get

$$\frac{y(b)}{w(b)} = \frac{y(a_0)}{w(a_0)}(= k). \tag{3.6}$$

Hence, the function $y(x)$ satisfies $y(x) = k\sqrt{1 + y'(x)^2}$. Thus, Proposition 1.1 implies that the function $y(x)$ is a catenary.

Case 2. $J = I$. In this case, from (3.5) we get

$$y(x) = kx, \quad k = \frac{y(a_0)}{a_0}, \tag{3.7}$$

which leads a contradiction.

Case 3. $J \neq \phi$ and $J \neq I$. In this case, it follows from (3.5) that for all $x \in J$, $y(x) = kx$, where $k = y(a_0)/a_0$. We fix a point $x_0 \in J$ and denote by $K = (x_1, x_2)$ the maximal open interval containing x_0 such that $y(x) = kx$. If the maximal interval K satisfies $K = I$, then the function $y(x)$ is a linear function. This contradiction yields $K \neq I$, and hence one of x_1 and x_2 belongs to I. Thus we may assume that $x_2 \in I$. For a sufficiently small $\epsilon > 0$, the interval $(x_2, x_2 + \epsilon)$ does not intersect J. Hence, it follows from (3.4) with $a = x_2$ that for all $b \in (x_2, x_2 + \epsilon)$

$$\frac{y(b)}{w(b)} = \frac{y(x_2)}{w(x_2)}\,(= l)\,, l = \frac{kx_2}{\sqrt{1 + k^2}}, \tag{3.8}$$

where we use $y(x_2) = kx_2, y'(x_2) = k$ and $w(x_2) = \sqrt{1 + k^2}$. Hence, on the interval $(x_2, x_2 + \epsilon)$ the function $y(x)$ satisfies $y(x) = l\sqrt{1 + y'(x)^2}$. Therefore, Proposition 1.1 shows that

$$y(x) = \begin{cases} kx, & \text{if } x \in (x_1, x_2], \\ l \quad \text{or} \quad l\cosh\left(\frac{x - x_2}{l}\right), & \text{if } x \in (x_2, x_2 + \epsilon), \end{cases} \tag{3.9}$$

which cannot be a C^2 function.

Due to the above three cases, we see that (1) $\Rightarrow$ (2).

Conversely, it follows from Proposition 1.2 that (2) $\Rightarrow$ (1). This completes the proof of Theorem 1.4.

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